ON THE LATTICE EQUIVALENCE OF TOPOLOGICAL SPACES

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1. Introduction

A topology on a set $\mathscr{X}$ is defined by specifying a family $\mathscr{C}$ of its subsets which has the properties (i) arbitrary set intersections of members of $\mathscr{C}$ belong to $\mathscr{C}$, (ii) finite set unions of members of $\mathscr{C}$ belong to $\mathscr{C}$ and (iii) the empty set $\square$ and the set $\mathscr{X}$ each belong to $\mathscr{C}$. The members of $\mathscr{C}$ are called the closed subsets of $\mathscr{X}$. If X is any subset of $\mathscr{X}$ then $\bar{X}$ denotes the closure of X, that is, the set intersection of all closed subsets which contain X, however when $X = \{x\}$ contains one point only we will denote $\bar{X}$ by $\bar{x}$. The pair $(\mathscr{X}, \mathscr{C})$ is called a topological space or, in what follows, a T-space. By a T-lattice we mean a complete distributive lattice of sets in which arbitrary g.l.b. means arbitrary set intersection, finite l.u.b. means finite set union and which contains the empty set $\square$. It is well-known, for example Birkhoff [1], that if $(\mathscr{X}, \mathscr{C})$ is a T-space and the members of $\mathscr{C}$ are partially ordered by set inclusion then $\mathscr{C}$ is a T-lattice.

Two T-spaces $(\mathscr{X}, \mathscr{C})$ and $(\mathscr{Y}, \mathscr{D})$ are said to be homeomorphic when there exists a one to one mapping f of $\mathscr{X}$ onto $\mathscr{Y}$ which preserves the operation of closure, that is, for each subset X of $\mathscr{X}$ we have $f(\bar{X}) = \overline{f(X)}$. If f is a homeomorphism so is the inverse mapping f^{-1}. Equivalently a homeomorphism between the two T-spaces may be defined as a one to one mapping f of $\mathscr{X}$ onto $\mathscr{Y}$ such that

$$C \in \mathscr{C} \Rightarrow f(C) \in \mathscr{D}$$

and

$$D \in \mathscr{D} \Rightarrow f^{-1}(D) \in \mathscr{C}.$$

(Here and in what follows $\Rightarrow$ stands for implies and $\Leftrightarrow$ for logical equivalence.) A homeomorphism $f : \mathscr{X} \to \mathscr{Y}$ induces a one to one mapping of the lattice $\mathscr{C}$ onto the lattice $\mathscr{D}$ which, together with its inverse, is order preserving; that is, the homeomorphism induces a lattice isomorphism between the T-lattices $\mathscr{C}$ and $\mathscr{D}$. Two T-spaces are said to be lattice equivalent when their T-lattices of closed subsets are isomorphic.

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2. Lattice equivalence and homeomorphisms

We prove

THEOREM (2.1). *In order that two T-spaces* $(\mathscr{X}, \mathscr{C})$ *and* $(\mathscr{Y}, \mathscr{D})$ *be homeomorphic it is necessary and sufficient that there exist a lattice equivalence* $\phi : \mathscr{C} \to \mathscr{D}$ *with the following properties,*

(i) *to each* x *in* $\mathscr{X}$ *there is at least one* y *in* $\mathscr{Y}$ *such that*

$$\bar{y} = \phi(\bar{x}),$$

(ii) *to each* y *in* $\mathscr{Y}$ *there is at least one* x *in* $\mathscr{X}$ *such that*

$$\bar{x} = \phi^{-1}(\bar{y})$$

(iii) *if, for each* x *in* $\mathscr{X}$ *and each* y *in* $\mathscr{Y}$,

$$X_x = \{\xi : \bar{\xi} = \bar{x}, \xi \in \mathscr{X}\}$$
$$Y_y = \{\eta : \bar{\eta} = \bar{y}, \eta \in \mathscr{Y}\}$$

then

$$\bar{y} = \phi(\bar{x}) \Rightarrow |X_x| = |Y_y|$$

and

$$\bar{x} = \phi^{-1}(\bar{y}) \Rightarrow |X_x| = |Y_y|.$$

(Here $|\mathscr{S}|$ denotes the cardinal number of the set $\mathscr{S}$).

REMARK. The second implication in (iii) is logically equivalent to the first since, when ϕ is a lattice equivalence

$$\bar{x} = \phi^{-1}(\bar{y}) \Leftrightarrow \bar{y} = \phi(\bar{x}).$$

PROOF OF THEOREM (2.1). (a) Necessity. Let f be a homeomorphism of $\mathscr{X}$ onto $\mathscr{Y}$. The lattice equivalence induced by f, namely that defined by $\phi(C) = f(C)$ for each C in $\mathscr{C}$, has the stated properties. In the first place (i) holds, since if $y = f(x)$ then

$$\bar{y} = \overline{f(x)} = f(\bar{x}) = \phi(\bar{x}),$$

and similarly (ii) holds. Finally (iii) holds since if $y = \phi(\bar{x})$ and η belongs to Y_y then

$$\overline{f^{-1}(\eta)} = f^{-1}(\bar{\eta}) = \phi^{-1}(\bar{\eta}) = \phi^{-1}(\bar{y}) = \bar{x},$$

whence

$$Y_y \subseteqq \{f(\xi) : \xi \in X_x\}.$$

Conversely if ξ belong X_x then

$$\overline{f(\xi)} = f(\bar{\xi}) = f(\bar{x}) = \phi(\bar{x}) = \bar{y}$$

whence

$$\{f(\xi) : \xi \in X_x\} \subseteqq Y_y.$$

Thus

$$Y_y = \{f(\xi) : \xi \in X_x\}$$

and since f is one to one this implies that $|Y_y| = |X_x|$.

(b) Sufficiency. Suppose that between the two T-spaces there exists a lattice equivalence ϕ which has the stated properties. The relation $\bar{x}_1 = \bar{x}_2$ is an equivalence relation on $\mathscr{X}$ and so the sets X_x, for x in $\mathscr{X}$, form a partition of $\mathscr{X}$. Let this partition be denoted by

$$\mathscr{A} = \{X_\alpha : \alpha \in A\},$$

where A is an indexing set which indexes the elements of the partition.

