

THE WIELANDT SUBALGEBRA OF A LIE ALGEBRA

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(Received 9 October 2000; revised 6 February 2002)

Communicated by R. B. Howlett

Abstract

Following the analogy with group theory, we define the Wielandt subalgebra of a finite-dimensional Lie algebra to be the intersection of the normalisers of the subnormal subalgebras. In a non-zero algebra, this is a non-zero ideal if the ground field has characteristic 0 or if the derived algebra is nilpotent, allowing the definition of the Wielandt series. For a Lie algebra with nilpotent derived algebra, we obtain a bound for the derived length in terms of the Wielandt length and show this bound to be best possible. We also characterise the Lie algebras with nilpotent derived algebra and Wielandt length 2.

2000 *Mathematics subject classification*: primary 17B30, 17B50.

Keywords and phrases: Lie algebras, subnormal subalgebras.

1. Introduction

The Wielandt subgroup of a group G , denoted by $\omega(G)$, is the intersection of the normalisers of all the subnormal subgroups of G . An automorphism of G permutes the subnormal subgroups, so leaves $\omega(G)$ fixed. Thus $\omega(G)$ is a characteristic subgroup of G .

The Wielandt subgroup was first studied by Wielandt [16] in 1958. He proved that, if G is a non-trivial finite group, then $\omega(G)$ is non-trivial. This makes it possible to define the Wielandt series by $\omega_1(G) = \omega(G)$ and $\omega_{i+1}(G)/\omega_i(G) = \omega(G/\omega_i(G))$ for $i \geq 1$. For a finite group G , there is some n such that $\omega_n(G) = G$. The least such n is called the Wielandt length of G .

There have been a number of investigations of the structure of groups with respect to the Wielandt series (see [3, 4, 5, 6, 7, 8]). In [4], Bryce and Cossey show that a finite soluble group of Wielandt length n has derived length at most $5n/3 + 1$.

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There is a tradition of investigating analogies between group theory and the theory of Lie algebras. There are myriad correspondences between these two fields. In this paper, we develop a theory of the Wielandt series for a finite-dimensional Lie algebra L with nilpotent derived algebra, analogous to the above cited theory for finite groups. As we shall see (Example 5.4 and Example 7.2 below), the restriction L' nilpotent is needed. In non-zero characteristic, solubility is not enough.

2. Preliminaries

Let L be a finite-dimensional Lie algebra over the field F . The Wielandt subalgebra $\omega(L)$ of L is the intersection of the normalisers of the subnormal subalgebras of L . The Wielandt series of L is defined by $\omega_{i+1}(L)/\omega_i(L) = \omega(L/\omega_i(L))$ and the Wielandt length of L is the length of this series. This definition of $\omega_{i+1}(L)$, to be meaningful, needs the preceding terms to be ideals. For the definition of the Wielandt length to be meaningful, we need the series to reach L , that is, we need $\omega(L) \neq 0$ whenever $L \neq 0$. As we shall see in Section 3, these always hold if the field F has characteristic 0. In Section 5, we show that they also hold for algebras L whose derived algebra L' is nilpotent. We shall make frequent reference to Lie algebras with nilpotent derived algebra, so it is convenient to have a term for them. We adopt the terminology used by Strade and Farnsteiner [14].

DEFINITION 2.1. A Lie algebra L is called *completely soluble* if its derived algebra L' is nilpotent.

All finite-dimensional soluble Lie algebras over a field of characteristic 0 are completely soluble. (For example, see Jacobson [11, page 51].) Thus our results on completely soluble Lie algebras apply to all finite-dimensional soluble Lie algebras over fields of characteristic 0.

All vector spaces and algebras considered in this paper are assumed to be finite-dimensional over a field which we denote by F . For any Lie algebra L , we denote by L^i the terms of the descending central series of L , defined by $L^1 = L$ and $L^{i+1} = [L^i, L]$. We denote by L^∞ the intersection of the L^i and the centre of L by $\mathcal{Z}(L)$. We denote by ad_u the adjoint derivation of L given by $u \in L$, defined by $\text{ad}_u(x) = [u, x]$ for all $x \in L$. We denote the subspace spanned by a set U by $\langle U \rangle$. We write $U < L$ if U is a subalgebra of L , $U \triangleleft L$ if U is an ideal of L and $U \triangleleft\triangleleft L$ if U is a subideal (subnormal subalgebra) of L . We denote the normaliser of U in L by $\mathcal{N}_L(U)$ and the centraliser by $\mathcal{C}_L(U)$. For any subset $U \subseteq L$, we denote by U^L the normal closure of U in L , that is, the smallest ideal of L which contains U . We denote the subnormal closure by U^{*L} .

We define inductively the series of subalgebras $U^{L,i}$ by $U^{L,0} = L$ and $U^{L,i+1} = U^{U^{L,i}}$. Since L is assumed finite-dimensional, we have $U^{L,i} = U^{sL}$ for all sufficiently large i . We will find it convenient to have another description of U^{sL} when $U = \{u\}$ has only one element. Let X be the free Lie algebra over F on the generators x, x_1, x_2, \dots . We shall regard X as a set of Lie polynomials in the variables x, x_1, x_2, \dots . Put $X_i = X' \cap x^{X,i}$. Thus X_1 is spanned by all products with at least one factor x and X_2 is spanned by certain of the products with 2 factors x . For example, $[[[x_1, x], x_2], x] \in X_2$ but $[[[x_1, x], x], x_2] \notin X_2$. For any Lie algebra L and $u, a_1, \dots \in L$, we have the substitution homomorphism $X \rightarrow L$ in which $x \mapsto u$ and $x_i \mapsto a_i$ for all i . This maps each polynomial in X_i into $L' \cap u^{L,i}$. We denote by $X_i(u, L)$ the space spanned by all the elements so obtained from all possible choices of the a_i . Clearly, $u^{L,i} = \langle u, X_i(u, L) \rangle$ and, if $u \notin L', X_i(u, L) = L' \cap u^{L,i}$.

3. The special case of characteristic 0

When the field F has characteristic 0, we have available Tuck's Theorem [15].

THEOREM 3.1. *Let L be a finite-dimensional Lie algebra over a field of characteristic 0, and let G be the group of all automorphisms of L . If H is a subalgebra of L which is invariant under G , then H is a characteristic ideal of L .*

Tuck's Theorem is important since it means that any 'natural' subalgebra will be a characteristic ideal, and in particular, $\omega(L)$ is a characteristic ideal. We shall also use the fundamental result of Schenkman [12]:

THEOREM 3.2. *Let L be a finite-dimensional Lie algebra over any field and let $U \triangleleft\triangleleft L$. Then $U^\infty \triangleleft L$.*

We now show that the theory of the Wielandt series in characteristic 0 essentially reduces to that for soluble algebras. We denote the (soluble) radical of L by $R(L)$.

