Proceedings of the Edinburgh Mathematical Society (2024) **67**, 1196–1211
doi:10.1017/S0013091524000531

SECOND HANKEL DETERMINANT OF LOGARITHMIC COEFFICIENTS OF INVERSE STRONGLY STARLIKE FUNCTIONS

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(Received 6 June 2023)

Abstract The sharp bound of the second Hankel determinant of logarithmic coefficients of inverse functions of strongly starlike functions is computed.

Keywords: univalent function; starlike function; strongly starlike function; inverse function; logarithmic coefficients; Hankel determinant

2010 Mathematics subject classification: Primary 30C50
Secondary 30C45

1. Introduction

For $r > 0$, let $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$, $\mathbb{D} := \mathbb{D}_1$, $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leqslant 1\}$ and let $\mathbb{R}T := \{z \in \mathbb{C} : |z| = 1\}$. Let $\mathcal{H}(\mathbb{D}_r)$ denote the class of all analytic functions f in $\mathbb{D}_r$ and let $\mathcal{H} := \mathcal{H}(\mathbb{D})$. Then $f \in \mathcal{H}(\mathbb{D}_r)$ has the following representation

$$f(z) = \sum_{n=0}^{\infty} a_n(f) z^n, \quad z \in \mathbb{D}_r. \tag{1.1}$$

Let $\mathcal{A}(\mathbb{D}_r)$ be the subclass of $\mathcal{H}(\mathbb{D}_r)$ of all f normalized by $f(0) = 0 = f'(0) - 1$ and let $\mathcal{A} := \mathcal{A}(\mathbb{D})$. By $\mathcal{S}$ we denote the subclass of all univalent (i.e. analytic and injective in $\mathbb{D}$) functions in $\mathcal{A}$.

Given $\alpha \in (0, 1]$, let $\mathcal{S}^*_\alpha$ denote class of all functions $f \in \mathcal{A}$ such that

$$\left| \operatorname{Arg} \frac{zf'(z)}{f(z)} \right| < \alpha \frac{\pi}{2}, \quad z \in \mathbb{D}, \tag{1.2}$$

and the so-called *strongly starlike of order* α. For $\alpha := 1$ the class $\mathcal{S}^*_1 =: \mathcal{S}^*$ is the well-known class of *starlike functions*, i.e. functions f which map univalently $\mathbb{D}$ onto a set which is star-shaped with respect to the origin. Then, the condition (1.2) can be written as

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in \mathbb{D}.$$

The class of strongly starlike functions was introduced by Stankiewicz [18] and [19] and independently by Brannan and Kirwan [1] (see also [6, Vol. I, pp. 137–142]). Stankiewicz [19] presented an external geometrical characterization of strongly starlike functions. Brannan and Kirwan found a geometrical condition called *δ-visibility* which is sufficient for functions to be strongly starlike. In turn, Ma and Minda [15] gave the internal characterization of functions in $\mathcal{S}^*_\alpha$ basing on the concept of k-starlike domains. Further results regarding the geometry of strongly starlike functions were presented in [13, Chapter IV], [14] and [20]. Since $\mathcal{S}^* \subset \mathcal{S}$ (cf. [5, pp. 40–41]) and $\mathcal{S}^*_\alpha \subset \mathcal{S}^*$ for every $\alpha \in (0,1]$, it follows that $\mathcal{S}^*_\alpha \subset \mathcal{S}$ for every $\alpha \in (0,1]$.

If $f \in \mathcal{S}$, then the inverse function $F := f^{-1}$ is well-defined and analytic in $\mathbb{D}_{r(f)}$, where $r(f) := \sup(\{r > 0 : \mathbb{D}_r \subset f(\mathbb{D})\})$. Thus

$$F(w) = w + \sum_{n=2}^{\infty} A_n w^n, \quad w \in \mathbb{D}_{r(f)}, \tag{1.3}$$

where $A_n := a_n(F)$. By Koebe one-quarter theorem (e.g. [5, p. 31]), it follows that $r(f) \geqslant 1/4$ for every $f \in \mathcal{S}$.

For $f \in \mathcal{S}$ define

$$F_f(z) := \frac{1}{2} \log \frac{f(z)}{z} = \sum_{n=1}^{\infty} \gamma_n z^n, \quad z \in \mathbb{D},$$

a logarithmic function associated with f. The numbers $\gamma_n := a_n(F_f)$ are called the *logarithmic coefficients* of f. It is well-known that the logarithmic coefficients play a crucial role in Milin's conjecture (see [16], [5, p. 155]).

Referring to the above idea, for $f \in \mathcal{S}$, there exists the unique function $F_{f^{-1}}$ analytic in $\mathbb{D}_{r(f)}$ such that

$$F_{f^{-1}}(w) := \frac{1}{2} \log \frac{f^{-1}(w)}{w} = \sum_{n=1}^{\infty} \Gamma_n w^n, \quad w \in \mathbb{D}_{r(f)}, \tag{1.4}$$

where $\Gamma_n := a_n\left(F_{f^{-1}}\right)$ are logarithmic coefficients of the inverse function f^{-1}.