Similarly let

$$\mathscr{B} = \{Y_\beta : \beta \in B\}$$

denote the partition of $\mathscr{Y}$ which is formed by the sets Y_y for y in $\mathscr{Y}$. The lattice equivalence ϕ induces a one to one mapping ψ of $\mathscr{A}$ onto $\mathscr{B}$, namely that given by

$$\psi X_\alpha = \{y : \exists\, x \in X_\alpha,\ \phi(\bar{x}) = \bar{y}\}.$$

To prove this assertion note firstly that by (i) no ψX_α is empty and that if y_1, y_2 are in ψX_α there are elements x_1, x_2 in X_α such that

$$\bar{y}_1 = \phi(\bar{x}_1) = \bar{y} = \phi(\bar{x}_2) = \bar{y}_2.$$

Thus ψX_α is contained in some Y_β. To show that ψX_α belongs to $\mathscr{B}$ observe that if y is in ψX_α and η is in Y_y then there is an x in X_α with

$$\phi(\bar{x}) = \bar{y} = \bar{\eta},$$

and so η is in ψX_α. This establishes that ψ maps $\mathscr{A}$ into $\mathscr{B}$. To prove that the mapping is onto $\mathscr{B}$ we argue as follows. If Y_β belongs to $\mathscr{B}$ and y is in Y_β there is, by (ii), at least one element x in $\mathscr{X}$ such that $\bar{x} = \phi^{-1}(\bar{y})$. But there is a unique X_α in $\mathscr{A}$ which contains this element x and since $\bar{y} = \phi(\bar{x})$ it follows that $Y_\beta = \psi X_\alpha$.

Because of the result just established there is no loss of generality in supposing that $B = A$, that is, that $\mathscr{B}$ and $\mathscr{A}$ have the same indexing set. We shall suppose therefore that this is so and without further comment write

$$Y_\alpha = \{y : \exists\, x \in X_\alpha,\ \phi(\bar{x}) = \bar{y}\}$$

and

$$X_\alpha = \{x : \exists\, y \in Y_\alpha,\ \phi^{-1}(\bar{y}) = \bar{x}\}.$$

From (iii) it follows that $|X_\alpha| = |Y_\alpha|$ for each α in A.

For each α in A let F_α be the set of one to one mappings of X_α onto Y_α and let

$$\mathscr{X} = \{F_\alpha : \alpha \in A\}.$$

By (iii) none of the sets F_α is empty and so, by the axiom of choice, there exists a non-empty set

$$f = \{f_\alpha : f_\alpha \in F_\alpha, \alpha \in A\}.$$

Since $\mathscr{A}$ is a partition of $\mathscr{X}$ and since $\mathscr{B}$ is a partition of $\mathscr{Y}$, f is the one to one mapping of $\mathscr{X}$ onto $\mathscr{Y}$ given by

$$f(x) = f_\alpha(x) \text{ when } x \in X_\alpha.$$

If f_α^{-1} is the inverse mapping of f_α then

$$f^{-1}(y) = f_\alpha^{-1}(y) \text{ when } y \in Y_\alpha$$

is the inverse mapping of f. Moreover this one to one mapping f of $\mathscr{X}$ onto $\mathscr{Y}$ is such that for each x in $\mathscr{X}$ and each y in $\mathscr{Y}$,

$$\overline{f(x)} = \phi(\bar{x})$$

and

$$\overline{f^{-1}(y)} = \phi^{-1}(\bar{y}).$$

To prove this we observe that to each x in $\mathscr{X}$ there is a unique X_α in $\mathscr{A}$ which contains x and then

$$\overline{f(x)} = \overline{f_\alpha(x)} = \phi(\bar{x}).$$

The second equation is proved in the same way.

Finally we show that the one to one mapping f of $\mathscr{X}$ onto $\mathscr{Y}$ which we have just defined is a homeomorphism between the two T-spaces and this will complete the proof of the theorem. To do so we establish that for each C in $\mathscr{C}$ we have $f(C) = \phi(C)$, and similarly, that for each D in $\mathscr{D}$ we have $f^{-1}(D) = \phi^{-1}(D)$. Suppose then that C belongs to $\mathscr{C}$ and let y be any element of $f(C)$. There is a unique x in $\mathscr{X}$ for which $y = f(x)$ and since,

$$\overline{f(x)} = \phi(\bar{x}) \leqq \phi(C),$$

we obtain $f(C) \leqq \phi(C)$. Conversely if y is in $\phi(C)$ then $\phi(C)$ contains $\bar{y}$ and

$$\overline{f^{-1}(y)} = \phi^{-1}(\bar{y}) \leqq C.$$

Thus $f^{-1}(y)$ belongs to C, that is, y belongs to $f(C)$. This shows that $\phi(C) \leqq f(C)$ and hence, because of the reverse order established above, that $f(C) = \phi(C)$. The proof that $f^{-1}(D) = \phi^{-1}(D)$ for each D in $\mathscr{D}$ is similar. This concludes the proof of the theorem.

We recall that a T_0-space is a T-space $(\mathscr{X}, \mathscr{C})$ in which $\bar{x}_1 = \bar{x}_2$ implies $x_1 = x_2$. In such a space $|X_x| = 1$ for each x in $\mathscr{X}$. It follows that in the statement of theorem (2.1) the condition (iii) is superfluous when each of the spaces is a T_0-space. We may state therefore,

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COROLLARY (2.1).[1] *In order that two T_0-spaces $(\mathscr{X}, \mathscr{C})$ and $(\mathscr{Y}, \mathscr{D})$ be homeomorphic it is necessary and sufficient that there exist a lattice equivalence $\phi : \mathscr{C} \to \mathscr{D}$ with the properties* (i) *and* (ii) *above.*

Another form of this result may be stated as follows.