THEOREM 3.3. *Let L be a finite-dimensional Lie algebra over a field F of characteristic 0. Then $\omega(L) = \bigcap_{U \triangleleft\triangleleft R(L)} \mathcal{N}_L(U)$ and $\omega(L) \cap R(L) = \omega(R(L))$.*

PROOF. If $U \triangleleft L$, then $R(U) = U \cap R(L)$, and it follows that if $U \triangleleft\triangleleft L$, then $R(U) = U \cap R(L)$. Let $U \triangleleft\triangleleft L$ and let S be a Levi factor of U . Put $R_0 = R(U)$. Let $x \in \mathcal{N}_L(U)$, and put $R_1 = [U, R(U)]$. Then x normalises R_1 . Since $(S + R_1)/R_1$ is a characteristic ideal of $U/R_1 = (S + R_1)/R_1 \oplus R(U)/R_1$, x normalises $S + R_1$. Conversely, any x which normalises both $U_1 := S + R_1$ and R_0 also normalises U . Defining $R_{i+1} = [U_i, R_i]$ and $U_{i+1} = S + R_{i+1}$, we see that any x which normalises U_i and all the R_i also normalises U . However, for some n , we have $[U_n, R_n] = R_n$

and $(U_n)^\infty = U_n$. Therefore $U_n \triangleleft L$ and any element $x \in L$ which normalises all the R_i also normalises U . Thus any element $x \in L$ which normalises every subnormal subalgebra of R also normalises every subnormal subalgebra of L . \square

In [10], Hartley proved that, if L is a soluble Lie algebra over a field of characteristic 0 and if A is a minimal ideal of L , then $A \leq \omega(L)$. (In Section 5 below, we shall prove this for completely soluble algebras L over any field.) Since for a semi-simple algebra S , $\omega(S) = S$, it follows for algebras over a field of characteristic 0, that $L \neq 0$ implies $\omega(L) \neq 0$. Thus the Wielandt series reaches L and the Wielandt length is defined. From Theorem 3.3, it follows that $\omega_i(L) \cap R(L) = \omega_i(R(L))$. It follows that the Wielandt length of L is either equal to or one more than the Wielandt length of $R(L)$.

4. Subnormal subalgebras in characteristic p

In [12], Schenkman also proved that, if $\text{char } F = 0$ and if $A, B \triangleleft\triangleleft L$, then their join $A \cup B \triangleleft\triangleleft L$. His proof depends on two special features of Lie algebras in characteristic 0, namely that if $R = R(L)$, then L/R is a direct sum of simple algebras and that R' is nilpotent. In non-zero characteristic, his proof works if we restrict ourselves to completely soluble Lie algebras. We begin with a useful result which is implicit in Schenkman's proof.

LEMMA 4.1. *Let U be a nilpotent subnormal subalgebra of the completely soluble Lie algebra L , and let N be the nil radical of L . Then $U \leq N$.*

PROOF. Since $U \triangleleft\triangleleft L$, U acts nilpotently on the U -module L/U . As U is nilpotent, it acts nilpotently on U , and so on L . It follows that every chief factor of $U + N$ is central, so $U + N$ is nilpotent. But $N \geq L'$, so $U + N \triangleleft L$ and therefore $U + N \leq N$. \square

LEMMA 4.2. *Suppose L is completely soluble and that $A \triangleleft\triangleleft L$ and $B \triangleleft\triangleleft L$. Then $A \cup B \triangleleft\triangleleft L$.*

PROOF. If there exists an ideal $K \triangleleft L$, $K \leq A$, then we can work in L/K and the result follows by induction over the dimension of L . If no such K exists, then by Theorem 3.2, A is nilpotent. Thus we may assume both A and B nilpotent. By Lemma 4.1, both A and B are in the nil radical N . It follows that $A \cup B \triangleleft\triangleleft N \triangleleft L$. \square

The condition that L is completely soluble is needed in Lemma 4.2. Assuming L soluble is not enough as is shown by the following example.

EXAMPLE 4.3. Let $V = \langle v_0, \dots, v_{p-1} \rangle$ where $p = \text{char } F$ and the indices are integers mod p . Let $L = \langle x, y, z, V \rangle$ with multiplication defined by $[x, y] = z$, $[x, z] = [y, z] = [v_i, v_j] = 0$, and $[x, v_n] = nv_{n-1}$, $[y, v_n] = v_{n+1}$, $[z, v_n] = v_n$. Let $A = \langle x \rangle$ and $B = \langle y - z \rangle$. Then $A \triangleleft L$, $B \triangleleft L$ but $A \cup B = \langle x, y, z \rangle$ is not subnormal in L .

PROOF. Since L/V is nilpotent, both $A + V$ and $B + V$ are subnormal in L . As x acts nilpotently on V , $A + V$ is nilpotent, so $A \triangleleft (A + V) \triangleleft L$. V is an irreducible L/V -module. By Zassenhaus [17, Lemma 1], all the composition factors of V as B -module are isomorphic. Now $[(y - z), (v_0 + \dots + v_{p-1})] = 0$. It follows that $(y - z)$ acts nilpotently on V . Hence $B + V$ is nilpotent and it follows that $B \triangleleft L$. Since $A \cup B$ is nilpotent, if $A \cup B \triangleleft L$, then every subalgebra of $A \cup B$ is subnormal in L , in particular $\langle z \rangle$, is subnormal. But $\langle z \rangle$ is not subnormal in $\langle z \rangle + V$. Therefore $A \cup B$ is not subnormal. \square

5. Some basic properties of the Wielandt subalgebra

In this section, we establish some basic properties of the Wielandt subalgebra $\omega(L)$ of a completely soluble Lie algebra L .

THEOREM 5.1. *Let L be a soluble Lie algebra and let $w \in \omega(L)$. Then there exists $\lambda \in F$ such that for all nilpotent subalgebras $U \triangleleft L$ and all $u \in U$, $[w, u] = \lambda u$. If for any such U , $U' \neq 0$ or $U \not\triangleleft L'$, then $\lambda = 0$.*

PROOF. Let $u \in U \triangleleft L$ and suppose U is nilpotent. Then $\langle u \rangle \triangleleft L$. Since $w \in \mathcal{N}_L(\langle u \rangle)$, $[w, u] = \lambda_u u$ for some $\lambda_u \in F$. We have to show that λ_u is independent of the choice of u . Let A be a minimal ideal of L and let $a \in A$, $a \neq 0$. Since $\langle u \rangle \triangleleft L$, $\text{ad}_u^r(A) \leq \langle u \rangle$ for some r and $\text{ad}_u^{r+1}(A) = 0$. It follows that $\langle u \rangle + A$ is nilpotent and subnormal. Therefore $\lambda_{u+a}(u + a) = [w, u + a] = [w, u] + [w, a] = \lambda_u u + \lambda_a a$. It follows that $\lambda_u = \lambda_a$.

Now suppose we have $U \triangleleft L$, U nilpotent and $U' \neq 0$. Take $u, v \in U$ with $x = [u, v] \neq 0$. Then

$$\lambda x = [w, [u, v]] = [[w, u], v] + [u, [w, v]] = [\lambda u, v] + [u, \lambda v] = 2\lambda x,$$

so $\lambda = 0$.

Suppose $U \triangleleft L$, U nilpotent and $U \not\triangleleft L'$. Take $u \in U$, $u \notin L'$. We then have $\lambda u = [w, u] \in L'$, so $\lambda = 0$. \square

COROLLARY 5.2. *Let N be a nilpotent Lie algebra. Then $\omega(N) = \mathcal{Z}(N)$.*

PROOF. If $w \in \omega(N)$ and N is non-abelian, then $w \in \mathcal{Z}(N)$ by Theorem 5.1. \square

LEMMA 5.3. *Suppose L is completely soluble and let A be a minimal ideal of L . Then $A \leq \omega(L)$.*

PROOF. Since A is a minimal ideal, either $A \cap L' = A$ or $A \cap L' = 0$. If $A \cap L' = 0$, then for all $x \in L$ and $a \in A$, we have $[x, a] \in A \cap L' = 0$ so $a \in \mathcal{Z}(L) \leq \omega(L)$. Suppose $A \leq L'$. Let $U \triangleleft\triangleleft L$. We show that $A \leq \mathcal{N}_L(U)$. This holds if $U \triangleleft L$, so suppose U is not an ideal. There exists a maximal ideal $M > U$. But $M \geq L'$. Hence $U + L'$ is a proper ideal of L . Since L' is nilpotent, $A \leq \mathcal{Z}(L')$ and we can consider A as an irreducible module for the abelian algebra L/L' . As $(U + L')/L'$ -module, A is a direct sum of copies A_i of an irreducible submodule A_1 . Now A_i is a minimal ideal of $U + L'$, so by induction over $\dim L$, we have $A_i \leq \mathcal{N}_{U+L'}(U) \leq \mathcal{N}_L(U)$. Thus $A = \bigoplus_i A_i \leq \mathcal{N}_L(U)$. \square

The condition that L is completely soluble is needed in Lemma 5.3.