It follows from Equation (1.3) that (e.g. [6, Vol. I, p. 57])

$$A_2 = -a_2, \quad A_3 = -a_3 + 2a_2^2 \quad \text{and} \quad A_4 = -a_4 + 5a_2a_3 - 5a_2^3, \tag{1.5}$$

where $a_n := a_n(f)$. Thus from Equation (1.4) we derive that

$$\Gamma_1 = \frac{1}{2}A_2, \quad \Gamma_2 = \frac{1}{2}A_3 - \frac{1}{4}A_2^2, \quad \Gamma_3 = \frac{1}{2}A_4 - \frac{1}{2}A_2A_3 + \frac{1}{6}A_2^3,$$

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and next using Equation (1.5) we obtain

$$\Gamma_1 = -\frac{1}{2}a_2, \quad \Gamma_2 = -\frac{1}{2}a_3 + \frac{3}{4}a_2^2 \quad \text{and} \quad \Gamma_3 = -\frac{1}{2}a_4 + 2a_2a_3 - \frac{5}{3}a_2^3. \tag{1.6}$$

For $q, n \in \mathbb{N}$, the Hankel matrix $H_{q,n}(f)$ of $f \in \mathcal{A}$ of the form (1.1) is defined as

$$H_{q,n}(f) := \begin{bmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{bmatrix}. \tag{1.7}$$

In recent years, there has been a great deal of attention devoted to finding bounds for the modulus of the second and third Hankel determinants $\det H_{2,2}(f)$ and $\det H_{3,1}(f)$, when f belongs to various subclasses of $\mathcal{A}$ (see [2, 10, 11] for further references).

Based on these ideas, in [8] and [9], the authors started the study the Hankel determinant $\det H_{q,n}(F_f)$ whose entries are logarithmic coefficients of $f \in \mathcal{S}$, that is, a_n in Equation (1.7) are replaced by γ_n. In this paper, we continue analogous research considering the Hankel determinant $\det H_{q,n}(F_{f^{-1}})$ whose entries are logarithmic coefficients of inverse functions, i.e. a_n in Equation (1.7) are now replaced by Γ_n. We demonstrate the sharp estimates of

$$\left|\det H_{2,1}\left(F_{f^{-1}}\right)\right| = \left|\Gamma_1\Gamma_3 - \Gamma_2^2\right| = \frac{1}{48}\left|13a_2^4 - 12a_3^2 + 12a_2a_4 - 12a_2^2a_3\right|$$

in the classes $\mathcal{S}_\alpha^*$.

2. Preliminary lemmas

Denote by $\mathcal{P}$ the class of analytic functions $p \in \mathcal{H}$ with positive real part given by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{2.1}$$

where $c_n := a_n(p)$.

In the proof of the main result, we will use the following lemma which contains the well-known formula for c_2 (see, e.g. [17, p. 166]) and the formula for c_3 (see [3, Lemma 2.4] with further remarks related to extremal functions).

Lemma 1. *If $p \in \mathcal{P}$ is of the form (2.1), then*

$$c_1 = 2\zeta_1, \tag{2.2}$$

$$c_2 = 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2 \tag{2.3}$$

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and

$$c_3 = 2\zeta_1^3 + 2(1-|\zeta_1|^2)(2\zeta_1 - \overline{\zeta_1}\zeta_2)\zeta_2 + 2(1-|\zeta_1|^2)(1-|\zeta_2|^2)\zeta_3 \tag{2.4}$$

for some $\zeta_1, \zeta_2, \zeta_3 \in \overline{\mathbb{D}}$.

For $\zeta_1 \in \mathbb{T}$, *there is a unique function* $p \in \mathcal{P}$ *with* c_1 *as in Equation (2.2), namely,*

$$p(z) = \frac{1+\zeta_1 z}{1-\zeta_1 z}, \quad z \in \mathbb{D}.$$

For $\zeta_1 \in \mathbb{D}$ *and* $\zeta_2 \in \mathbb{T}$, *there is a unique function* $p \in \mathcal{P}$ *with* c_1 *and* c_2 *as in Equations (2.2) and (2.3), namely,*

$$p(z) = \frac{1+(\overline{\zeta}_1\zeta_2+\zeta_1)z+\zeta_2 z^2}{1+(\overline{\zeta}_1\zeta_2-\zeta_1)z-\zeta_2 z^2}, \quad z \in \mathbb{D}. \tag{2.5}$$

Lemma 2. ([4]). *For real numbers* A, B, C, *let*

$$Y(A,B,C) := \max\left(\left\{|A+Bz+Cz^2| + 1 - |z|^2 : z \in \overline{\mathbb{D}}\right\}\right).$$

I. *If* $AC \geqslant 0$, *then*

$$Y(A,B,C) = \begin{cases} |A|+|B|+|C|, & |B| \geqslant 2(1-|C|), \\ 1+|A|+\dfrac{B^2}{4(1-|C|)}, & |B| < 2(1-|C|). \end{cases}$$

II. *If* $AC < 0$, *then*

$$Y(A,B,C) = \begin{cases} 1-|A|+\dfrac{B^2}{4(1-|C|)}, & -4AC(C^{-2}-1) \leqslant B^2 \wedge |B| < 2(1-|C|), \\ 1+|A|+\dfrac{B^2}{4(1+|C|)}, & B^2 < \min\left\{4(1+|C|)^2, -4AC(C^{-2}-1)\right\}, \\ R(A,B,C), & \text{otherwise}, \end{cases}$$

where

$$R(A,B,C) := \begin{cases} |A|+|B|-|C|, & |C|(|B|+4|A|) \leqslant |AB|, \\ -|A|+|B|+|C|, & |AB| \leqslant |C|(|B|-4|A|), \\ (|C|+|A|)\sqrt{1-\dfrac{B^2}{4AC}}, & \text{otherwise.} \end{cases}$$