COROLLARY (2.1)$'$. *In order that the T_0-identifications of two T-spaces $(\mathscr{X}, \mathscr{C})$ and $(\mathscr{Y}, \mathscr{D})$ be homeomorphic it is necessary and sufficient that there exist a lattice equivalence $\phi : \mathscr{C} \to \mathscr{D}$ with the properties* (i) *and* (ii) *above.*

We remark that condition (i) above is logically equivalent to

(i)$'$ for each x in $\mathscr{X}$

$$\{\phi^{-1}(\bar{y}) : y \in \phi(\bar{x})\} = \bar{x}.$$

To see this note that (i) clearly implies (i)$'$. Conversely if (i)$'$ is true there is at least one y in $\phi(\bar{x})$ such that $\phi^{-1}(\bar{y})$ contains x, whence $\bar{x} \leqq \phi^{-1}(\bar{y})$. Since however $\phi^{-1}(\bar{y}) \leqq \bar{x}$ by (i)$'$ we obtain (i). Similarly the condition (ii) is logically equivalent to

(ii)$'$ $$\{\phi(\bar{x}) : x \in \phi^{-1}(\bar{y})\} = \bar{y}.$$

We say that a T-space $(\mathscr{X}, \mathscr{C})$ is a T_D-space when the set $\bar{x} \backslash X_x$ is closed for each x in $\mathscr{X}$. When $(\mathscr{X}, \mathscr{C})$ is also a T_0-space we say that it is a T_{0D}-space. In this case the definition reduces to that in Thron [4], since a T_0-space $(\mathscr{X}, \mathscr{C})$ is a T_{0D}-space if and only if the derived set $\{x\}' = \bar{x} \backslash \{x\}$ is closed for each x in $\mathscr{X}$. We prove

LEMMA (2.1). *A lattice equivalence ϕ between two T_D-spaces $(\mathscr{X}, \mathscr{C})$ and $(\mathscr{Y}, \mathscr{D})$ has the properties* (i) *and* (ii) *above.*

PROOF. If ϕ does not have the property (i)$'$ then there is at least one point x in $\mathscr{X}$ such that

$$\{\phi^{-1}(\bar{y}) : y \in \phi(\bar{x})\}$$

is a proper subset of $\bar{x}$. It follows that this set can contain no element of X_x and hence that

$$\bigcup \{\phi^{-1}(\bar{y}) : y \in \phi(\bar{x})\} \subseteqq \bar{x} \backslash X_x.$$

Since $(\mathscr{X}, \mathscr{C})$ is a T_D-space it follows that the closure of the set union on the left of the above relation is a subset of $\bar{x} \backslash X_x$, that is,

$$V\{\phi^{-1}(\bar{y}) : y \in \phi(\bar{x})\} \leqq \bar{x} \backslash X_x$$

Hence

$$\begin{aligned} \bar{x} &= \phi^{-1} \cdot \phi(\bar{x}) \\ &= \phi^{-1}[V\{\bar{y} : y \in \phi(\bar{x})\}] \\ &= V[\phi^{-1}(\bar{y}) : y \in \phi(\bar{x})] \\ &\leqq \bar{x} \backslash X_x \end{aligned}$$

[1] Mr. R. S. Buckdale also obtained this result independently of the author.

which is absurd since $X_x \neq \square$. Similarly ϕ has the property (ii)′ and this concludes the proof of the lemma.

From the Lemma and Theorem (2.1) we deduce

THEOREM (2.2). *In order that two T_D-spaces $(\mathscr{X}, \mathscr{C})$ and $(\mathscr{Y}, \mathscr{D})$ be homeomorphic it is necessary and sufficient that there exist a lattice equivalence $\phi : \mathscr{C} \to \mathscr{D}$ with the property* (iii) *above.*

Similarly from corollary (2.1) we obtain the following result due to Thron [4].

COROLLARY (2.2). *Every lattice equivalence between two T_{0D}-spaces is induced by a homeomorphism between the two spaces.*

It is of some interest to note that corollary (2.1) leads to a simple proof of the following result in Thron [4], namely

COROLLARY (2.3). *If $\mathscr{X}, \mathscr{C})$ is not a T_D-space there exists a lattice equivalence between $(\mathscr{X}, \mathscr{C})$ and some other space $(\mathscr{Y}, \mathscr{D})$ which is not induced by a homeomorphism.*

PROOF. As noted by Thron if $(\mathscr{X}, \mathscr{C})$ is not a T_0-space the lattice equivalence between the space and its T_0-identification gives the desired result. Suppose then that $(\mathscr{X}, \mathscr{C})$ is a T_0-space but not a T_D-space so that there is at least one point ξ in $\mathscr{X}$ for which $\{\xi\}'$ is not closed. Write $\mathscr{Y} = \mathscr{X} \backslash \{\xi\}$ and let $\mathscr{D}$ be the family of closed sets in $\mathscr{Y}$ defined by

$$\mathscr{D} = \{D : D = C \backslash \{\xi\}, C \in \mathscr{C}\}.$$

As is proved by Thron and may be verified easily the mapping $\phi : C \to C \backslash \{\xi\}$ is a lattice equivalence. Further if y is any element of $\mathscr{Y}$ then

$$\bar{y}^{\mathfrak{Y}} = \bar{y}^{\mathfrak{X}} \backslash \{\xi\}. \tag{2.1}$$

If there were a homeomorphism f of $\mathscr{X}$ onto $\mathscr{Y}$ which induced the lattice equivalence ϕ then condition (i) of theorem (2.1) would hold and there would be an η in $\mathscr{Y}$ such that

$$\bar{\eta}^{\mathfrak{Y}} = \bar{\xi}^{\mathfrak{X}} \backslash \{\xi\}.$$

But by (2.1) this would imply

$$\bar{\eta}^{\mathfrak{X}} \backslash \{\xi\} = \bar{\xi}^{\mathfrak{X}} \backslash \{\xi\}. \tag{2.2}$$

If $\xi \in \bar{\eta}^{\mathfrak{X}}$ then (2.2) implies that $\bar{\eta}^{\mathfrak{X}} = \bar{\xi}^{\mathfrak{X}}$ and since $(\mathscr{X}, \mathscr{C})$ is a T_0-space it follows that $\eta = \xi$ contradicting the fact that η is in $\mathscr{Y}$. On the other hand if $\xi \notin \bar{\eta}^{\mathfrak{X}}$ then

$$\{\xi\}' = \bar{\xi}^{\mathfrak{X}} \backslash \{\xi\} = \bar{\eta}^{\mathfrak{X}}$$

and this contradicts the fact that $\{\xi\}'$ is not closed. Thus there is no homeomorphism which induces the lattice equivalence and this is the desired result.

3. Representations of abstract lattices

According to theorem (2.1), two T-spaces are homeomorphic if and only if between their lattices of closed subsets there is a lattice equivalence with certain specified properties. This result does not characterise those complete distributive lattices which are isomorphic to a lattice of closed subsets of some T-space, nor does it indicate how, in such lattices, one can characterise lattice theoretically those elements which are the images of point closures. To discuss these problems we introduce some further terminology.