EXAMPLE 5.4. Let L be the algebra of Example 4.3 above. We have $A + V$ and $B + V$ both nilpotent, subnormal and not in L' . By Theorem 5.1, $\omega(L) \leq \mathcal{C}_L(A + V)$ and $\omega(L) \leq \mathcal{C}_L(B + V)$. Since $A + V$ and $B + V$ generate L , $\omega(L) \leq \mathcal{Z}(L) = 0$.

By Theorem 5.1, for each $w \in \omega(L)$, we have $\lambda(w) \in F$ such that $[w, u] = \lambda(w)u$ for every element u which lies in some nilpotent subnormal subalgebra. This is clearly a linear map $\lambda : \omega(L) \rightarrow F$.

LEMMA 5.5. *Suppose L is completely soluble. Let $N = N(L)$ be the nil radical of L . Let $w \in \omega(L)$. If $w \in N$ then $\lambda(w) = 0$ and $w \in \mathcal{Z}(N)$. Conversely, if $\lambda(w) = 0$, then $w \in N$.*

PROOF. Suppose $w \in N$. Let $z \in \mathcal{Z}(N)$, $z \neq 0$. Then $\lambda(w)z = [w, z] = 0$, so $\lambda(w) = 0$ and for all $x \in N$, we have $[w, x] = \lambda(w)x = 0$ and $w \in \mathcal{Z}(N)$.

Suppose conversely that $\lambda(w) = 0$. Then $\langle w \rangle + N$ is nilpotent. As $\langle w \rangle + N \geq L'$, $\langle w \rangle + N \triangleleft L$, so $\langle w \rangle + N \leq N$, that is, $w \in N$. \square

We shall have occasion later to consider the algebra direct sum $L \oplus A$ of L , assumed completely soluble, with an abelian algebra A . Since clearly A normalises every subnormal subalgebra of L , it is tempting to expect $\omega(L \oplus A) = \omega(L) \oplus A$, but this is not always true as there are subnormal subalgebras of $L \oplus A$ which do not lie in either direct summand. Indeed, if $\omega(L)$ is non-abelian, then so is $\omega(L) \oplus A$, but $\omega(L \oplus A)$ is abelian by Theorem 5.1 and Lemma 5.5. If however, $\omega(L)$ is abelian, then the expected result holds. More generally, we have the following result.

THEOREM 5.6. *Suppose A and B are non-trivial completely soluble Lie algebras. Let L be the algebra direct sum $L = A \oplus B$. Then $\omega(L) = N(\omega(A)) \oplus N(\omega(B))$ and $\omega(L)$ is abelian.*

PROOF. Let $a \in A$ and $b \in B$. Suppose $U \lll A$. Then $(a + b) \in \mathcal{N}_L(U)$ if and only if $a \in \mathcal{N}_A(U)$. Thus $\omega(L) \leq \omega(A) \oplus \omega(B)$. It follows that $N(\omega(L)) \leq N(\omega(A)) \oplus N(\omega(B))$. Now suppose $a \in N(\omega(A))$ and $b \in N(\omega(B))$. We have to show that $a + b \in \omega(L)$. Let $u = c + d$ where $c \in A$ and $d \in B$. Put $U = u^{sL}$, $V = c^{sA}$ and $W = d^{sB}$. We have $U = \langle u, X_i(u, L) \rangle$ for some i . We have to show that $[a + b, u] \in U$. But $X_i(u, L) = X_i(c, A) + X_i(d, B)$ and $V = \langle c, X_i(c, A) \rangle$. Thus $[a, c] \in X_i(c, A)$, (0 if $c \in A'$) and similarly $[b, d] \in X_i(d, B)$. Hence $a + b \in \omega(L)$.

If $a + b \in \omega(L)$, then $[a + b, c + d] = \lambda c + \lambda d$ for all $c \in N(A)$ and all $d \in N(B)$ and some $\lambda \in F$. If $\omega(A)$ is abelian, then $\lambda = 0$ and so $a + b \in N(L)$ by Lemma 5.5. Thus $\omega(L)$ is abelian if $\omega(A)$ is abelian.

Now suppose $\omega(L)$ is non-abelian. Then $\omega(A)$ and, by symmetry, $\omega(B)$ are non-abelian. There exists $a + b \in \omega(L)$ such that $[a + b, v] = v$ for all $v \in N(L)$. Since $A \neq A'$, there exists $c \in A$, $c \notin A'$. Since B is non-abelian, we can take $d \in B'$, $d \neq 0$, and put $u = c + d$. As $\langle d \rangle \lll L$, $[a + b, d] = d$. Since $a + b \in \omega(L)$, $[a + b, u] = [a, c] + d \in u^{sL}$. But $u^{sL} = \langle c + d, X_i(c, A) \rangle$ for some i . Thus $X_i(c, A)$ has codimension 1 in u^{sL} whence it follows that $d \notin u^{sL}$. Since $a \in \omega(A)$, $[a, c] \in X_i(c, A)$ contrary to $[a, c] + d \in u^{sL}$. Thus $\omega(L)$ is abelian and it follows that $\omega(L) = N(\omega(A)) \oplus N(\omega(B))$. □

Since $(A \oplus B)/\omega(A \oplus B) \simeq A/N(\omega(A)) \oplus B/(N(\omega(B)))$, we can readily determine $\omega_i(A \oplus B)$ by repeated application of Theorem 5.6 with the help of Theorem 6.2 below to deal with any non-abelian cases which arise.

6. Non-abelian Wielandt subalgebras

In this section, we determine the structure of completely soluble Lie algebras L with $\omega(L)$ non-abelian.

LEMMA 6.1. *Suppose L is completely soluble. Then $\omega(L)$ is abelian if and only if the map λ is the zero map.*

PROOF. If $\lambda = 0$, then $\omega(L) \leq \mathcal{Z}(N)$, so $\omega(L)$ is abelian. If $\lambda(w) \neq 0$ and A is a minimal ideal of L , then $\langle w, A \rangle$ is a subalgebra by Theorem 5.1, clearly non-abelian and $\langle w, A \rangle \leq \omega(L)$ by Lemma 5.3. □

THEOREM 6.2. *Suppose L is completely soluble and $\omega(L)$ is non-abelian. Then $N(L) = L'$ is abelian and $L = A \oplus L'$ (vector space direct sum) where $A < L$. There*

exists $w \in A \cap \omega(L)$ with $\lambda(w) = 1$ and $\omega(L) = \langle w, L' \rangle$ which is a characteristic ideal of L . The Wielandt series of L exists and $\omega_2(L) = L$. If $c \in A$, $c \notin \langle w \rangle$ then the linear transformation $\bar{c} : L' \rightarrow L'$ given by the action of c is invertible and has no eigenvalues in F .

