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ON THE SINGULAR BEHAVIOUR OF THE TITCHMARSH-WEYL *m*-FUNCTION FOR THE PERTURBED HILL'S EQUATION ON THE LINE

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ABSTRACT. For the perturbed Hill's equation on the line,

$$-\frac{d^2y}{dx^2} + [P(x) + V(x)]y = Ey, \quad -\infty < x < \infty,$$

we study the behaviour of the matrix m-function at the spectral gap endpoints. In particular, we extend the result of Hinton, Klaus and Shaw that E_n, a gap endpoint, is a half-bound state (HBS) if and only if $(E - E_n)^{\frac{1}{2}} m(E)$ approaches a nonzero constant as $E \longrightarrow E_n$, to the present case.

1. Introduction.

In this short note we study the behaviour of the Titchmarsh-Weyl m-function for the equation

$$-\frac{d^2y}{dx^2} + [P(x) + V(x)] = Ey, \quad -\infty < x < \infty. \tag{1.1}$$

Under the assumption that $P(x)$ and $V(x)$ are real-valued potentials with $P(x) \in L_1([0,1])$, $P(x+1) = P(x)$ and

$$\int_{-\infty}^{\infty} (1 + |x|)|V(x)|\, dx < \infty,$$

the spectrum of the operator H induced by (1.1) on $L_2(\mathbf{R})$ is well known. In particular, it consists of an absolutely continuous part which is the union of closed intervals of type $[E_{2n}, E_{2n+1}]$, $-\infty < E_0 < E_1 \leq E_2 < E_3 \cdots$ and may have at most a finite number of eigenvalues in any of the spectral gaps (E_{2n+1}, E_{2n+2}). Information about eigenvalues of H is readily available in the literature (see [10] for example).

Our concern in this article is the Titchmarsh-Weyl m-function associated to (1.1), in particular its behaviour at the spectral gap endpoints. Specifically, we extend the four-part m-function spectral characterization of Hinton and Shaw [9] to the case when a spectral point E_n is a so-called half-bound state (HBS), by which we mean that the equation $Hy = E_n y$ has a nontrivial bounded solution which is not square integrable.

The problem we study here has been studied by Hinton, Klaus and Shaw [7] for the operator H restricted to $L_2\big([0,\infty)\big)$, and as such our result here is an extension of that paper. Similar results have been obtained in [8] and [1] for the case where $P(x) \equiv 0$ in the Dirac counterpart of (1.1) as well as for the periodic Dirac case [2] on $[0,\infty)$. The methods used in all the above-mentioned papers are similar, and we continue in the same

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spirit in the present article. As a result, we shall only provide outlines of our proofs and refer the reader accordingly for details; in particular, we rely heavily on the analysis of [4]. Let us point out that the analysis presented here also works for the Dirac System, in view of [3].

This paper is organized as follows. In the next section we introduce all the pertinent solutions of (1.1), relabel the spectral parameter by the so-called quasimomentum, and express the *m*-function in terms of Jost-type functions. Then in Section 3 we present the asymptotic behaviour of the *m*-function, which we obtain via the asymptotic behaviour of our Jost-type functions.

2. **Preliminaries.** To begin with, we want to regard (1.1) as a perturbation of the equation

$$-\frac{d^2y}{dx^2} + P(x)y = Ey, \quad -\infty < x < \infty, \tag{2.1}$$

with $P(x)$ as in (1.1). Now, let $\phi_0(x, E)$ and $\theta_0(x, E)$ be the solutions of (2.1) satisfying the conditions

$$\theta_0(0, E) = \theta_0'(0, E) = 1 \quad \text{and} \quad \phi_0(0, E) = \phi_0'(0, E) = 0. \tag{2.2}$$

Further denote $\phi_0(E) = \phi_0(1, E)$, $\theta_0(E) = \theta_0(1, E)$, and recall the definition of the quasi-momentum k [6]:

$$k = k(E) = \cos^{-1}[\triangle(E)], \tag{2.3}$$

where $\triangle(E) = \frac{1}{2}[\phi_0'(E) + \theta_0(E)]$. The properties of k are well documented in [6] and recaptured in [4]. In the sequel, our spectral parameter will be k, and hence we shall write $\phi_0(x, k)$ in place of $\phi_0(x, E)$, *etc*.