We recall now Laguerre's rule of counting zeros of polynomials in an interval (see [7], [12], [21, pp. 19–20]). We will apply Laguerre's algorithm in the proof of the main

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theorem. Given a real polynomial

$$Q(u) := d_0u^n + d_1u^{n-1} + \cdots + d_{n-1}u + d_n, \quad u \in \mathbb{R},\ d_0, \ldots, d_n \in \mathbb{R},$$

consider a finite sequence $(q_k), k = 0, 1, \ldots, n$, of polynomials of the form

$$q_k(u) = \sum_{j=0}^{k} d_j u^{k-j}, \quad u \in \mathbb{R}.$$

For each $u_0 \in \mathbb{R}$, let $N(Q; u_0)$ denote the number of sign changes in the sequence $(q_k(u_0)), k = 0, 1, \ldots, n$. Given an interval $I \subset \mathbb{R}$, denote by $Z(Q; I)$ the number of zeros of Q in I counted with their orders. Then the following theorem due to Laguerre holds.

Theorem 1. *If $a < b$ and $Q(a)Q(b) \neq 0$, then*

$$Z(Q; (a, b)) = N(Q; a) - N(Q; b)$$

or

$$N(Q; a) - N(Q; b) - Z(Q; (a, b))$$

is an even positive integer.

Note that

$$q_k(0) = d_k, \quad q_k(1) = \sum_{j=0}^{k} d_j.$$

Thus, when $[a, b] := [0, 1]$, Theorem 1 reduces to the following useful corollary.

Corollary 1. *If $Q(0)Q(1) \neq 0$, then*

$$Z(Q; (0, 1)) = N(Q; 0) - N(Q; 1)$$

or

$$N(Q; 0) - N(Q; 1) - Z(Q; (0, 1))$$

is an even positive integer, where $N(Q; 0)$ and $N(Q; 1)$ are the numbers of sign changes in the sequence of polynomial coefficients (d_k) and in the sequence of sums $(\sum_{j=0}^{k} d_j)$, where $k = 0, 1, \ldots, n$, respectively.

3. Main result

The main result of this paper is the following.

Theorem 2. *Let $\alpha \in (0,1]$. If $f \in \mathcal{S}^*_\alpha$, then*

$$\left|\det H_{2,1}\left(F_{f-1}\right)\right| = \left|\Gamma_1\Gamma_3 - \Gamma_2^2\right| \leqslant \begin{cases} \frac{1}{4}\alpha^2, & 0<\alpha<\frac{1}{5}, \\ \frac{\alpha^2(15\alpha^2+5\alpha+2)}{(35\alpha^2+30\alpha+7)}, & \frac{1}{5}\leqslant\alpha\leqslant\alpha_0, \\ \frac{1}{36}\alpha^2(35\alpha^2+4), & \alpha_0<\alpha\leqslant 1, \end{cases} \tag{3.1}$$

where $\alpha_0 \approx 0.39059$ is the unique root in $(0,1]$ of the equation

$$1225\alpha^4 + 1050\alpha^3 - 155\alpha^2 - 60\alpha - 44 = 0. \tag{3.2}$$

All inequalities are sharp.

Proof. Let $f \in \mathcal{S}^*_\alpha$ be of the form (1.1). Then by Equation (1.2), there exists $p \in \mathcal{P}$ of the form (2.1) such that

$$\frac{zf'(z)}{f(z)} = (p(z))^\alpha, \quad z \in \mathbb{D}. \tag{3.3}$$

Putting the series (1.1) and (2.1) into (3.3), by equating the coefficients we get

$$\begin{aligned} a_2 &= \alpha c_1, \quad a_3 = \frac{1}{2}\alpha\left(c_2 + \frac{3\alpha-1}{2}c_1^2\right), \\ a_4 &= \frac{1}{3}\alpha\left(c_3 + \frac{5\alpha-2}{2}c_1c_2 + \frac{17\alpha^2-15\alpha+4}{12}c_1^3\right). \end{aligned} \tag{3.4}$$

Hence and from Equation (1.6) we obtain

$$\begin{aligned} \Gamma_1 &= -\frac{1}{2}\alpha c_1, \quad \Gamma_2 = -\frac{1}{8}\alpha(2c_2 - (3\alpha+1)c_1^2), \\ \Gamma_3 &= -\frac{1}{72}\alpha\left(12c_3 - (42\alpha+12)c_1c_2 + (29\alpha^2+21\alpha+4)c_1^3\right), \end{aligned}$$

and therefore

$$\Gamma_1\Gamma_3 - \Gamma_2^2 = \frac{1}{576}\alpha^2(c_1^4(35\alpha^2+30\alpha+7) - 12(5\alpha+1)c_1^2c_2 + 48c_1c_3 - 36c_2^2). \tag{3.5}$$

Since both the class $\mathcal{S}^*_\alpha$ and $|\det H_{2,1}\left(F_{f-1}\right)|$ are rotationally invariant, without loss of generality we may assume that $a_2 \geqslant 0$, which in view of Equation (3.4) yields $c_1 \geqslant 0$,