PROOF. By Theorem 5.1 and Lemma 6.1, $N(L) = L'$ is abelian. There exists $w \in \omega(L)$ with $\lambda(w) \neq 0$, so we may choose w with $\lambda(w) = 1$. Put $V = L'$. We consider V as an L/L' -module. We have $V^{L/L'} = 0$. By Barnes [1, Theorem 1], $H^2(L/L', V) = 0$. Thus there exists a subalgebra A which complements L' in L . Since every minimal ideal of L is contained in $N(L)$ and $A \cap N(L) = 0$, A contains no non-trivial ideal of L . Thus $\mathcal{C}_A(L') = 0$ and V is a faithful A -module.

For any $v \in V$, the map $\alpha_v = 1 - \text{ad}_v : L \rightarrow L$ is an automorphism of L since

$$\begin{aligned} [\alpha_v(x), \alpha_v(y)] &= [x - [v, x], y - [v, y]] \\ &= [x, y] - [[v, x], y] - [x, [v, y]] + [[v, x], [v, y]] \\ &= [x, y] - [v, [x, y]] = \alpha_v[x, y] \end{aligned}$$

as $[v, x], [v, y] \in V$ which is abelian. Let w' be the element of the coset $w + L'$ which lies in A . Then $w' = w + v_0$ for some $v_0 \in V$. Since $w' = \alpha_{v_0}(w)$, $w' \in \omega(L)$ and we may replace w with w' . Thus we obtain $w \in A$ with the required properties. Further, for every $v \in V$, $w + v = w - [v, w] = \alpha_v(w) \in \omega(L)$, so $v \in \omega(L)$ and $\omega(L) = \langle w, L' \rangle$. Let d be any derivation of L . Then $dL' \subseteq L'$. Also, for any $v \in V$, we have $dv = d[w, v] = [dw, v] + [w, dv] = [dw, v] + dv$, so $dw \in \mathcal{C}_L(L') = L'$. Thus $d\omega(L) \subseteq \omega(L)$. Since $L/\omega(L)$ is abelian, $\omega_2(L) = L$ and L has Wielandt length at most 2.

Now take $c \in A$, $c \notin \langle w \rangle$. Then we can decompose V with respect to \bar{c} into the direct sum $V = V_0 \oplus V_1$ where \bar{c} acts nilpotently on V_0 and invertibly on V_1 . As A is abelian, this decomposition is A -invariant, that is, V_0 and V_1 are ideals of L . Suppose $V_0 \neq 0$. Put $V_0^i = \bar{c}^i V_0$. For some r , we have $V_0^r \neq 0$ but $V_0^{r+1} = 0$. Let $v \in V_0$ and put $u = c + v$. Then

$$\langle u, V_1 \rangle \triangleleft \langle u, V_0^r, V_1 \rangle \triangleleft \langle u, V_0^{r-1}, V_1 \rangle \triangleleft \dots \triangleleft \langle u, V_0, V_1 \rangle \triangleleft L$$

and $\langle u, V_1 \rangle \triangleleft L$. But $[w, u] = [w, c] + [w, v] = v$, so $v \in \langle u, V_1 \rangle \cap V_0 = 0$. Therefore $V_0 = 0$ and \bar{c} is invertible. Since for all $\mu \in F$, $\mu w - c$ acts invertibly on V , \bar{c} has no eigenvalues in F . □

7. Normality of the Wielandt subalgebra

We are now in a position to prove the main result about the Wielandt subalgebra.

THEOREM 7.1. *Suppose L is completely soluble. Then $\omega(L) \triangleleft L$.*

PROOF. By Theorem 6.2, we may suppose $\omega(L) \leq \mathcal{Z}(N)$. We then have

$$\omega(L) = \bigcap_{U \triangleleft\triangleleft L} \mathcal{N}_L(U) = \bigcap_{U \triangleleft\triangleleft L} (N \cap \mathcal{N}_L(U)).$$

We consider subnormal subalgebras U . If U is not nilpotent, then there exists an ideal $A \triangleleft L, 0 \neq A \leq U$. Now $\mathcal{N}_{L/A}(U/A) = \mathcal{N}_L(U)/A$. By induction over $\dim L$,

$$\bigcap_{U \geq A} \mathcal{N}_L(U) \triangleleft L.$$

If U is nilpotent, then $U \leq N$ by Lemma 4.1, and we have

$$\bigcap_{U \text{ nilp}} (N \cap \mathcal{N}_L(U)) = \mathcal{Z}(N) \triangleleft L.$$

Hence

$$\omega(L) = \bigcap_{U \triangleleft\triangleleft L} \mathcal{N}_L(U) = \bigcap_{A \triangleleft L} \left(\bigcap_{U \geq A} \mathcal{N}_L(U) \right) \cap \bigcap_{U \text{ nilp}} \mathcal{N}_L(U) \triangleleft L. \quad \square$$

The following example shows that the condition that L is completely soluble cannot be omitted from Theorem 7.1.

EXAMPLE 7.2. Let $V = \langle v_0, \dots, v_{p-1} \rangle$ where $p = \text{char } F$ and the subscripts are integers mod p . Put $L = \langle a, b, c, V \rangle$ with multiplication given by $[a, b] = b$, $[a, c] = -c$, $[b, c] = 0$, $[v_i, v_j] = 0$ and $[a, v_i] = i v_i$, $[b, v_i] = v_{i+1}$ and $[c, v_i] = v_{i-1}$. Then $\omega(L) = \langle v_0 + \dots + v_{p-1} \rangle$ which is not an ideal of L .

PROOF. We determine the subnormal subalgebras U which are not ideals. Any such U is contained in some maximal ideal which must contain L' . But $L' = \langle b, c, V \rangle$ is a maximal ideal, so $U \leq L'$ and so $U \leq M$ for some maximal ideal M of L' . As $L'' = V$, $M = \langle \lambda b - \mu c, V \rangle$ for some $\lambda, \mu \in F$. If $\lambda = 0$ or $\mu = 0$, then $M' = V$ and it follows that $U \leq V$. If $\lambda \neq 0$, we may take it to be 1. Then we have $[b - \mu c, v_i] = v_{i+1} - \mu v_{i-1}$. Thus for all i , we have $v_i - \mu v_{i-2} \in M'$. It follows that $v_i - \mu^r v_{i-2r} \in M'$ for all r . Taking $r = p$, we get $(1 - \mu^p)v_i \in M'$. Thus $M' = V$ and $U \leq V$ unless $\mu^p = 1$, that is, unless $\mu = 1$ and $M = \langle b - c, V \rangle$. Since $\langle b - c \rangle \triangleleft\triangleleft \langle a, b, c \rangle$ and V is an irreducible $\langle a, b, c \rangle$ -module, it contains only one type of composition factor as $\langle b - c \rangle$ -module. As $[b - c, v_0 + \dots + v_{p-1}] = 0$, $b - c$ acts nilpotently on V and $M = \langle b - c, V \rangle$ is nilpotent. Thus $\omega(L) = \mathcal{Z}(M) = \langle v_0 + \dots + v_{p-1} \rangle$. □

If $\text{char } F = 0$, then $\omega(L)$ is a characteristic ideal. It is natural to ask if this holds for completely soluble Lie algebras L over a field of non-zero characteristic. Suppose L is a minimal counterexample. If $U \triangleleft L$ is not nilpotent, then U^∞ is a non-trivial characteristic ideal. It follows that the intersection of the $\mathcal{N}_L(U)$ of the non-nilpotent $U \triangleleft L$ is a characteristic ideal of L . Let d be any derivation of L and let M be the split extension of L by d . Then $d(N) + L' \leq M'$ and is an ideal of L . Now M is soluble. If M' is nilpotent, then $M' \leq N$, so $d(N) \subseteq N$. It then follows that $d(\mathcal{Z}(N)) \subseteq \mathcal{Z}(N)$ and so $d(\omega(L)) \subseteq \omega(L)$. Hence our minimal counterexample must have a derivation d such that M' is not nilpotent. We are now in a position to construct such an example. We have incidently given another proof that $\omega(L)$ is a characteristic ideal if L is a soluble algebra over a field of characteristic 0 as then, M' is always nilpotent.