Firstly we recall that in section 1. we said that a T-lattice is a complete lattice of subsets of a set $\mathscr{X}$, which contains $\mathscr{X}$ and the empty set $\square$ and in which finite l.u.b. and arbitrary g.l.b. mean finite set union and arbitrary set intersection respectively. Note that a T-lattice is necessarily distributive. If $\mathscr{C}$ is a T-lattice on a set $\mathscr{X}$ then $(\mathscr{X}, \mathscr{C})$ is a T-space and $\mathscr{C}$ is its lattice of closed subsets. A lattice homomorphism $\phi : \mathscr{L} \rightarrow \mathscr{L}'$ of a complete lattice $\mathscr{L}$ onto a complete lattice $\mathscr{L}'$ will be said to be lower complete when is preserves arbitrary g.l.b., that is, if and only if

$$\phi \Lambda\{L_\gamma : \gamma \in \Gamma\} = \Lambda\{\phi(L_\gamma\} : \gamma \in \Gamma\},$$

for any system $\{L_\gamma : \gamma \in \Gamma\}$ of elements of $\mathscr{L}$. We say that a complete lattice admits a T-respresentation if there is a T-lattice $\mathscr{C}$ and a lower complete lattice homomorphism ϕ of $\mathscr{L}$ onto $\mathscr{C}$. When ϕ is an isomorphism we say that the T-representation is faithful. Our first result is

LEMMA (3.1). *A complete lattice $\mathscr{M}$ of subsets of a set $\mathscr{X}$ in which finite l.u.b. and finite g.l.b mean finite set union and finite set intersection respectively and which contains $\mathscr{X}$ and the empty set $\square$, is a T-lattice if and only if, for each x in $\mathscr{X}$*

$$x \in \Lambda\{M : x \in M, M \in \mathscr{M}\}. \tag{3.1}$$

PROOF. The only if part of the lemma is obvious. To prove the converse we must show that when (3.1) holds arbitrary g.l.b. in $\mathscr{M}$ means arbitrary set intersection. But (3.1) implies

$$\bigcap \{M_\gamma : \gamma \in \Gamma\} \subseteqq \Lambda\{M_\gamma : \gamma \in \Gamma\}$$

for any family $\{M_\gamma : \gamma \in \Gamma\}$ of elements of $\mathscr{M}$.

Since we have also

$$\Lambda\{M_\gamma : \gamma \in \Gamma\} \subseteqq \bigcap \{M_\gamma : \gamma \in \Gamma\}$$

we obtain the desired result.

We recall that a lattice $\mathscr{L}$ is said to be a subdirect union of a family of lattices $\{\mathscr{L}_\alpha : \alpha \in A\}$ when (i) $\mathscr{L}$ is a sublattice of the cartesian product of the family and (ii) for each α in A the projection of $\mathscr{L}$ into $\mathscr{L}_\alpha$ is in fact onto $\mathscr{L}_\alpha$. It is known, Birkhoff [1], that any distributive lattice with more than one element is isomorphic with a subdirect union of replicas of 2, the two element chain $\{0, 1\}$, $0 < 1$. Thus each element L of $\mathscr{L}$ may be represented, under an isomorphism, by

$$L \to \{l_\alpha : \alpha \in A\}$$

where each l_α is either 0 or 1. This representation leads, in the usual way, to a mapping θ of $\mathscr{L}$ onto a set $\mathscr{A}$ of subsets of A, namely that defined by

$$\theta(L) = \{\alpha : l_\alpha = 1\}.$$

The mapping $\theta : \mathscr{L} \to \mathscr{A}$ is an isomorphism in which finite l.u.b. and finite g.l.b. correspond to finite set union and finite set intersection respectively. Since the projection of $\mathscr{L}$ into $\mathscr{L}_\alpha$ is onto $\mathscr{L}_\alpha$ there is, for each α in A, at least one element L in $\mathscr{L}$ with $l_\alpha = 1$ and at least one L' in $\mathscr{L}$ with $l'_\alpha = 0$. If $\mathscr{L}$ is a complete lattice so is $\mathscr{A}$ and it then follows that, if 0, 1 are least and greatest elements of $\mathscr{L}$ then $\theta(0) = \square$ and $\theta(1) = A$. Conversely, of course, any isomorphism of a distributive lattice $\mathscr{L}$ onto a distributive lattice of sets, in which finite l.u.b and finite g.l.b. correspond to finite set union and finite set intersection respectively, leads to a representation of $\mathscr{L}$ as a subdirect union of replicas of 2.

We prove now a result which provides much of the motivation in what follows.

THEOREM (3.1). *A complete distributive lattice $\mathscr{L}$ admits a faithful T-representation if and only if it has a representation as a subdirect union of a family $\{\mathscr{L}_\alpha : \alpha \in A\}$ of replicas of 2 in which each of the projections*

$$\pi_\alpha : \mathscr{L} \to \mathscr{L}_\alpha$$

is a lower complete lattice homomorphism.

PROOF. Let $\mathscr{L}$ be a complete distributive lattice, which has a representation as a subdirect union with the stated properties. Write

$$L_{*\alpha} = \Lambda\{L : \pi_\alpha L = 1, L \in \mathscr{L}\}.$$

Each π_α is of course, a lattice homomorphism but because of the additional property of lower completenes we have

$$\pi_\alpha\{L_{*\alpha}) = 1$$

for each α in A. Let $\mathscr{A}$ be the complete lattice of sets defined above and θ the defined isomorphism of $\mathscr{L}$ onto $\mathscr{A}$. We prove that for each α in A

$$\alpha \in \Lambda\{B : \alpha \in B, B \in \mathscr{A}\}, \tag{3.3}$$

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and hence, by lemma (3.1), that $\mathscr{A}$ is a T-lattice. This will establish that $\mathscr{A}$ is a faithful T-representation of $\mathscr{L}$. To prove (3.3) we note that

$$\begin{aligned} &\Lambda\{B : \alpha \in B,\, B \in \mathscr{A}\} \\ (3.4)\qquad &= \Lambda\{\theta(L) : l_\alpha = 1,\, L \in \mathscr{L}\} \\ &= \theta[\Lambda\{L : \pi_\alpha(L) = 1,\, L \in \mathscr{L}\}] \\ &= \theta(L_{*\alpha}) \end{aligned}$$

But since $\pi_\alpha(L_{*\alpha}) = 1$, $l_{*\alpha} = 1$ and so $\theta(L_{*\alpha})$ contains α and this proves (3.3). Thus $\mathscr{A}$ is a faithful T-representation of $\mathscr{L}$.