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i.e. by Equation (2.2) that $\zeta_1 \in [0,1]$. Thus substituting Equations (2.2)–(2.4) into Equation (3.5), we obtain

$$\begin{aligned} \mathit{\Gamma}_1\mathit{\Gamma}_3 - \mathit{\Gamma}_2^2 = \frac{\alpha^2}{36}\big(& (35\alpha^2+4)\zeta_1^4 - 30\alpha(1-\zeta_1^2)\zeta_1^2\zeta_2 - 3(1-\zeta_1^2)(\zeta_1^2+3)\zeta_2^2 \\ & +12\zeta_1(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_3\big) \end{aligned} \tag{3.6}$$

for some $\zeta_1 \in [0,1]$ and $\zeta_2, \zeta_3 \in \overline{\mathbb{D}}$.

A. Suppose that $\zeta_1 = 0$. Then from Equation (3.6),

$$\left|\mathit{\Gamma}_1\mathit{\Gamma}_3 - \mathit{\Gamma}_2^2\right| = \frac{\alpha^2}{4}|\zeta_2|^2 \leqslant \frac{\alpha^2}{4}.$$

B. Suppose that $\zeta_1 = 1$. Then from Equation (3.6),

$$\left|\mathit{\Gamma}_1\mathit{\Gamma}_3 - \mathit{\Gamma}_2^2\right| = \frac{1}{36}\alpha^2(35\alpha^2+4).$$

C. Suppose that $\zeta_1 \in (0,1)$. Since $\zeta_3 \in \overline{\mathbb{D}}$, from Equation (3.6) we get

$$\left|\mathit{\Gamma}_1\mathit{\Gamma}_3 - \mathit{\Gamma}_2^2\right| \leqslant \frac{1}{3}\alpha^2\zeta_1(1-\zeta_1^2)\mathit{\Phi}(A,B,C),$$

where

$$\mathit{\Phi}(A,B,C) := \left|A + B\zeta_2 + C\zeta_2^2\right| + 1 - |\zeta_2|^2,$$

with

$$A := \frac{\zeta_1^3(35\alpha^2+4)}{12(1-\zeta_1^2)}, \quad B := \frac{-5\alpha\zeta_1}{2}, \quad C := \frac{-(\zeta_1^2+3)}{4\zeta_1}.$$

Observe that $AC < 0$ and therefore we apply only the part II of Lemma 2.

C1. Let's consider the condition $|B| < 2(1-|C|)$, i.e.

$$\frac{5\alpha\zeta_1}{2} < 2\left(1 - \frac{\zeta_1^2+3}{4\zeta_1}\right).$$

The above inequality is equivalent to

$$\frac{\zeta_1^2(5\alpha+1) - 4\zeta_1 + 3}{2\zeta_1} < 0, \tag{3.7}$$

which is equivalent to $(5\alpha+1)\zeta_1^2 - 4\zeta_1 + 3 < 0$. However

$$(5\alpha+1)\zeta_1^2 - 4\zeta_1 + 3 = 5\alpha\zeta_1^2 + (1-\zeta_1)(3-\zeta_1) > 0$$

for $\zeta_1 \in (0,1)$, which shows that the inequality (3.7) is false.

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C2. Since

$$-4AC\left(\frac{1}{C^2}-1\right)=-\frac{(9-\zeta_1^2)(35\alpha^2+4)\zeta_1^2}{12(\zeta_1^2+3)}<0$$

for $\zeta_1 \in (0,1)$, we deduce that the condition $B^2 < \min\{4(1+|C|)^2, -4AC(C^{-2}-1)\}$ is equivalent to

$$\frac{\zeta_1^2[(10\alpha^2-1)\zeta_1^2+135\alpha^2+9]}{3(\zeta_1^2+3)}<0, \tag{3.8}$$

which is equivalent to $(10\alpha^2-1)\zeta_1^2+135\alpha^2+9<0$ for $\zeta_1 \in (0,1)$. However, in the case when $10\alpha^2-1 \geqslant 0$ we have

$$(10\alpha^2-1)\zeta_1^2+135\alpha^2+9 \geqslant 135\alpha^2+9>0,$$

and in the case when $10\alpha^2-1<0$ we have

$$(10\alpha^2-1)\zeta_1^2+135\alpha^2+9 \geqslant 145\alpha^2+8>0,$$

for all $\zeta_1 \in (0,1)$. Thus the inequality (3.8) is false.

C3. The inequality $|C|(|B|+4|A|) \leqslant |AB|$ is equivalent to

$$\frac{(175\alpha^3-70\alpha^2+35\alpha-8)\zeta_1^4-6(35\alpha^2-5\alpha+4)\zeta_1^2-45\alpha}{24(1-\zeta_1^2)} \geqslant 0$$

which is equivalent to

$$\varphi_\alpha(\zeta_1^2) \geqslant 0, \tag{3.9}$$

where for $t \in \mathbb{R}$,

$$\varphi_\alpha(t):=(175\alpha^3-70\alpha^2+35\alpha-8)t^2-6(35\alpha^2-5\alpha+4)t-45\alpha.$$

Observe that the equation $175\alpha^3-70\alpha^2+35\alpha-8=0$ has only one real root α_1 in $(0,1]$, where

$$\alpha_1:=\frac{1}{105}(13769+882\sqrt{445})^{1/3}-\frac{77}{15(13769+882\sqrt{445})^{1/3}}+\frac{2}{15}\approx 0.2758,$$

and that the inequality (3.9) is false for $\alpha := \alpha_1$. Let now $\alpha \in (0,1] \setminus \{\alpha_1\}$. For φ_α, we have $\Delta := 144(525\alpha^4-175\alpha^3+120\alpha^2-20\alpha+4)>0$, which is true for all $\alpha \in (0,1]\setminus\{\alpha_1\}$.