EXAMPLE 7.3. Suppose $\text{char } F = p \neq 0$. Let $L = \langle y, z, v_0, \dots, v_{p-1} \rangle$, where the subscripts are integers $\pmod p$, be the Lie algebra with multiplication given by $[y, z] = 0$, $[y, v_i] = i v_{i+1}$, $[z, v_i] = v_i$ and $[v_i, v_j] = 0$. Then $\omega(L)$ is not a characteristic ideal of L .

PROOF. We have $N = \langle y, v_0, \dots, v_{p-1} \rangle > L'$, so $\omega(L) \leq \mathcal{Z}(N) = \langle v_0 \rangle$. As $\langle v_0 \rangle$ is a minimal ideal, it follows that $\omega(L) = \langle v_0 \rangle$. Let $d : L \rightarrow L$ be given by $d(y) = z$, $d(z) = 0$ and $d(v_i) = v_{i-1}$. Then d is a derivation of L , (the split extension M of L by d is a familiar example of a soluble algebra with non-nilpotent derived algebra) and $d(\omega(L)) = \langle v_{p-1} \rangle \not\subseteq \omega(L)$. □

8. Wielandt and derived lengths

A T -algebra is a Lie algebra in which the relation of normality is transitive, thus an algebra in which every subideal is an ideal, or in the terminology we have introduced, an algebra of Wielandt length 1. We require the following theorem of Stewart [13] giving the structure of such algebras:

THEOREM 8.1. *L is a soluble T -algebra if and only if L is either abelian or the split extension of an abelian algebra by its algebra of scalar linear transformations.*

PROOF. Let L be a soluble T -algebra. We use induction over $\dim(L)$. Since L is soluble, there exists a 1-dimensional subnormal subalgebra A of L . Since L is a T -algebra, $A \triangleleft L$. As L/A is also a T -algebra, by induction, we have that any ideal of L/A of codimension 1 is abelian. Now consider A as an L -module. Then $L/\mathcal{C}_L(A)$ has dimension at most 1 and it follows that there exists $N \triangleleft L$ of codimension 1 in L , such that $A \leq N \leq \mathcal{C}_L(A)$. We have N/A abelian and $A \leq \mathcal{Z}(N)$. Thus N is

nilpotent. Take $x \in L, x \notin N$. Since $L = \omega(L)$, by Theorem 5.1, for some $\lambda \in F$, we have $[x, n] = \lambda n$ for all $n \in N$.

If $\lambda = 0$, then $x \in \mathcal{Z}(L)$ and L is nilpotent. We then have $L = \omega(L) = \mathcal{Z}(L)$ by Corollary 5.2 and L is abelian.

If $\lambda \neq 0$, then N is abelian by Theorem 5.1 and L has the asserted structure.

Conversely, suppose L is the split extension of an abelian algebra A by its algebra of scalar linear transformations. Then $A = L'$ and has codimension 1. Thus A is the only maximal ideal of L , so every proper subideal is a subalgebra of A . Clearly, every subspace of A is an ideal of L . Thus L is a T -algebra. \square

If L is completely soluble, $\omega(L) \triangleleft L$ and so every subideal of $\omega(L)$ is also a subideal of L , so normalised by $\omega(L)$. Thus $\omega(L)$ is a T -algebra. By Theorem 8.1, $N(\omega(L))$ is either equal to $\omega(L)$ or is an abelian ideal of it of codimension 1. In either case, it is a characteristic ideal of $\omega(L)$ and so is an ideal of L .

LEMMA 8.2. *Suppose L is completely soluble. Then*

$$[L', N(\omega(L))] = 0 \quad \text{and} \quad L' \cap \omega(L) \leq N(\omega(L)).$$

PROOF. Let $x \in L'$. If $x \in N(\omega(L))$, then $[x, N(\omega(L))] = 0$ since $N(\omega(L))$ is abelian. Suppose $x \notin N(\omega(L))$. Since L' is nilpotent, every subalgebra of L' is subnormal in L , in particular, $\langle x \rangle \triangleleft L$ and is normalised by $\omega(L)$. Thus $[x, N(\omega(L))] \subseteq \langle x \rangle$. But $N(\omega(L)) \triangleleft L$, so $[x, N(\omega(L))] \subseteq N(\omega(L))$. Thus $[x, N(\omega(L))] \subseteq \langle x \rangle \cap N(\omega(L)) = 0$ since $x \notin N(\omega(L))$. Thus $[L', N(\omega(L))] = 0$. As $L' \cap \omega(L)$ is a nilpotent ideal of $\omega(L)$, it is contained in $N(\omega(L))$. \square

LEMMA 8.3. *If L is completely soluble and has Wielandt length k , then L' has class at most k .*

PROOF. If $k = 1$, then L is a T -algebra and L' is abelian. Suppose $k > 1$. $L/\omega(L)$ has Wielandt length $k - 1$, so using induction over k , we may suppose $(L/\omega(L))'$ has class at most $k - 1$. Thus $(L')^k \leq \omega(L)$. Now using Lemma 8.2, we have

$$(L')^{k+1} = [L', (L')^k] \subseteq [L', L' \cap \omega(L)] \subseteq [L', N(\omega(L))] = 0. \quad \square$$

THEOREM 8.4. *Suppose L is completely soluble. Let k be the Wielandt length of L and let l be the least integer greater than or equal to $\log_2(k + 1)$. Then L has derived length at most $l + 1$.*

PROOF. For any n , we have $L^{(n+1)} = (L')^{(n)} \leq (L')^{2^n}$. But $(L')^{k+1} = 0$. Thus $L^{(n+1)} = 0$ if $2^n \geq k + 1$, that is, if $n \geq \log_2(k + 1)$. \square

We now show that this bound is best possible. To do this, we first consider some nilpotent algebras with some special structure.

Let N be a positively graded finite-dimensional Lie algebra over the field F . Thus $N = N_1 \oplus N_2 \oplus \dots \oplus N_k$ as vector space and $[N_i, N_j] \subseteq N_{i+j}$. Then N is nilpotent of class at most k . To simplify notation, we put $N_r^* = \sum_{i \geq r} N_i$. We shall suppose that N is generated by N_1 . In this case, if $N_k \neq 0$, N has class precisely k , the descending central series of N being $N = N_1^* \triangleright N_2^* \triangleright \dots \triangleright N_k^* \triangleright N_{k+1}^* = 0$. The terms $\mathcal{Z}_i(N)$ of the ascending central series of N satisfy $\mathcal{Z}_i(N) \supseteq N_{k-i+1}^*$. To make an inductive argument using the quotient by the centre work smoothly, we suppose that $\mathcal{Z}_i(N) = N_{k-i+1}^*$ for all i . Such an algebra, one in which the terms of the ascending central series are as small as possible, we shall call mini-central. Such algebras do exist as is shown by the following example.