The converse result is immediate. If $\mathscr{L}$ has a faithful T-representation this leads to a representation of $\mathscr{L}$ as a subdirect union of replicas of 2. Since $\mathscr{A}$ is a T-lattice (3.3) must hold and the implications in (3.4) may be reversed to establish that each of the projection π_α is lower complete. This concludes the proof of the theorem.

The following corollary is worth mentioning, its proof is straight forward and is omitted.

Corollary (3.1). *Under the conditions of theorem* (3.1) *those elements of $\mathscr{L}$ which correspond to the point closures in $\mathscr{A}$ are precisely the elements $L_{*\alpha}$. In fact, for each α in A we have*

$$\overline{\{\alpha\}} = \theta(L_{*\alpha}).$$

It is known, Birkhoff [1], that the representation of an abstract algebra as a subdirect union correspond one to one to those sets of congruence relations on the algebra whose g.l.b is the null congruence. Thus identifying the projections π_α with their associated congruences we are led to study the set of lower complete lattic homomorphisms of a complete lattice onto the two element chain. This study is carried out in the next section where however, it turns out to be more convenient to introduce the concept of a lattice bisector.

4. Lattice bisectors

Let $\mathscr{L}$ be an arbitrary lattice. We say that a non-zero element L_* is a lattice bisector of $\mathscr{L}$ if and only if

$$(4.1)\qquad L_1 \ngeqq L_* \text{ and } L_2 \ngeqq L_* \Rightarrow L_1 \vee L_2 \ngeqq L_*$$

for any elements L_1, L_2 in $\mathscr{L}$. An equivalent formulation of (4.1) is

$$(4.2)\qquad L_1 \vee L_2 \geqq L_* \Rightarrow L_1 \geqq L_* \text{ or } L_2 \geqq L_*.$$

We establish

Lemma (4.1). *In a complete lattice $\mathscr{L}$ there is a one to one correspondence* *between lattice bisectors and lower complete lattice homomorphisms of $\mathscr{L}$ onto the two element chain.*

Proof. Suppose that $\pi : \mathscr{L} \to 2$ is a lower complete lattice homomorphism of the complete lattice $\mathscr{L}$ onto the two element chain. Write

$$L_* = \Lambda\{L : \pi(L) = 1, L \in \mathscr{L}\}. \tag{4.3}$$

Since π maps $\mathscr{L}$ onto 2 and since $\mathscr{L}$ is complete L_* exists. Since π is lower complete $\pi(L_*) = 1$, it follows that $L_* \neq 0$ (for π is onto 2) and that

$$\pi(L) = \begin{cases} 1 & \text{if} \quad L \geqq L_* \\ 0 & \text{if} \quad L \ngeqq L_*. \end{cases} \tag{4.4}$$

This implies that L_* is a lattice bisector of $\mathscr{L}$ for if $L_1 \ngeqq L_*$ and $L_2 \ngeqq L_*$ we have

$$\pi(L_1 \vee L_2) = \pi(L_1) \vee \pi(L_2) = 0 \vee 0 = 0$$

and consequently $L_1 \vee L_2 \ngeqq L_*$.

Conversely if L_* is a lattice bisector of $\mathscr{L}$ then π defined by (4.4) is a lower complete lattice homomorphism of $\mathscr{L}$ onto the two element chain. The fact that π maps $\mathscr{L}$ onto 2 follows from $L_* \neq 0$ so that there is in $\mathscr{L}$ an element L such that $\pi(L) = 0$. To show that π is a lower complete lattice homomorphism note that by (4.2),

$$\begin{aligned} \pi(L_1 \vee L_2) = 1 &\Leftrightarrow L_1 \vee L_2 \geqq L_* \\ &\Leftrightarrow L_1 \geqq L_* \text{ or } L_2 \geqq L_* \\ &\Leftrightarrow \pi(L_1) \vee \pi(L_2) = 1. \end{aligned}$$

Further if $\{L_\gamma : \gamma \in \Gamma\}$ is any family of elements of $\mathscr{L}$

$$\begin{aligned} \pi(\Lambda L_\gamma) = 1 &\Leftrightarrow \Lambda L_\gamma \geqq 1 \\ &\Leftrightarrow L_\gamma \geqq L_*, \ \gamma \in L \\ &\Leftrightarrow \Lambda \pi(L_\gamma) = 1. \end{aligned}$$

The proof of the lemma is concluded by the observation that distinct lattice bisectors correspond to distinct lower complete lattice homomorphisms of onto 2. To see this note that L_*, L'_* are distinct lattice bisectors of such that

$$L \geqq L_* \Leftrightarrow L \geqq L'_*$$

then $L_* \geqq L'_*$ and $L'_* \geqq L_*$. Thus $L_* = L'_*$ and this establishes the desired result.

Lemma (4.1) shows that the results which follow may be expressed in terms of subdirect unions of replicas of the two element chain. We prove now

THEOREM (4.1). *Let $\mathscr{L}$ be a complete lattice and let $\mathscr{B}$ be its set of lattice bisectors. Suppose that $\mathscr{B} \neq \square$ and let $\mathscr{X}$ be any non-empty subsetof $\mathscr{B}$. Let $\mathscr{C} = \phi \cdot \mathscr{L}$ be the family of subsets C of $\mathscr{X}$ defined by*

$$\phi(L) = C = \{x : x \leqq L,\ x \in \mathscr{X}\},\ L \in \mathscr{L}. \tag{4.5}$$

Then $\mathscr{C}$ is a complete lattice of subsets of $\mathscr{X}$ in which finite l.u.b. and arbitrary g.l.b. mean finite set union and arbitrary set intersection respectively. Further the mapping $\phi : \mathscr{L} \to \mathscr{C}$ is a lower complete lattice homomorphism of $\mathscr{L}$ onto $\mathscr{C}$.

