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# ADMISSIBLE LIMITS OF BLOCH FUNCTIONS ON BOUNDED STRONGLY PSEUDOCONVEX DOMAINS

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Let $\mathcal{D} \subseteq \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with $C^2$ boundary $\partial\mathcal{D}$. In this paper we prove that for a Bloch function in $\mathcal{D}$ the existance of radial limits at almost all $\zeta \in \partial\mathcal{D}$ implies the existence of admissible limits almost everywhere on $\partial\mathcal{D}$.

Let $\mathcal{D} \subseteq \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with $C^2$ boundary $\partial\mathcal{D}$ (for the definition we refer to [**1**]). For $z \in \mathcal{D}$ the Euclidean distance from $z$ to $\partial\mathcal{D}$ is denoted by $\delta(z)$ and for $\zeta \in \partial\mathcal{D}$ the unit outward normal of $\partial\mathcal{D}$ at $\zeta$ is denoted by $v_\zeta$. If $\zeta \in \partial\mathcal{D}$ and $\alpha > 0$ we define the admissible approach region $\mathcal{U}_\alpha(\zeta)$ with the vertex $\zeta$ by

$$\mathcal{U}_\alpha(\zeta) = \{z \in \mathcal{D};\ |(z-\zeta)\cdot \bar{v}_\zeta| < (1+\alpha)\delta_\zeta(z),\ |z-\zeta|^2 < \alpha\delta_\zeta(z)\},$$

where $\delta_\zeta(z) = \min\{\delta(z), \operatorname{dist}(z, T_\zeta(\partial\mathcal{D}))\}$ and $T_\zeta(\partial\mathcal{D})$ is the real tangent space to $\partial\mathcal{D}$ at $\zeta$. A function $f$ on $\mathcal{D}$ is called to have an admissible limit at $\zeta$ if $\lim\limits_{z\to\zeta,\, z\in\mathcal{U}_\alpha(\zeta)} f(z)$ exists for all $\alpha > 0$.

Let $\mathcal{B}(\mathcal{D})$ be the space of all Bloch functions $f$ which are holomorphic in $\mathcal{D}$ with $\sup\{|\nabla f(z)|\cdot\delta(z) : z \in \mathcal{D}\} < \infty$. It is well known that each function in $H^p$ has an admissible limit at almost every $\zeta \in \partial\mathcal{D}$. But for Bloch functions generally we can say nothing on the existence of admissible limits. For example, Ullrich constructed a Bloch function in the unit ball (a typical model of a strongly pseudoconvex domain) in $\mathbb{C}^n$ which has a radial limit at no point of the boundary (see [**2**]). On the other hand, Lehto and Virtanen proved in [**3**] that the existence of radial limits implies the existence of angular limits for Bloch functions in the unit disc $D = \{z \in \mathbb{C}:\ |z| < 1\}$.

The purpose of this paper is to extend this result to the setting of strongly pseudoconvex domains. Our approach will be very different from that of [**3**] and our result also generalises [**4**].

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**Theorem.** *Let $\mathcal{D} \subseteq \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with $C^2$ boundary, and let $f$ be a Bloch function in $\mathcal{D}$. If the limit $\lim_{t \to +0} f(\zeta - tv_\zeta)$ exists at almost all $\zeta \in \partial\mathcal{D}$, then $f$ has admissible limits almost everywhere on $\partial\mathcal{D}$.*

Proof: First, we do some estimates on $\mathcal{U}_\alpha(\zeta)$ for $\zeta \in \partial\mathcal{D}$. Without loss of the generality we may assume that $\zeta$ is the origin, that $v_\zeta$ is in the negative $y_1$ direction (here $z_1 = x_1 + y_1 i$) and thus the complex normal space $\mathcal{N}_\zeta$ and the complex tangential space $\mathcal{T}_\zeta$ are

$$\mathcal{N}_\zeta = \{(z_1, 0, \ldots, 0);\ z_1 \in \mathbb{C}\},\ \mathcal{T}_\zeta = \{(0, z_2, \ldots, z_n);\ z_j \in \mathbb{C},\ j = 2, \ldots, n\}.$$

For $z = (z_1, \ldots, z_n) \in \mathcal{U}_\alpha(\zeta)$, by definition

$$\begin{aligned} |z_1| = |\langle z - \zeta, v_\zeta\rangle| < (1+\alpha)y_1, \\ |z_2|^2 + \cdots + |z_n|^2 \leqslant |z - \zeta|^2 < \alpha y_1. \end{aligned} \tag{1}$$

Put $z' = (y_1 i, 0, \ldots, 0)$ and let $P_{z'}(r_1, r_2)$ be the polydisc centred at $z'$ with radius $r_1$ in the complex normal direction and $r_2$ in each complex tangential direction [**1**, p.55]. Then we know from (1) that $z \in \mathcal{U}_\alpha(\zeta)$ implies $z \in P_{z'}\ \left(\sqrt{3\alpha}\ y_1, \sqrt{\alpha}\sqrt{y_1}\right)$. Notice that Lemma 6 of [**5**] is still valid under the weaker assumption that $\mathcal{D}$ is a bounded strongly pseudoconvex domain with $C^2$ boundary in $\mathbb{C}^n$, with the same proof to the case of $C^\infty$ boundary. By applying this lemma we can take $\alpha > 0$ so small that

$$P_{z'}\left(\sqrt{3\alpha}y_1, \sqrt{\alpha}\sqrt{y_1}\right) \subseteq \{w \in \mathcal{D};\ \beta(z', w) < 1\},$$

where $\beta(z', w)$ is the Kobayashi distance from $z'$ to $w$. Hence, for $z \in \mathcal{U}_\alpha(\zeta) \cap \{w : |w - \zeta| < \varepsilon\}$ with $\varepsilon$ small enough we have

$$\beta(z', z) < 1. \tag{2}$$

Now suppose $f \in \mathcal{B}(\mathcal{D})$ and for almost all $\zeta \in \partial\mathcal{D}$ the limit $\lim_{t \to +0} f(\zeta - tv_\zeta)$ exists. Set $E = \{\zeta \in \partial\mathcal{D} : \lim_{t \to +0} f(\zeta - tv_\zeta) \text{ exists}\}$. If $\zeta \in E$, then $f$ is bounded on $\{\zeta - tv_\zeta : 0 < t \leqslant \varepsilon\}$, say

$$|f(\zeta - tv_\zeta)| \leqslant M \quad t \in (0, \varepsilon]. \tag{3}$$

Then for $z \in \mathcal{U}_\alpha(\zeta) \cap \{w : |w - \zeta| < \varepsilon\}$, from the estimate on [**6**, p.150] and (2), (3) we obtain

$$\begin{aligned} |f(z)| &\leqslant |f(z) - f(z')| + |f(z')| \\ &\leqslant C\beta(z', z) + M \leqslant C + M. \end{aligned}$$

That $f$ is bounded on $\mathcal{U}_\alpha(\zeta) \setminus \{w : |w - \zeta| < \varepsilon\}$ is obvious. This means that $f$ is admissible bounded at $\zeta \in E$. Theorem 12 of [**1**] tells us that $f$ has admissible limits at almost all $\zeta \in \partial\mathcal{D}$. The proof is complete. ∎

REMARK. We have actually proved the following: If $E \subseteq \partial\mathcal{D}$ is measurable and $f$ is a Bloch function which has radial limits at each $\zeta \in E$, then $f$ has admissible limits at almost all $\zeta \in E$.

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