Next, let us recall that the *m*-functions $m \pm (k)$ associated with (1.1) are defined by

$$m \pm (k) = \lim_{x \to \pm\infty} -\frac{\theta(x, k)}{\phi(x, k)}, \tag{2.4}$$

where $\theta(x, k)$ and $\phi(x, k)$ are solutions of (1.1) satisfying condition (2.2), with a similar definition for $m_0 \pm (k)$ associated with (2.1). Then we know from the Titchmarsh-Weyl theory that for $\Im k > 0$, we have that

$$\psi_0^+(x, k) \equiv \theta_0(x, k) + m_0^+(k)\phi_0(x, k) \in L_2(0, \infty), \tag{2.5}$$

$$\psi_0^-(x, k) \equiv \theta_0(x, k) + m_0^-(k)\phi_0(x, k) \in L_2(-\infty, 0). \tag{2.6}$$

Further, the Floquet theory provides us with functions $\xi^{\pm}(x, k)$ with $\xi^{\pm}(x+1, k) = \xi^{\pm}(x, k)$, $\xi^{\pm}(0, k) = 1$, such that

$$\psi_0^{\pm}(x, k) = \xi_0^{\pm}(x, k)e^{\pm ikx}. \tag{2.7}$$

From (2.3), (2.5)–(2.7), we arrive at

$$[\psi_0^+(\cdot, k); \psi_0^-(\cdot, k)] = m_0^-(k) - m_0^+(k) = -\frac{2i \sin k}{\phi(k)}, \tag{2.8}$$

where $[f(\cdot); g(\cdot)]$ denotes the Wronskian of $f(\cdot)$ and $g(\cdot)$. In addition to the solutions $\theta(x,k)$ and $\phi(x,k)$ introduced above, we also have the Jost solutions $F^{\pm}(x,k)$ of (1.1), which are defined by the integral equations

$$F^+(x,k) = \psi_0^+(x,k) - \int_x^\infty A(x,t;k)V(t)F^+(t,k)\,dt, \tag{2.9}$$

$$F^-(x,k) = \psi_0^-(x,k) + \int_{-\infty}^x A(x,t;k)V(t)F^-(t,k)\,dt, \tag{2.10}$$

where

$$A(x,t;k) \equiv -[\psi_0^+(\cdot,k);\psi_0^-(\cdot,k)]^{-1}[\psi_0^+(x,k)\psi_0^-(t,k) - \psi_0^-(x,k)\psi_0^+(t,k)]. \tag{2.11}$$

Let us define the following functions, which we call Jost functions. For any solution y of (1.1) we define

$$F_y^+(k) = \left(-m_0^+(k), 1\right)\begin{pmatrix} y(0,k) \\ y'(0,k)\end{pmatrix} + \int_0^\infty \psi_0^+(t,k)V(t)y(t,k)\,dt. \tag{2.12}$$

and

$$F_y^-(k) = \left(-m_0^-(k), 1\right)\begin{pmatrix} y(0,k) \\ y'(0,k)\end{pmatrix} + \int_0^\infty \psi_0^-(t,k)V(t)y(t,k)\,dt. \tag{2.13}$$

It is then a straightforward exercise (see [9]) to show that

$$y(x,k) = \frac{\xi_0^+(x,k)e^{ikx}}{m_0^-(k) - m_0^+(k)}[F_y^+(k) + o(1)] \quad \text{as } x \longrightarrow +\infty \tag{2.14}$$

and

$$y(x,k) = \frac{\xi_0^-(x,k)e^{-ikx}}{m_0^-(k) - m_0^+(k)}[F_y^-(k) + o(1)] \quad \text{as } x \longrightarrow -\infty. \tag{2.15}$$