EXAMPLE 8.5. Let N be the Lie algebra of strictly upper triangular $(k + 1) \times (k + 1)$ matrices. Put $N_r = \langle e_{i,i+r} \mid i = 1, \dots, k - r + 1 \rangle$ where e_{ij} denotes the matrix with 1 in the ij position and all other entries 0. This algebra is clearly mini-central.

A mini-central graded nilpotent algebra N has a derivation $d : N \rightarrow N$ of degree 0 given by $da = ia$ for $a \in N_i$. Let $L = \langle d, N \rangle$ be the split extension of N by $\langle d \rangle$. Thus $[d, a] = ia$ for $a \in N_i$.

LEMMA 8.6. *Let L be as above with $k > 1$. Then $\omega(L) = N_k$.*

PROOF. Since L is soluble, any maximal ideal of L must contain the derived subalgebra L' . But $L' = N$. Thus N is the only maximal ideal and the subnormal subalgebras of L are precisely the subalgebras of N . From Theorem 5.1, it follows that $\omega(L) = \mathcal{C}_L(N)$. We have only to show that $\omega(L) \leq N$ and the result follows.

Suppose $w \in \omega(L)$ but $w \notin N$. Then without loss of generality, we may suppose $w = d + a$ for some $a \in N$. If $a \notin \mathcal{Z}(N)$, then there exists $b \in N_r$ for some r with $[a, b] \neq 0$. We then have $[w, b] = rb + [a, b] \notin \langle b \rangle$ contrary to $w \in \omega(L)$. If $a \in \mathcal{Z}(N)$, we take non-zero elements $b \in N_1$ and $c \in N_2$. Then $[w, b + c] = b + 2c \notin \langle b + c \rangle$ contrary to assumption. □

THEOREM 8.7. *The algebra L constructed above has Wielandt length k and derived length $l + 1$ where l is the least integer greater than or equal to $\log_2(k + 1)$ so attaining the bound given in Theorem 8.4. Hence that bound is best possible.*

PROOF. We take L the algebra constructed as above from the minicentral graded nilpotent algebra N of class k . From Lemma 8.6, it follows that $\omega_{k-1}(L) = N_2^*$. As clearly $\omega(L/N_2^*) = L/N_2^*$, the Wielandt length of L is k . The derived length of N is l , so the derived length of L is $l + 1$. □

9. Lie algebras with Wielandt length 2

In this section, we describe the structure of the completely soluble Lie algebras which have Wielandt length precisely 2. We divide them into three cases according to which, if any, of $\omega(L)$ and $L/\omega(L)$ are non-abelian, both non-abelian being excluded by Theorem 6.2. We have already given the structure for the case $\omega(L)$ non-abelian in Theorem 6.2. We now consider the cases in which $\omega(L)$ is abelian. Some are obtained as direct sums $L = M \oplus A$ where $\omega(M)$ is non-abelian and A is an abelian algebra. The structure of these follows from Theorem 6.2. If L is an algebra with $\omega(L)$ abelian and A is abelian, then $\omega(L \oplus A) = \omega(L) \oplus A$ and $(L \oplus A)/\omega(L \oplus A) \simeq L/\omega(L)$. Thus we need only consider algebras L which have no non-trivial abelian direct summand.

THEOREM 9.1. *Suppose $\text{char } F \neq 2$. Let L be a completely soluble Lie algebra over F with Wielandt length 2. Suppose $L/\omega(L)$ is non-abelian. Suppose L has no non-trivial abelian direct summand. Then there exist subspaces W, A and an element b such that the following conditions are satisfied.*

- (1) As vector space, $L = W \oplus A \oplus \langle b \rangle$.
- (2) $[W, W + A] = 0$.
- (3) Setting $\beta(w) = [w, b]$ defines an invertible linear map $\beta : W \rightarrow W$.
- (4) Put $W_1 = \{w \in W \mid (\beta - 1)^n w = 0 \text{ for some } n\}$. Then for all $a \in A$, $[a, b] = a + \mu(a)$ where $\mu(a) \in W_1$.
- (5) $[A, A] = L' \leq W$ and $[w, b] = 2w$ for all $w \in L'$.
- (6) For all $a \in A$, $a \neq 0$, there exists $a' \in A$ with $[a, a'] \neq 0$.

PROOF. Put $W = \omega(L)$. Then by assumption, $[W, W] = 0$. Also $L' \not\subseteq W$. We choose some subspace A complementary to W in $L' + W$. Then $(L/W)' = (A + W)/W$. Since L/W is non-abelian of Wielandt length 1, $A + W$ has codimension 1 in L , and there exists $b \in L$ such that $[a, b] = a + \mu(a)$ with $\mu(a) \in W$ for all $a \in A$. Also $[A, A] \subseteq W$. Let $f(x) = x^r(x - 1)^s g(x)$, where x and $(x - 1)$ do not divide $g(x)$, be the minimum polynomial of the linear map $\beta : W \rightarrow W$ given by $\beta(w) = [w, b]$ for all $w \in W$. Put $W_0 = (\beta - 1)^s g(\beta)W$, $W_1 = \beta^r g(\beta)W$ and $W_* = \beta^r (\beta - 1)^s W$. Then β is nilpotent on W_0 and invertible on $W_1 + W_*$. We now adjust our choice of A . For any linear map $\alpha : A \rightarrow W$, we can replace A by $\{a + \alpha(a) \mid a \in A\}$. Observe that if $a_1 = a + \alpha(a)$, then

$$[a_1, b] = [a, b] + [\alpha(a), b] = a + \mu(a) + \beta\alpha(a) = a_1 + (\mu + (\beta - 1)\alpha)(a).$$

We thus obtain $A_1 = (1 + \alpha)A$ and $\mu_1 : A_1 \rightarrow W$ such that $[a_1, b] = a_1 + \mu_1(a_1)$ with $\mu_1(a_1) = \mu(a) + (\beta - 1)\alpha(a)$. Regarding α, μ, μ_1 as maps $(A + W)/W \rightarrow W$, we have $\mu_1 = \mu + (\beta - 1)\alpha$. By making a suitable choice of subspace A_1 , we can thus

replace μ by $\mu + \gamma$ for any $\gamma : A \rightarrow W$ for which there exists α with $(\beta - 1)\alpha = \gamma$, that is, any γ such that $\gamma(A) \subseteq (\beta - 1)W$. Since $(\beta - 1)$ acts invertibly on $W_0 + W_*$, we can choose A such that $\mu(A) \subseteq W_1$. As $W_1 = [W_1, b] \subseteq L'$, for this choice of A , we have $a = [a, b] - \mu(a) \in L'$ for all $a \in A$, so $A \subseteq L'$ and A complements $W \cap L'$ in L' . By Lemma 6.1, $[W, L'] = 0$. We have

$$[[a, a'], b] = [[a, b], a'] + [a, [a', b]] = [a + \mu(a), a'] + [a, a' + \mu(a')] = 2[a, a']$$

since $[W, A] = 0$. Since $\text{char } F \neq 2$, $[A, A] \leq W_*$.