PROOF. Clearly

$$\phi(L_1) \cup \phi(L_2) \subseteqq \phi(L_1 \vee L_2).$$

However, since

$$x \leqq L_1 \vee L_2 \Rightarrow x \leqq L_1 \text{ or } x \leqq L_2,$$

when x is a lattice bisector, we have in fact

$$\phi(L_1 \vee L_2) = \phi(L_1) \cup \phi(L_2).$$

If $\{L_\gamma : \gamma \in \Gamma\}$ is any family of elements of $\mathscr{L}$ then, for x in $\mathscr{X}$,

$$\begin{aligned} x \leqq \Lambda\{L_\gamma : \gamma \in \Gamma\} &\Leftrightarrow x \leqq L_\gamma,\ \gamma \in \Gamma \\ &\Leftrightarrow x \in \phi(\Gamma_\gamma),\ \gamma \in \Gamma \\ &\Leftrightarrow x \in \bigcap \{\phi(L_\gamma) : \gamma \in \Gamma\}. \end{aligned}$$

Hence

$$\phi[\Lambda\{L_\gamma : \gamma \in \Gamma\}] = \cap \{\phi(L_\gamma) : \gamma \in \Gamma\}.$$

This shows that ϕ is a lower complete lattice homomorphism of $\mathscr{L}$ onto $\mathscr{C}$. However since arbitrary g.l.b. exist in $\mathscr{C}$ and $\mathscr{C}$ contains a unit, namely $\mathscr{X} = \phi(1)$, it follows, Birkhoff [1], p. 49, that $\mathscr{C}$ is a complete lattice. This concludes the proof of the theorem.

This theorem shows that every complete lattice with lattice bisectors admits as a T-representation a T-lattice on any non-empty subset of its lattice bisectors and whose elements are of the form (4.5). Such a T-representation we call a canonical T-representation. Note that a canonical T-representation $(\mathscr{X}, \mathscr{C})$ is always a T_0-space, for if the point closure $\bar{x}$ of a point x in $\mathscr{X}$ is just the set $\phi(\bar{x})$ and for x_1, x_2 in

$$\phi(x_1) = \phi(x_2) \Rightarrow x_1 = x_2.$$

Our next result establishes that not only is every T-representation of a complete lattice a canonical T-representation to within a lattice isomorphism but further that the T_0-identification of any T-representation is homomorphic to a canonical T-representation.

THEOREM (4.2). *If $\mathscr{D}$ is a T-representation of a complete lattice $\mathscr{L}$ on a* *set $\mathscr{Y}$ then there is a canonical T-representation $\mathscr{C}$ on a non-empty set $\mathscr{X}$ of lattice bisectors of $\mathscr{L}$ such that $(\mathscr{X}, \mathscr{C})$ is homeomorphic to the T_0-identification of $(\mathscr{Y}, \mathscr{D})$.*

Proof. Let $\phi : \mathscr{L} \to \mathscr{D}$ be the lower complete homomorphism of $\mathscr{L}$ onto $\mathscr{D}$ which determines the T-representation as a family $\mathscr{D}$ of subsets of the set $\mathscr{Y}$. For each y in $\mathscr{Y}$ write

$$L_*(\bar{y}) = \Lambda\{L : \bar{y} \subseteqq \phi(L)\}.$$

Since $\mathscr{L}$ is complete and since ϕ is onto $\mathscr{D}$, $L_*(\bar{y})$ exists for each y in $\mathscr{Y}$. Since ϕ is lower complete we have

$$\phi\{L_*(\bar{y})\} = \bar{y}. \tag{4.6}$$

We prove that $L_*(\bar{y})$ is a lattice bisector of $\mathscr{L}$. To do so observe that $L_*(\bar{y})$ is non-zero and

$$L_1 \vee L_2 \geqq L_*(\bar{y}) \Rightarrow \phi(L_1) \cup \phi(L_2) \supseteqq \bar{y}.$$

Hence y belongs to at least one of the sets $\phi(L_1)$ and $\phi(L_2)$. If $\phi(L_1)$ contains y then $\phi(L_1) \supseteqq \bar{y}$ and hence $L_1 \geqq L_*(\bar{y})$. Similarly if $\phi(L_2)$ contains y then $L_2 \geqq L_*(\bar{y})$. This establishes that $L_*(\bar{y})$ is a lattice bisector of $\mathscr{L}$ for each y in $\mathscr{Y}$. Moreover it is an immediate consequence of (4.6) that for y_1, y_2 in $\mathscr{Y}$

$$L_*(\bar{y}_1) = L_*(\bar{y}_2) \Rightarrow \bar{y}_1 = \bar{y}_2. \tag{4.7}$$

There is, therefore, a one to one correspondence between the point closure of $\mathscr{D}$ and a subset of the lattice bisectors of $\mathscr{L}$. Let

$$\mathscr{X} = \{L_*(\bar{y});\ \bar{y} \in \mathscr{D}\},$$

and let $\mathscr{C} = \theta \cdot \mathscr{L}$ be the family of subsets $\mathscr{C}$ of $\mathscr{X}$ defined by

$$C = \theta(L) = \{x : x \leqq L,\ x \in \mathscr{X}\}$$

for L in $\mathscr{L}$. By theorem (4.1) $\mathscr{C}$ is a canonical T-representation of $\mathscr{L}$ on the subset $\mathscr{X}$ of the set of all lattice bisectors of $\mathscr{L}$. Note that for y in $\mathscr{Y}$ and L in $\mathscr{L}$.

$$L_*(\bar{y}) \leqq L \Leftrightarrow \bar{y} \leqq \phi(L), \tag{4.7}$$

the forward implication following from (4.6) and the reverse implication following from the definition of $L_*(y)$ It is an immediate consequence of (4.7) that

$$\theta(L_1) = \theta(L_2) \Leftrightarrow \phi(L_1) = \phi(L_2) \tag{4.8}$$

for if $\theta(L_1) = \theta(L_2)$ then

$$\begin{aligned}\bar{y} \leqq \phi(L_1) &\Leftrightarrow L_*(\bar{y}) \leqq L_1 \\ &\Leftrightarrow L_*(\bar{y}) \leqq L_2 \\ &\Leftrightarrow \bar{y} \leqq \phi(L_2),\end{aligned}$$

and, if $\phi(L_1) = \phi(L_2)$ then,

$$\begin{aligned}L_*(\bar{y}) \leqq L_1 &\Leftrightarrow \bar{y} \leqq \phi(L_1) \\ &\Leftrightarrow \bar{y} \leqq \phi(L_2) \\ &\Leftrightarrow L_*(\bar{y}) \leqq L_2.\end{aligned}$$