In view of (2.4), we therefore arrive at the m-function representations

$$m^+(k) = -\frac{F_\theta^+(k)}{F_\phi^+(k)} \quad \text{and} \quad m^-(k) = -\frac{F_\theta^-(k)}{F_\phi^-(k)}. \tag{2.16}$$

Recalling that the whole-line m-function for (1.1) is (suppressing the k-dependence)

$$M(k) = (m^- - m^+)^{-1}\begin{pmatrix} 1 & \frac{1}{2}(m^- + m^+) \\ \frac{1}{2}(m^- + m^+) & m^- + m^+ \end{pmatrix},$$

we therefore arrive at the representation, by (2.16),

$$M(k) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{pmatrix}, \tag{2.17}$$

where $m_{11} = \frac{F_\phi^+(k)F_\phi^-(k)}{F(k)}$, $m_{22} = \frac{F_\theta^+(k)F_\theta^-(k)}{F(k)}$ and $m_{12} = m_{21} = \frac{F_\theta^+(k)F_\phi^-(k)+F_\phi^+(k)F_\theta^-(k)}{2F(k)}$ with $F(k) \equiv F_\phi^+(k)F_\phi^-(k) - F_\theta^+(k)F_\phi^-(k)$. It is easy to check that

$$F(k) = [F^+(\cdot,k); F^-(\cdot,k)], \tag{2.18}$$

$$F_y^+(k) = [F^+(\cdot,k); y(\cdot,k)] \text{ and } F_y^-(k) = [F^-(\cdot,k); y(\cdot,k)]. \tag{2.19}$$

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3. **Asymptotic behaviour of $M(E)$.** The asymptotic behaviour of the m-function at the gap endpoints k_n, which is our aim in this note, is now easily deduced from that of the Jost-type functions $F^{\pm}_{\phi}(k), F^{\pm}_{\theta}(k)$ and $F(k)$.

First, let us note that the numerators in the expression for $M(k)$, (2.17), do not simultaneously vanish at $k = k_n$. This is due to the well-known [5] behaviour of the solutions of (1.1) at $k = k_n$, in particular that one solution is bounded while another is unbounded, and the following lemma, whose proof we omit.

LEMMA 1 (SEE [4] LEMMA (2.1)). *Let $Z(x, k_n)$ be a solution of (1.1) for $k = k_n$. Then $Z(x, k_n)$ is bounded for $x \geq 0$ (resp., $x \leq 0$) if and only if $F^{+}_{z}(k_n) = 0$ (resp., $F^{-}_{z}(k_n) = 0$).*

In particular, Lemma 1 tells us, since $\phi(x, k_n)$ and $\theta(x, k_n)$ cannot be simultaneously bounded as either $x \longrightarrow +\infty$ or $x \longrightarrow -\infty$, that the pairs $\big(F^{+}_{\theta}(k_n), F^{+}_{\phi}(k_n)\big)$, and $\big(F^{-}_{\theta}(k_n), F^{-}_{\phi}(k_n)\big)$ are non-vanishing.

It therefore only remains to compute the asymptotic behaviour of $F(k)$ as $k \longrightarrow k_n$. In the case we do not have a HBS at $k = k_n$, then $F^{+}(x, k_n)$ and $F^{-}(x, k_n)$ are linearly independent and hence, by (2.18), $F(k_n)$ is nonzero. Therefore in this case $M(k)$ approaches a nonzero constant matrix as $k \longrightarrow k_n$.