We now prove $r \leq 1$. Let $U = b^{sL}$. Since $\beta^i(W_1 + W_*) = W_1 + W_*$ for all i , $W_1 + W_* \leq U$. Since $[\dots[A, b], b] \dots, b] + W_1 = A + W_1$ whatever the number of factors, $A \subseteq U$. But as $\langle b, A, \beta^r W \rangle \triangleleft \langle b, A, \beta^{r-1} W \rangle \triangleleft \dots \triangleleft \langle b, A, W \rangle = L$, it follows that $U = \langle b, A, W_1, W_* \rangle$. Therefore $[W, b] \subseteq W_1 + W_*$, so $r \leq 1$.

We now clearly have $L' = A + W_1 + W_*$ and $L'' = [A, A]$. Also, $W_0 \triangleleft L$ and $L = W_0 \oplus (W_1 + W_* + A + \langle b \rangle)$. Since L has no abelian direct summand, $W_0 = 0$ and $W = W_1 + W_*$.

Now suppose $a \in A$, $a \neq 0$. Since $a \notin \omega(L)$, there exists $x \in L$ such that $[a, x] \notin x^{sL}$. If $x \notin W + A$, then $x^{sL} = W + A + \langle x \rangle = L$, so $x \in W + A$ and $x^{sL} = \langle x \rangle$. We have $x = w + a'$ for some $w \in W$ and $a' \in A$. Now $[a, x] = [a, a']$, so $[a, a'] \notin x^{sL}$ which implies $[a, a'] \neq 0$. □

In the above, we still have some freedom of choice for the subspace A and the corresponding map $\mu : A \rightarrow W_1$. For any linear map $\alpha : A \rightarrow W_1$, the subspace $A_1 = (1 + \alpha)A$ also satisfies our requirements with μ replaced by $\mu_1 = \mu + (\beta - 1)\alpha$. Let $\bar{\mu}, \bar{\mu}_1$ denote the composites of μ, μ_1 with the natural map $W_1 \rightarrow W_1/(\beta - 1)W_1$. Then $\bar{\mu} = \bar{\mu}_1$. Conversely, for any linear map $\mu_1 : A \rightarrow W_1$ such that $\bar{\mu}_1 = \bar{\mu}$, there exists a linear map $\alpha : A \rightarrow W_1$ such that $\mu_1 - \mu = (\beta - 1)\alpha$. Thus by suitable choice of A_1 , we may replace μ by any μ_1 with $\bar{\mu}_1 = \bar{\mu}$.

Any Lie algebra L with the structure described has Wielandt length 2 and $L/\omega(L)$ non-abelian. Indeed, given the above data, we can construct a Lie algebra, uniquely determined by the data, which has those properties.

THEOREM 9.2. *Let F be a field with $\text{char } F \neq 2$. Let W, A be vector spaces over F with $\dim A \geq 2$. Let $\beta : W \rightarrow W$ be a non-singular linear transformation with 2 as an eigenvalue. Let $W_1 = \{w \in W \mid (\beta - 1)^r w = 0 \text{ for some } r\}$, let W_* be the complementary invariant subspace and let $W_2 = \{w \in W \mid b(w) = 2w\}$. Let $m : A \times A \rightarrow W_2$ be a non-singular alternating bilinear map and let $\tau : A \rightarrow W/(\beta - 1)W$ be any linear map. Take $B = \langle b \rangle$ a 1-dimensional vector space. Let μ be any lifting of τ to a linear map into W_1 . Put $L = W \oplus A \oplus B$ and define the multiplication $[\ , \] : L \wedge L \rightarrow L$ by*

$$[w, w'] = [w, a] = 0, \quad [a, a'] = m(a, a'), \quad [w, b] = \beta(w), \quad [a, b] = a + \mu(a)$$

for all $w, w' \in W$ and $a, a' \in A$. Then L is a Lie algebra with no non-trivial abelian direct summand, $L' = W + A$ is nilpotent of class 2, $\omega(L) = W = \mathcal{Z}(L')$, L has Wielandt length 2 and $L/\omega(L)$ is non-abelian.

PROOF. By definition, $[x, y] = -[y, x]$ for all $x, y \in L$. The Jacobi identity is easily verified. Thus L is a Lie algebra. Clearly, $L' = W + A$. Since m is non-singular, for any $a \in A, a \neq 0$, there exists $a' \in A$ with $[a, a'] \neq 0$. Thus $L'' = [A, A] \neq 0$. But $[A, A] \leq W$ and $[W, W + A] = 0$. Thus L' is nilpotent of class 2. If $a \in A, a \neq 0$ and $w \in W$, then for some $a' \in A$, we have $[a + w, a'] \neq 0$. Hence $W = \mathcal{Z}(L')$. Thus $\omega(L) \leq W$. If $x = b + w + a$ for some $w \in W$ and $a \in A$, then $[W, x] = [W, b] = W$, so $W \leq x^{sL}$. Since $[a + W, x] = a + W$ for all $a \in A, A \leq x^{sL}$. Thus $x^{sL} \geq L'$ so $x^{sL} \triangleleft L$. It follows that $W \leq \mathcal{N}_L(U)$ for every $U \triangleleft L$ with $U \not\leq L'$. But $W = \mathcal{Z}(L')$, so $W \leq \mathcal{N}_L(U)$ for $U \triangleleft L$ with $U \leq L'$. Thus $W = \omega(L)$. As L/W has the abelian ideal $A + W/W$ of codimension 1 with $b + W$ acting as the identity, L/W is non-abelian of Wielandt length 1. Finally, as $\mathcal{Z}(L) = 0$, L has no non-trivial abelian direct summand. \square

The situation in characteristic 2 is a little different.

THEOREM 9.3. Suppose $\text{char } F = 2$. Let L be a completely soluble Lie algebra over F with Wielandt length 2. Suppose $L/\omega(L)$ is non-abelian. Suppose L has no non-trivial abelian direct summand. Then there exist subspaces W_0, W_*, A and an element b such that, putting $W = W_0 \oplus W_*$, the following conditions are satisfied.

- (1) As vector space, $L = W \oplus A \oplus \langle b \rangle$.
- (2) $[W, W + A] = 0$.
- (3) Setting $\beta(w) = [w, b]$ defines a linear map $\beta : W \rightarrow W$ with $\beta^2(W_0) = 0$ and $\beta(W_*) = W_*$.
- (4) Put $W_1 = \{w \in W \mid (\beta - 1)^n w = 0 \text{ for some } n\}$. Then for all $a \in A, [a, b] = a + \mu(a)$ where $\mu(a) \in W_1$.
- (5) $[A, A] = L'' = \{w \in W \mid \beta(w) = 0\}$.
- (6) For all $a \in A, a \neq 0$, there exists $a' \in A$ with $[a, a'] \neq 0$.

PROOF. As in the proof of Theorem 9.1, we obtain spaces W, A and an element b giving maps $\beta : W \rightarrow W$ and $\mu : A \rightarrow W$. As before, we get a decomposition $W = W_0 \oplus W_1 \oplus W_*$. We put $W_* = W_1 + W_*$. As before, we can adjust the choice of A so that we have $\mu(A) \subseteq W_1$. Also as before, we have $[[a, a'], b] = 2[a, a']$, but as $\text{char } F = 2$, we have $[A, A] \subseteq \{w \in W \mid \beta(w) = 0\} \subseteq W_0$.

We now prove $\beta(W_0) \subseteq [A, A]$. Let $U = b^{sL}$. As before, we see that $A + W_* \subseteq U$, and it follows that $U = \langle A, [A, A], W_*, b \rangle$. Hence, if $w \in W_0$, then $\beta(w) \in [A, A]$.