Hence the square trinomial φ_α has two roots

$$t_{1,2} := \frac{3(35\alpha^2 - 5\alpha + 4) \pm 6\sqrt{525\alpha^4 - 175\alpha^3 + 120\alpha^2 - 20\alpha + 4}}{175\alpha^3 - 70\alpha^2 + 35\alpha - 8}.$$

Note that for all $\alpha \in (0,1] \setminus \{\alpha_1\}$ we have $-6(35\alpha^2 - 5\alpha + 4) < 0$. Now for $\alpha \in (\alpha_1, 1]$ we have $175\alpha^3 - 70\alpha^2 + 35\alpha - 8 > 0$. Hence $t_2 < 0$ because the inequality

$$3(35\alpha^2 - 5\alpha + 4) - 6\sqrt{525\alpha^4 - 175\alpha^3 + 120\alpha^2 - 20\alpha + 4} < 0$$

is equivalent to

$$-45\alpha(175\alpha^3 - 70\alpha^2 + 35\alpha - 8) < 0,$$

which is true for all $\alpha \in (\alpha_1, 1]$. On the other hand, the inequality $t_1 > 1$ is equivalent to

$$6\sqrt{525\alpha^4 - 175\alpha^3 + 120\alpha^2 - 20\alpha + 4} > 5(35\alpha^3 - 35\alpha^2 + 10\alpha - 4), \tag{3.10}$$

which is evidently true for $\alpha \in (\alpha_1, \alpha_2]$, where $\alpha_2 \approx 0.82155$, since then the right hand side of Equation (3.10) is non-positive. For $\alpha \in (\alpha_2, 1]$ by squaring both sides of Equation (3.10), we equivalently get the inequality

$$(5\alpha - 8)(6125\alpha^5 - 2450\alpha^4 + 1925\alpha^3 - 560\alpha^2 + 140\alpha - 32) < 0$$

which is true for $\alpha \in (\alpha_2, 1]$. Thus we conclude that for $\alpha \in (\alpha_1, 1]$ the inequality (3.9) is false.

Let $\alpha \in (0, \alpha_1)$. Then $175\alpha^3 - 70\alpha^2 + 35\alpha - 8 < 0$ and therefore $t_1 < 0$ evidently. Moreover, the inequality $t_2 < 0$ is equivalent to

$$3(35\alpha^2 - 5\alpha + 4) - 6\sqrt{525\alpha^4 - 175\alpha^3 + 120\alpha^2 - 20\alpha + 4} > 0$$

which is equivalent to

$$-45\alpha(175\alpha^3 - 70\alpha^2 + 35\alpha - 8) > 0,$$

which is true for all $\alpha \in (0, \alpha_1)$. Thus we conclude that for $\alpha \in (0, \alpha_1)$ the inequality (3.9) is false.

C4. The inequality $|C|(|B| - 4|A|) \geqslant |AB|$ is equivalent to

$$\frac{(175\alpha^3 + 70\alpha^2 + 35\alpha + 8)\zeta_1^4 + 6\zeta_1^2(35\alpha^2 + 5\alpha + 4) - 45\alpha}{24(1 - \zeta_1^2)} \leqslant 0$$

which is equivalent to

$$\gamma_\alpha(\zeta_1^2) \leqslant 0, \tag{3.11}$$

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where for $t \in \mathbb{R}$,

$$\gamma_\alpha(t) := (175\alpha^3 + 70\alpha^2 + 35\alpha + 8)t^2 + 6(35\alpha^2 + 5\alpha + 4)t - 45\alpha.$$

For γ_α we have $\Delta := 144(525\alpha^4 + 175\alpha^3 + 120\alpha^2 + 20\alpha + 4) > 0$ for all $\alpha \in (0, 1]$. Hence the square trinomial γ_α has two roots

$$t_{3,4} := \frac{-3(35\alpha^2 + 5\alpha + 4) \pm 6\sqrt{525\alpha^4 + 175\alpha^3 + 120\alpha^2 + 20\alpha + 4}}{175\alpha^3 + 70\alpha^2 + 35\alpha + 8}.$$

Note that $t_4 < 0$ evidently. Observe now that $t_3 > 0$. Indeed, this inequality is equivalent to

$$-3(35\alpha^2 + 5\alpha + 4) + 6\sqrt{525\alpha^4 + 175\alpha^3 + 120\alpha^2 + 20\alpha + 4} > 0$$

which is equivalent to the evidently true inequality

$$45\alpha(175\alpha^3 + 70\alpha^2 + 35\alpha + 8) > 0, \quad \alpha \in (0, 1].$$

Moreover, $t_3 < 1$ is equivalent to

$$6\sqrt{525\alpha^4 + 175\alpha^3 + 120\alpha^2 + 20\alpha + 4} < 5(35\alpha^3 + 35\alpha^2 + 10\alpha + 4)$$

that after squaring both sides is equivalent to

$$(5\alpha + 8)(6125\alpha^5 + 2450\alpha^4 + 1925\alpha^3 + 560\alpha^2 + 140\alpha + 32) > 0,$$

which is true for all $\alpha \in (0, 1]$. Therefore the inequality (3.11) is true for $\zeta_1 \in (0, \zeta_1^0]$, where $\zeta_1^0 := \sqrt{t_3}$.