In case we have a HBS at $k = k_n$, so that there is a constant a_n with $F^{+}(x, k_n) = a_n F^{-}(x, k_n)$, we proceed as follows. Define a solution $z(x, k)$ by

$$z(x, k) = F^{+}(0, k_n)\theta(x, k) + F^{+\prime}(0, k_n)\phi(x, k), \tag{3.1}$$

where we assume, without loss, that $F^{+}(0, k_n) \neq 0$. It is then a straightforward calculation to arrive at the identity

$$F^{+}(0, k_n)[F^{+}(\cdot, k); F^{-}(\cdot, k)] = F^{-}(0, k)[F^{+}(\cdot, k); z(\cdot, k)] - F^{+}(0, k)[F^{-}(\cdot, k); z(\cdot, k)]. \tag{3.2}$$

Using (2.19) and (3.1), we easily arrive at the identities

$$[F^{+}(\cdot, k); z(\cdot, k)] = -m^{+}_{0}(k)F^{+}(0, k_n) + F^{+\prime}(0, k_n) + \int_{0}^{\infty} \psi^{+}_{0}(t, k)V(t)z(t, k)\, dt$$

and

$$[F^{-}(\cdot, k); z(\cdot, k)] = -m^{-}_{0}(k)F^{+}(0, k_n) + F^{+\prime}(0, k_n) + \int_{-\infty}^{0} \psi^{-}_{0}(t, k)V(t)z(t, k)\, dt.$$

Writing, in the preceding identities,

$$\begin{aligned} \psi^{\pm}_{0}(t, k_n)V(t)z(t, k) = \psi^{\pm}_{0}(t, k_n)V(t)z(t, k) + [\psi^{\pm}_{0}(t, k) - \psi^{\pm}_{0}(t, k_n)]V(t)z(t, k_n) \\ + \psi^{\pm}_{0}(t, k)V(t)[z(t, k) - z(t, k_n)] \end{aligned}$$

and using standard bounds on the bracketed terms as well as the boundedness of $z(t, k_n)$, we finally obtain (see [4] for details, and [3] for the Dirac case), as $k \longrightarrow k_n$ through real values,

$$[F^{+}(\cdot, k); z(\cdot, k)] = (-1)^{n+1} i[\phi_0(k_n)]^{-1}(k - k_n) + o(k - k_n) \tag{3.3}$$

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and

$$[F^{-}(\cdot,k); z(\cdot,k)] = (-1)^n a_n i[\phi_0(k_n)]^{-1}(k-k_n) + o(k-k_n). \tag{3.4}$$

Combining (3.2)–(3.4) we hence obtain that as $k \longrightarrow k_n$ through real values,

$$F(k) = \frac{(-1)^{n+1} i(a_n^2+1)}{\phi_0(k_n)a_n}(k-k_n) + o(k-k_n). \tag{3.5}$$

To extend the validity of (3.5) to complex values, we note the bound

$$|F^{\pm}(x,k)| \leq Ce^{\mp\Im(k-k_n)x}(1+\max\{\mp x, 0\}), \tag{3.6}$$

which follows from (2.9), (2.10) and the bound

$$|A(x,t)| \leq Ce^{\mp\Im(k-k_n)x}(1+|x-t|).$$

In view of (3.6) and (2.18), we may therefore apply the Phragmen-Lindelöf theorem to conclude validity of (3.5) in the sector

$$0 \leq \arg(k-k_n) \leq \pi.$$

Before we summarise our considerations in the form of a theorem, let us note that (2.3), by simple expansion, yields an analytic function $g(k)$ which does not vanish at $k = k_n$ such that

$$E - E_n = g(k_n)(k-k_n)^2 \quad \text{as } E \longrightarrow E_n.$$

We therefore have the following result.

THEOREM 2. *The point $E = E_n$ is an* HBS *if and only if there exists a non-zero constant matrix C_n such that*

$$\lim_{E\longrightarrow E_n}(E-E_n)^{\frac{1}{2}}M(E) = C_n.$$

Moreover, if E_n is not an HBS, *then $M(E)$ approaches a nonzero constant matrix as $E \longrightarrow E_n$.*

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