Thus the map σ defined by

$$\sigma\{\theta(L)\} = \phi(L)$$

is a one to one mapping of $\mathscr{C}$ into $\mathscr{D}$. This mapping is, in fact, onto $\mathscr{D}$ for, since ϕ maps $\mathscr{L}$ onto $\mathscr{D}$, there is to each D in $\mathscr{D}$ at least one element L in $\mathscr{L}$ such that $\phi(L) = D$. Thus there is a unique element in $\mathscr{C}$, namely $C = \theta(L)$, such that $\sigma(C) = D$. Further σ is an isomorphism of C onto $\mathscr{D}$ for if $C_1 = \theta(L_1)$ and $C_2 = \theta(L_2)$ are elements of C then, when $C_1 \leqq C_2$ we obtain from (4.7),

$$\bar{y} \leqq \phi(L_1) \Rightarrow L_*(\bar{y}) \leqq L_1 \Rightarrow L_*(\bar{y}) \leqq L_2 \Rightarrow \bar{y} \leqq \phi(L_2).$$

Conversely, when $\phi(L_1) \leqq \phi(L_2)$,

$$L_*(\bar{y}) \leqq L_1 \Rightarrow \bar{y} \leqq \phi(L_1) \Rightarrow \bar{y} \leqq \phi(L_2) \Rightarrow L_*(\bar{y}) \leqq L_2,$$

and hence

$$C_1 \leqq C_2 \Leftrightarrow \sigma(C_1) \leqq \sigma(C_2).$$

Thus the given T-representation $(\mathscr{Y}, \mathscr{D})$ is lattice equivalent to the canonical T-representation $(\mathscr{X}, \mathscr{C})$. However to each x in $\mathscr{X}$ there is at least one y in $\mathscr{Y}$ such that $x = L_*(\bar{y})$ and then

$$\sigma(\bar{x}) = \phi(\bar{x}) = \phi(L_*(\bar{y})) = \bar{y}.$$

Conversely to each y in $\mathscr{Y}$ there is an element $x = L_*(\bar{y})$ in $\mathscr{X}$ such that

$$\bar{x} = \sigma^{-1}(\bar{y}).$$

The desired result then follows from Corollary (2.2).

A subset $\mathscr{M}$ of a lattice $\mathscr{L}$ will be said to be a join basis for $\mathscr{L}$ when very non-zero element of $\mathscr{L}$ is the join of elements of $\mathscr{M}$. That is, for each $\neq 0$ in $\mathscr{L}$ there is a family $\{M_\gamma : \gamma \in \Gamma\}$ of elements of $\mathscr{M}$ such that

$$L = V\{M_\gamma : \gamma \in \Gamma\}.$$

he preceding results enable us to establish

THEOREM (4.3). *A complete lattice $\mathscr{L}$ admits a faithful T-representation and only if it has a non-empty subset of lattice bisectors which is a join asis for $\mathscr{L}$.*

PROOF. Suppose $\mathscr{B} \neq \square$ is the set of lattice bisectors of $\mathscr{L}$ and suppose that $\mathscr{X}$ is a non-empty subset of $\mathscr{B}$ which is a join basis for $\mathscr{L}$. By theorem (4.1) there is a canonical T-representation of $\mathscr{L}$ as a lattice $\mathscr{C}$ of closed subsets of $\mathscr{X}$. This T-representation is given explicitly by equation (4.5). But since $\mathscr{X}$ is a join basis for $\mathscr{L}$ and since $\mathscr{L}$ is complete

$$L = V(x : x \in \phi(L)\}$$

and consequently

$$\phi(L_1) = \phi(L_2) \Rightarrow L_1 = L_2.$$

This establishes that the canonical T-representation on $\mathscr{X}$ is faithful. Notice that the distributivity of $\mathscr{L}$ is a consequence of the existence of a faithful T-representation and is not part of the hypothesis of the theorem. Conversely if the complete lattice $\mathscr{L}$ admits a faithful T-representation, and is therefore distributive, then, by theorem (4.2), $\mathscr{L}$ admits a faithful canonical T-representation $\phi : \mathscr{L} \to \mathscr{C}$ as a lattice of closed subsets of a non-empty subset $\mathscr{X}$ of its lattice bisectors. But for each C in $\mathscr{C}$

$$C = V\{\phi(L_*) : L_* \leqq \phi^{-1}(C),\ L_* \in \mathscr{X}\}$$

and the elements $\phi(L_*)$ for L_* in $\mathscr{X}$ form a join basis for $\mathscr{C}$. Since ϕ is an isomorphism of the complete lattice $\mathscr{L}$ onto the complete lattice $\mathscr{C}$ it follows for each L in $\mathscr{L}$ we have

$$L = V\{L_* : L_* \leqq L,\ L_* \in \mathscr{X}\}$$

and hence, the non-empty set $\mathscr{X}$ of lattice bisectors is a join basis for $\mathscr{L}$. This concludes the proof of the theorem.

An element $L \neq 0$ in a lattice $\mathscr{L}$ is said to be join reducible if there are elements L_1, L_2 in $\mathscr{L}$ with $L_1 < L$, $L_2 < L$ and $L = L_1 \vee L_2$. An element $L \neq 0$ of a lattice $\mathscr{L}$ which is not join reducible is said to be join irreducible. Note that each lattice bisector L_* of a lattice $\mathscr{L}$ is join irreducible for $L_* = L_1 \vee L_2$ implies either $L_* \geqq L_*$ or $L_2 \geqq L_*$. Conversely if $\mathscr{L}$ is a distributive lattice any join irreducible element is a lattice bisector. For if L is a join irreducible element of $\mathscr{L}$ and

$$L_1 \vee L_2 \geqq L$$

then the distributivity of $\mathscr{L}$ implies that

$$(L_1 \wedge L) \vee (L_2 \wedge L) = L.$$

Since $L \neq 0$ we cannot have $L_1 \wedge L = L_2 \wedge L = 0$, thus if $L_1 \wedge L = 0$ then $L_2 \wedge L = L$ and $L \leqq L_2$. Similarly if $L_2 \wedge L = 0$ then $L \leqq L_1$. If however $L_1 \wedge L \neq 0$ and $L_2 \wedge L \neq 0$ then the join irreducibility of L implies that either $L_1 \wedge L = L$ or $L_2 \wedge L = L$, that is, either $L \leqq L_1$ or $L \leqq L_2$. This shows that L is a lattice bisector. We have established therefore