Applying Lemma 2 for $0 < \zeta_1 \leqslant \zeta_1^0$, we get

$$\left|\Gamma_1\Gamma_3 - \Gamma_2^2\right| \leqslant \frac{1}{3}\alpha^2\zeta_1(1 - \zeta_1^2)(-|A| + |B| + |C|) = \rho_\alpha(\zeta_1),$$

where

$$\rho_\alpha(t) := -\frac{1}{36}\alpha^2((35\alpha^2 + 30\alpha + 7)t^4 - 6(5\alpha - 1)t^2 - 9), \quad t \in \mathbb{R}.$$

We have

$$\rho_\alpha(0) = \frac{1}{4}\alpha^2$$

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and

$$\begin{aligned}\rho_\alpha(\zeta_1^0) =&\frac{2\alpha^2}{(175\alpha^3+70\alpha^2+35\alpha+8)^2}\\&\times\left[-18375\alpha^6-16625\alpha^5-10150\alpha^4-3775\alpha^3-1025\alpha^2-150\alpha-12\right.\\&+(1050\alpha^4+700\alpha^3+320\alpha^2+80\alpha+10)\\&\left.\times\sqrt{525\alpha^4+175\alpha^3+120\alpha^2+20\alpha+4}\right].\end{aligned}$$

Note that for $\alpha \in (0, 1/5]$ the equation

$$\rho'_\alpha(t) = -\frac{1}{9}\alpha^2 t((35\alpha^2+30\alpha+7)t^2 - 3(5\alpha-1)) = 0 \tag{3.12}$$

has no root in $(0, \zeta_1^0)$ and then evidently

$$\rho_\alpha(t) \leqslant \rho_\alpha(0) = \frac{1}{4}\alpha^2, \quad 0 \leqslant t \leqslant \zeta_1^0.$$

For $\alpha \in (1/5, 1]$, Equation (3.12) has a unique positive root, namely

$$t_5 := \sqrt{\frac{3(5\alpha-1)}{35\alpha^2+30\alpha+7}}. \tag{3.13}$$

It remains to check the condition $t_5 < \zeta_1^0$ equivalently written as

$$\frac{10(105\alpha^4+70\alpha^3+32\alpha^2+8\alpha+1)}{35\alpha^2+30\alpha+7} < \sqrt{525\alpha^4+175\alpha^3+120\alpha^2+20\alpha+4},$$

which is equivalent to

$$\frac{(175\alpha^3+70\alpha^2+35\alpha+8)(2625\alpha^5-175\alpha^4-925\alpha^3-425\alpha^2-80\alpha-12)}{(35\alpha^2+30\alpha+7)^2} < 0.$$

The last inequality is true for $\alpha \in (1/5, \alpha_3)$, where $\alpha_3 \approx 0.812678$ is the unique root in $(0, 1)$ of the equation

$$2625\alpha^5 - 175\alpha^4 - 925\alpha^3 - 425\alpha^2 - 80\alpha - 12 = 0.$$

Then ρ_α attains its maximum value on $(0, \zeta_1^0]$ at t_5 with

$$\rho_\alpha(t_5) = \frac{\alpha^2(15\alpha^2+5\alpha+2)}{35\alpha^2+30\alpha+7}.$$

If $\alpha \in [\alpha_3, 1]$, then evidently,

$$\rho_\alpha(t) \leqslant \max\left(\{\rho_\alpha(0), \rho_\alpha(\zeta_1^0)\}\right) = \rho_\alpha(\zeta_1^0), \quad 0 \leqslant t \leqslant \zeta_1^0.$$

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C5. Applying Lemma 2 for $\zeta_1^0 < \zeta_1 < 1$, we get

$$\left|\varGamma_1\varGamma_3 - \varGamma_2^2\right| \leqslant \frac{1}{3}\alpha^2\zeta_1(1-\zeta_1^2)\left(|A|+|C|\right)\sqrt{1-\frac{B^2}{4AC}} = \psi_\alpha(\zeta_1),$$

where for $t \in [0,1]$,

$$\psi_\alpha(t) := \frac{1}{18}\alpha^2((35\alpha^2+1)t^4 - 6t^2 + 9)\sqrt{\frac{-(10\alpha^2-1)t^2+45\alpha^2+3}{(35\alpha^2+4)(t^2+3)}}.$$