If $\beta(w) = 0$ and $w \notin [A, A]$, then $\langle w \rangle$ is an abelian direct summand of L . Therefore $\{w \in W \mid \beta(w) = 0\} = [A, A]$. As before, for any non-zero $a \in A$, there exists

$a' \in A$ with $[a, a'] \neq 0$. □

As in the case $\text{char } F \neq 2$, given the spaces and maps as above, we can construct a Lie algebra with the required properties. We leave the details to the reader.

We now consider algebras L with both $\omega(L)$ and $L/\omega(L)$ abelian.

THEOREM 9.4. *Suppose L has Wielandt length 2 and both $\omega(L)$ and $L/\omega(L)$ are abelian. Suppose L has no non-trivial abelian direct summand. Then there exist subspaces W_0, \dots, W_k, A of L , $W_* = W_1 \oplus \dots \oplus W_k$, such that*

- (1) $\omega(L) = W_0 \oplus W_*$.
- (2) $L = \omega(L) \oplus A$ as vector space.
- (3) For all $a \in A$ and all i , either $[W_i, a] = 0$ or $[W_i, a] = W_i$.
- (4) $[W_0, A] = 0$ and $[W_*, A] = W_*$.
- (5) $W_0 = [A, A]$.
- (6) If $a \in A$, $a \neq 0$, then $[L, a] \neq 0$.
- (7) Either $\dim A > 1$ or there exists $a \in A$ such that the action $w \mapsto [w, a]$ is not a scalar transformation of W .

PROOF. Put $W = \omega(L)$. Then W is an L/W -module. Since L/W is abelian, W is the direct sum $W = W_0 \oplus \dots \oplus W_k$ of submodules W_i , each of which has only one type of composition factor. (See for example, Barnes [2].) Let W_0 be the component of the trivial composition factors and let W_* be the sum of the components with non-trivial composition factors. By Barnes [1, Theorem 1], $H^2(L/A, W_*) = 0$, so L/W_0 splits over W/W_0 . Thus there exists a subspace A such that $L = W \oplus A$ as vector space and $[A, A] \subseteq W_0$. We have $[W_*, A] = W_*$. Suppose $a \in A$. Let $\bar{a} : W \rightarrow W$ be the linear transformation given by $\bar{a}(w) = [w, a]$ and let $W = W_0(a) \oplus W_*(a)$ be the decomposition of W into null and non-null components with respect to \bar{a} . Suppose $a \neq 0$ and $w \in W$. Put $u = w + a$. Then $\langle W, u \rangle \triangleleft L$ and $u^{sL} = u^{s(W,u)}$, so it follows that $u^{sL} = \langle W_*(a), u \rangle$. If $w \in W_0(a)$, then $[w, a] = [w, u] \in u^{sL}$, so $[w, a] = 0$. Thus for each W_i , either $[W_i, a] = 0$ or W_i has no trivial composition factor as $\langle a \rangle$ -module and so $[W_i, a] = W_i$. In particular, $[W_0, A] = 0$. For each of the other components, there is some $a \in A$ which acts non-trivially, so $[W_*, A] = W_*$.

If $w \in W_0$ and $w \notin [A, A]$, then $\langle w \rangle$ is an abelian direct summand of L . Therefore $W_0 = [A, A]$. If $a \in A$ and $[L, a] = 0$, then $\langle a \rangle$ is an abelian direct summand, so $a = 0$. Clearly, $L \neq W$, so $\dim A \geq 1$. If $A = \langle a \rangle$, and \bar{a} is scalar, then $\omega(L) = L$ contrary to assumption. □

Clearly, if we are given vector spaces W_0, \dots, W_k, A , $W = W_0 \oplus \dots \oplus W_k$, and, for all $a \in A$, a linear transformation $\bar{a} : W \rightarrow W$, and an alternating bilinear map $m : A \times A \rightarrow W_0$ such that

- (1) $W \neq 0$ and $A \neq 0$;

- (2) for all $a, a' \in A$, $\bar{a}\bar{a}' = \bar{a}'\bar{a}$;
- (3) for all $a \in A$ and all i , either $\bar{a}(W_i) = 0$ or $\bar{a}(W_i) = W_i$;
- (4) for all $a \in A$, $\bar{a}(W_0) = 0$;
- (5) for all $i \neq 0$, there exists $a \in A$ such that $\bar{a}(W_i) = W_i$;
- (6) $m(A, A) = W_0$;
- (7) for all $a \in A$, $a \neq 0$, either $\bar{a} \neq 0$ or there exists $a' \in A$ such that $m(a, a') \neq 0$;
- (8) if $\dim A = 1$ and $a \in A$, $a \neq 0$, then \bar{a} is not scalar

then we can construct a Lie algebra L as in Theorem 9.4 by setting $L = W \oplus A$, $[w, w'] = 0$, $[a, a'] = m(a, a')$ and $[w, a] = \bar{a}(w)$ for all $w, w' \in W$ and $a, a' \in A$.

10. Lie algebras with Wielandt length > 2

We have seen (Theorem 6.2) that in the Wielandt series

$$0 < \omega_1(L) < \dots < \omega_{n-1}(L) < \omega_n(L) = L,$$

only the last $L/\omega_{n-1}(L)$ or second last $\omega_{n-1}(L)/\omega_{n-2}(L)$, but not both, of the quotients $\omega_i(L)/\omega_{i-1}(L)$ can be non-abelian. By Theorem 6.2, the second last non-abelian can only occur if the field F is not algebraically closed. In proving Theorem 8.7, we showed that we can have an arbitrarily long Wielandt series with non-abelian final quotient. We now show that we can have arbitrarily long Wielandt series with non-abelian penultimate quotient.

If S is the free Lie algebra generated by x_1, \dots, x_k , then S is positively graded with S_i the space spanned by the products of i factors. Any linear transformation $\alpha_1 : S_1 \rightarrow S_1$ has a unique extension to a derivation $\alpha : S \rightarrow S$ of degree 0. We may form the quotient by the ideal $\sum_{i>n} S_i$ to form the free nilpotent algebra N of class n which inherits the derivation α . We use this to produce our example. Put $N_r^* = \sum_{i \geq r} N_r$.

EXAMPLE 10.1. Let N be the free nilpotent Lie algebra of class n on the generators x_1, \dots, x_k . Let α, β be the derivations of N obtained by extending the maps $\alpha_1, \beta_1 : N_1 \rightarrow N_1$ where α_1 is any map having no eigenvalue in F , and β_1 is the identity. Let L be the split extension of N by the abelian algebra $A = \langle \alpha, \beta \rangle$. Then $\omega_{n-1}(L) = \mathcal{Z}_{n-1}(N) = N_2^*$, $\omega_n(L)/\omega_{n-1}(L) = \langle \beta, N_1 \rangle$ is non-abelian, and $\omega_{n+1}(L) = L$.

PROOF. Suppose $U \triangleleft L$ and that U is not an ideal. Then $U < M$ for some maximal ideal M . But $M = \langle (\lambda\alpha + \mu\beta), N \rangle$ for some $\lambda, \mu \in F$, not both zero, since $L' = N$. But $\lambda\alpha + \mu\beta$ acts invertibly on N_1 , so $M' = N$ and it follows that $U \leq N$. It follows that $\omega_i(L) = \mathcal{Z}_i(N) = N_{n-i+1}^*$ for $i = 1, \dots, n - 1$. The result follows by Theorem 6.2. □

Acknowledgements This work began as an MSc project [9] at the Australian National University by the second author, who thanks R. Bryce and J. Cossey for their continued help for the duration of the project. It has been generalised and extended by the first author.

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