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THEOREM (4.4). *In a distributive lattice an element is a lattice bisector if and only if it is join irreducible.*

An immediate consequence of theorems (4.3) and (4.4) is the following result in Thron [4], namely

THEOREM (4.5). *A complete distributive lattice $\mathscr{L}$ admits a faithful T-representation if and only if it has a join basis of join irreducible elements.*

However this result and our earlier results on faithful T-representations are in fact, a consequence of a more general result of Büchi [2] which has been partly rediscovered by several authors, for eaxmple Stelleckii [3] and Vaclav [4]. Büchi considered a complete lattice $\mathscr{L}$ with a family $\mathfrak{N}$ of subsets of $\mathscr{L}$ which contained all one element subsets of $\mathscr{L}$. An $\mathfrak{N}$-representation of $\mathscr{L}$ is a one to one mapping of $\mathscr{L}$ onto a lattice of subsets in which arbitrary g.l.b correspond to arbitrary set intersection and l.u.b of elements of $\mathfrak{N}$ correspond to set union. An element L in $\mathscr{L}$ is $\mathfrak{N}$-subirreducible when $L \leqq N \in \mathfrak{N}$ implies $a \geqq L$ for at least one element a in N. The result of Büchi referred to above is that $\mathscr{L}$ admits an $\mathfrak{N}$-representation if and only if there is a join basis of $\mathfrak{N}$-subirreducible elements. Theorem (4.5) is the particular case of this result obtained by taking for $\mathfrak{N}$ the set of all finite subsets of $\mathscr{L}$.

Because of theorem (2.1) and theorem (4.2) the investigation of the possibility of homeomorphism between two T-spaces each of which is lattice equivalent to a given lattice is reduced to the study of canonical faithful T-representations. Let $\mathscr{L}$ be a complete distributive lattice which admits a faithful T-representation, then $\mathscr{L}$ has a join basis $\mathscr{K}$ of join irreducible elements. Let $\mathscr{K}(\mathscr{L})$ be the set of join bases $\mathscr{K}$ of join irreducible elements. We introduce an equivalence relation into $\mathscr{K}(\mathscr{L})$ by writing

$$K_1 \equiv K_2$$

whenever there is an automorphism α of $\mathscr{L}$ onto itself such that $\alpha(K_1) = K_2$. We call the equivalence classes in $\mathscr{K}(\mathscr{L})$ the automorphic classes of $\mathscr{K}(\mathscr{L})$. We prove

THEOREM (4.6). *There is a one to one correspondence between the non-homeomorphic canonical faithful T-representations of a complete distributive lattice $\mathscr{L}$ and the automorphic classes of $\mathscr{K}(\mathscr{L})$.*

PROOF. Each K in $\mathscr{K}$ determines a canonical faithful T-representation of $\mathscr{L}$ as a T_0-space. If K_1, K_2 are elements of $\mathscr{K}$ the corresponding T_0-spaces are homeomorphic if and only if K_1 and K_2 belong to the same automorphic class. To see this let $\mathscr{X}, \mathscr{Y}$ be two elements of $\mathscr{K}(\mathscr{L})$ and let $(\mathscr{X}, \mathscr{C})$, $(\mathscr{Y}, \mathscr{D})$ be the corresponding canonical faithful T-representations of $\mathscr{L}$. Let

$$\theta(L) = \{x : x \leqq L, x \in \mathscr{X}\}$$

and

$$\psi(L) = \{y : y \leqq L, \ y \in \mathscr{Y}\}$$

be the isomorphisms of $\mathscr{L}$ onto $\mathscr{C}$ and $\mathscr{D}$ respectively.

The point closures in $(\mathscr{X}, \mathscr{C})$ are given by

$$\bar{x} = \theta(x), \quad x \in \mathscr{X}$$

and the point closures in $(\mathscr{Y}, \mathscr{D})$ are given by

$$\bar{y} = \psi(y), \quad y \in \mathscr{Y}.$$

Suppose firstly that $\mathscr{X}$, $\mathscr{Y}$ belong to the same automorphic class of $\mathscr{K}(\mathscr{L})$, then there is an automorphism α of $\mathscr{L}$ onto itself such that $\alpha(\mathscr{X}) = \mathscr{Y}$. The mapping

$$\phi = \psi\alpha\theta^{-1}$$

is an isomorphism of $\mathscr{C}$ onto $\mathscr{D}$. For x in $\mathscr{X}$ let $y = \alpha(x)$, then y is in $\mathscr{Y}$ and

$$\begin{aligned}\phi(\bar{x}) &= \psi\alpha\theta^{-1}(\bar{x}) \\ &= \psi\alpha(x) \\ &= \psi(y) \\ &= \bar{y}.\end{aligned}$$

Conversely for y in $\mathscr{Y}$, $x = \alpha^{-1}(y)$ is in $\mathscr{X}$ and

$$\begin{aligned}\bar{x} &= \theta(x) \\ &= \theta\alpha^{-1}(y) \\ &= \theta\alpha^{-1}\psi^{-1}(\bar{y}) \\ &= \phi^{-1}(\bar{y}).\end{aligned}$$

Since each of $(\mathscr{X}, \mathscr{C})$, $(\mathscr{Y}, \mathscr{D})$ is a T_0-space it follows from corollary (2.1) that the two T_0-spaces are homeomorphic. Conversely if there is a homeomorphism f of $\mathscr{X}$ onto $\mathscr{Y}$ this induces a lattice equivalence ϕ of $\mathscr{C}$ onto $\mathscr{D}$ The mapping

$$\alpha = \psi^{-1}\phi\theta$$

is an automorphism of $\mathscr{L}$ onto itself, and since for each x in $\mathscr{X}$ there is a unique y in $\mathscr{Y}$, and conversely, such that $\phi(\bar{x}) = \bar{y}$, we have

$$\begin{aligned}\alpha(x) &= \psi^{-1}\phi\theta(x) \\ &= \psi^{-1}\phi(\bar{x}) \\ &= \psi^{-1}(\bar{y}) \\ &= y\end{aligned}$$

and so $\mathscr{X}$ and $\mathscr{Y}$ belong to the same automorphic class. This concludes the proof of the theorem.

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