We have

$$\psi_\alpha(\zeta_1^0) = -\frac{2\alpha^2(35\alpha^2+4)}{(175\alpha^3+70\alpha^2+35\alpha+8)^2}H(\alpha)\sqrt{-\frac{G(\alpha)}{K(\alpha)}},$$

where

$$\begin{aligned} H(\alpha) :=& -1050\alpha^4 - 525\alpha^3 - 270\alpha^2 - 65\alpha - 10 + (35\alpha^2+10\alpha+3)\times \\ &\times \sqrt{525\alpha^4+175\alpha^3+120\alpha^2+20\alpha+4}, \\ G(\alpha) :=& -2625\alpha^5 - 1400\alpha^4 - 750\alpha^3 - 195\alpha^2 - 30\alpha - 4 + (20\alpha^2-2)\times \\ &\times \sqrt{525\alpha^4+175\alpha^3+120\alpha^2+20\alpha+4} \end{aligned}$$

and

$$K(\alpha) := (175\alpha^3+35\alpha^2+30\alpha+4+2\sqrt{525\alpha^4+175\alpha^3+120\alpha^2+20\alpha+4})(35\alpha^2+4).$$

Note that

$$\psi_\alpha(1) = \frac{1}{36}\alpha^2(35\alpha^2+4).$$

Differentiating ψ_α leads to the equation

$$\psi'_\alpha(t) = -\frac{1}{18}t\alpha^2\frac{Q(t^2)}{(35\alpha^2+4)(t^2+3)^2\sqrt{\dfrac{-(10\alpha^2-1)t^2+45\alpha^2+3}{(35\alpha^2+4)(t^2+3)}}} = 0,$$

where for $s \in [0,1]$,

$$\begin{aligned} Q(s) :=& 4(35\alpha^2+1)(10\alpha^2-1)s^3 + 3(175\alpha^4-315\alpha^2-4)s^2 \\ &- 18(1050\alpha^4+115\alpha^2-2)s + 2295\alpha^2+108. \end{aligned}$$

Now we describe the number of zeros of Q in the interval $(0,1)$ by combining Descartes' and Laguerre's rules. To apply Descartes' rule, we check the numbers of sign changes of coefficients of the polynomial Q. We have:

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- $d_0(\alpha) := q_0(0) = 4(35\alpha^2 + 1)(10\alpha^2 - 1) > 0$ iff $\alpha \in \left(1/\sqrt{10}, 1\right)$,
- $d_1(\alpha) := q_1(0) = 3(175\alpha^4 - 315\alpha^2 - 4) < 0$ iff $\alpha \in (0, 1)$,
- $d_2(\alpha) := q_2(0) = -18(1050\alpha^4 + 115\alpha^2 - 2) > 0$ iff $\alpha \in (0, \alpha_4)$, where

$$\alpha_4 := \frac{1}{2}\sqrt{\frac{1}{105}(\sqrt{865} - 23))} \approx 0.12355,$$

- $d_3(\alpha) := q_3(0) = 2295\alpha^2 + 108 > 0$ iff $\alpha \in (0, 1)$.

Thus there is one change of signs in $\left(0, 1/\sqrt{10}\right)$, i.e. $N(Q, 0) = 1$, and two changes of signs in $\left[1/\sqrt{10}, 1\right)$, i.e. $N(Q, 0) = 2$. According to Descartes' rule of signs, the polynomial Q has one positive real root in $\left(0, 1/\sqrt{10}\right)$ and zero or two positive real roots in $\left[1/\sqrt{10}, 1\right)$.

To apply Laguerres' rule, it remains to compute the number $N(Q, 1)$ of sign changes in the sequence of sums $\sum_{j=0}^{k} u_j(\alpha)$, where $k = 0, 1, 2, 3$. We have

- $d_0(\alpha) = 4(35\alpha^2 + 1)(10\alpha^2 - 1) > 0$ iff $\alpha \in \left(1/\sqrt{10}, 1\right)$,
- $d_0(\alpha) + d_1(\alpha) = 1925\alpha^4 - 1045\alpha^2 - 16 > 0$ iff $\alpha \in (\alpha_5, 1)$, where

$$\alpha_5 := \sqrt{(209 + 3\sqrt{5401})/770} \approx 0.74683,$$

- $d_0(\alpha) + d_1(\alpha) + d_2(\alpha) = -5(3395\alpha^4 + 623\alpha^2 - 4) > 0$ iff $\alpha \in (0, \alpha_6)$, where

$$\alpha_6 := \sqrt{(3\sqrt{49161} - 623)/6790} \approx 0.078806,$$

- $d_0(\alpha) + d_1(\alpha) + d_2(\alpha) + d_3(\alpha) = -(35\alpha^2 + 4)(485\alpha^2 - 32) > 0$ iff $\alpha \in (0, \alpha_7)$, where $\alpha_7 := 4\sqrt{2/485} \approx 0.25686$.

Thus there are no changes of signs in $\left(\alpha_7, 1/\sqrt{10}\right)$, i.e. $N(Q, 1) = 0$, and one change of sign in $(0, \alpha_7] \cup \left[1/\sqrt{10}, 1\right)$ i.e. $N(Q, 1) = 1$. According to Laguerre's rule, the polynomial Q has one root in $[0, 1]$ for $\alpha \in (\alpha_7, 1)$, and no roots in $[0, 1]$ for $\alpha \in (0, \alpha_7]$. Therefore, for $\alpha \in (0, \alpha_7]$, the function ψ_α is increasing for $\zeta_1^0 < t < 1$ and hence

$$\psi_\alpha(t) \leqslant \psi_\alpha(1), \quad \zeta_1^0 < t < 1.$$

In turn, for $\alpha \in (\alpha_7, 1)$, the function ψ_α has a unique critical point in $[0, 1]$, where by using jointly Descartes' and Laguerre's rules we state that ψ_α attains its minimum value. Thus

$$\psi_\alpha(t) \leqslant \max\left(\{\psi_\alpha(\zeta_1^0), \psi_\alpha(1)\}\right), \quad \zeta_1^0 < t < 1.$$

Now we summarize results of sections C4 and C5.

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(i) For $\alpha \in (0, 1/5)$, we compare $\psi_\alpha(1)$ and $\varrho_\alpha(0)$. Note that then $\varrho_\alpha(0) \geqslant \psi_\alpha(1)$ since it is equivalent to

$$\frac{1}{4}\alpha^2 - \frac{1}{36}\alpha^2(35\alpha^2+4) = \frac{1}{36}\alpha^2(5-35\alpha^2) \geqslant 0.$$

(ii) For $\alpha \in [1/5, \alpha_3)$, we compare $\psi_\alpha(1)$ and $\varrho_\alpha(t_5)$. Note that the inequality

$$\frac{\alpha^2(15\alpha^2+5\alpha+2)}{35\alpha^2+30\alpha+7} \geqslant \frac{1}{36}\alpha^2(35\alpha^2+4)$$

is equivalent to

$$\frac{1}{36}\alpha^2(1225\alpha^4+1050\alpha^3-155\alpha^2-60\alpha-44) \leqslant 0$$

which is true for $\alpha \in [1/5, \alpha_0]$, where $\alpha_0 \approx 0.390595$ is the unique root in $(0, 1]$ of Equation (3.2). Thus $\varrho_\alpha(t_5) \geqslant \psi_\alpha(1)$ for $\alpha \in [1/5, \alpha_0]$, and $\varrho_\alpha(t_5) < \psi_\alpha(1)$ for $\alpha \in (\alpha_0, \alpha_3)$.

(iii) For $\alpha \in [\alpha_3, 1)$, we compare $\psi_\alpha(1)$ and $\rho_\alpha(\zeta_1^0)$. Note that the inequality $\rho_\alpha(\zeta_1^0) \leqslant \psi_\alpha(1)$ is equivalent to

$$\begin{aligned}
&-18375\alpha^6-16625\alpha^5-10150\alpha^4-3775\alpha^3-1025\alpha^2-150\alpha-12\\
&+(1050\alpha^4+700\alpha^3+320\alpha^2+80\alpha+10)\sqrt{525\alpha^4+175\alpha^3+120\alpha^2+20\alpha+4}\\
\leqslant&\frac{1}{72}(35\alpha^2+4)(175\alpha^3+70\alpha^2+35\alpha+8)^2,
\end{aligned}$$

equivalently written as

$$\begin{aligned}
&(1050\alpha^4+700\alpha^3+320\alpha^2+80\alpha+10)\times\sqrt{525\alpha^4+175\alpha^3+120\alpha^2+20\alpha+4}\\
\leqslant&\frac{1}{72}\big(1071875\alpha^8+857500\alpha^7+2045750\alpha^6+1564500\alpha^5+881475\alpha^4+322200\alpha^3\\
&+85420\alpha^2+13040\alpha+1120\big),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\frac{25}{5184}(35\alpha^2+4)(175\alpha^3+70\alpha^2+35\alpha+8)^2\times\\
&\quad\times\big(42875\alpha^8+34300\alpha^7+134750\alpha^6+110460\alpha^5+17835\alpha^4-7344\alpha^3\\
&\quad-5036\alpha^2-1120\alpha-128\big)\geqslant 0
\end{aligned}$$

which is true for $\alpha \in [\alpha_3, 1)$.

D. We now show sharpness of all inequalities by using the formula (3.5). In the first inequality in Equation (3.1), the equality is attained by the function $f \in \mathcal{S}^*_\alpha$ given by

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Equation (3.3) with

$$p(z) := \frac{1-z^2}{1+z^2}, \quad z \in \mathbb{D},$$

for which $c_1 = c_3 = 0$ and $c_2 = -2$.

In the second inequality in Equation (3.1), the equality is attained by the function $f \in \mathcal{S}^*_\alpha$ given by Equation (3.3), where $p \in \mathcal{P}$ is defined by Equation (2.5) with $\zeta_1 = t_5 =: \tau$ and $\zeta_2 = 1$, i.e.

$$p(z) := \frac{1+2\tau z+z^2}{1-z^2}, \quad z \in \mathbb{D}.$$

Here t_5 is described by Equation (3.13).

In the third inequality in Equation (3.1), the equality is attained by the function $f \in \mathcal{S}^*_\alpha$ given by Equation (3.3), where $p \in \mathcal{P}$ is defined by

$$p(z) := \frac{1+z}{1-z}, \quad z \in \mathbb{D},$$

for which $c_1 = c_2 = c_3 = 2$.

This ends the proof of the theorem. □

For $\alpha = 1$, we have the following result:

Corollary 2. *If $f \in \mathcal{S}^*$, then*

$$\left|\det H_{2,1}\left(F_{f^{-1}}\right)\right| = \left|\Gamma_1\Gamma_3 - \Gamma_2^2\right| \leqslant \frac{13}{12}.$$

The inequality is sharp.

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