EQUIVALENT ABELIAN GROUPS

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1. Introduction. Throughout this note all groups are *abelian*, written additively. We refer to Kurosh **(8; 9)** for notation, terminology and theorems used without reference. We recall the notion of a *serving subgroup* (or pure subgroup) $\mathfrak{S}$ of a group $\mathfrak{G}$. This is a subgroup $\mathfrak{S}$ in which for every natural number n every equation $nx = s$, $s \in \mathfrak{S}$ can be solved provided that it can be solved in $\mathfrak{G}$. If $\mathfrak{G}$ is torsion-free, "linearly closed" subgroups coincide with serving subgroups and $\mathfrak{S}$ is a serving subgroup if and only if $\mathfrak{G}/\mathfrak{S}$ is torsion-free. Direct summands are serving subgroups but, in general, the converse is untrue (cf. **4**).

We call the groups $\mathfrak{G}$ and $\mathfrak{H}$ *equivalent* (or *equivalent by subgroups*) if each is isomorphic to a subgroup of the other, i.e.

$$(1)\qquad \begin{aligned} \mathfrak{G} &\to \phi\mathfrak{G} = \mathfrak{G}' \subset \mathfrak{H}, \\ \mathfrak{H} &\to \psi\mathfrak{H} = \mathfrak{H}' \subset \mathfrak{G}, \end{aligned}$$

where ϕ and ψ are isomorphic maps. If $\mathfrak{G}'$ and $\mathfrak{H}'$ are serving subgroups or direct summands of $\mathfrak{H}$ and $\mathfrak{G}$ respectively, we call $\mathfrak{G}$ and $\mathfrak{H}$ *equivalent by serving subgroups or equivalent by direct summands.*

We give a short survey of the main results obtained so far (a "positive answer" means that equivalence implies isomorphism of the groups under consideration, otherwise we speak of a "negative answer").

Equivalence by subgroups. In the main the problem is solved. In general, the answer is negative. De Bruijn (orally) and Kaplansky **(6)** gave a counter-example for periodic groups. Here is one for the case of torsion-free groups (by a slight alteration the groups can be made countable).

Example (i). $\mathfrak{G}$ is the additive group of real numbers, $\mathfrak{H} = \mathfrak{G} + \mathfrak{C}$, where $\mathfrak{C}$ is an infinite cyclic group. $\mathfrak{H}$ can be embedded isomorphically in $\mathfrak{G}$, just as any torsion-free group with at most continuously many elements. So $\mathfrak{G}$ and $\mathfrak{H}$ are equivalent but clearly not isomorphic.

Nevertheless, in each of the following cases, the answer is positive: the class of complete groups,[1] the class of groups with a finite number of generators (cf. **1**), the class of subgroups of the additive group of rationals.

Equivalence by serving subgroups. In the main the problem is solved. In general, the answer is negative. Indeed, in §**4**, we shall construct a counter-example. $\mathfrak{G}$ and $\mathfrak{H}$ are both countable, torsion-free groups, decomposable into groups of rank $\leqslant 2$.

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[1]A group $\mathfrak{G}$ is complete (or divisible) if each equation $nx = g$, $g \in \mathfrak{G}$ for every natural number n has at least one solution in $\mathfrak{G}$, that is if $n\mathfrak{G} = \mathfrak{G}$.

However, it is proved in **(5)**, using the results of §2, that for the class of completely decomposable groups[2] the answer is positive.

Equivalence by direct summands. In the main the problem is unsolved. The answer is positive for the class of countable periodic groups (Kaplansky **(6)**) and the class of completely decomposable groups **(5)**.

In general, the author expects a negative answer in the case of torsion-free groups. However, the structure of the torsion-free groups seems not yet sufficiently cleared to overcome the difficulties involved. The problem is difficult (Massey **(11)**); for its topological consequences and some other positive results, see Yang **(13)**.

In §3 we construct a simple example of an *indecomposable abelian group of arbitrary finite rank.*

2. Positive results.

THEOREM I.[3] *If $\mathfrak{G}$ and $\mathfrak{H}$ are equivalent by direct summands ($\phi\mathfrak{G}$ and $\psi\mathfrak{H}$ in* (1) *being direct summands) and if in one of the groups the sum of an ascending sequence of direct summands is again a (proper or improper) direct summand, then $\mathfrak{G}$ and $\mathfrak{H}$ are isomorphic.*

Moreover, it is possible to determine direct decompositions

$$\mathfrak{G} = \mathfrak{G}_1 + \mathfrak{G}_2 \tag{2}$$
$$\mathfrak{H} = \mathfrak{H}_1 + \mathfrak{H}_2 \tag{3}$$

such that the relations

$$\phi\mathfrak{G}_1 = \mathfrak{H}_1 \tag{4}$$
$$\psi^{-1}\mathfrak{G}_2 = \mathfrak{H}_2 \tag{5}$$

hold true and define an isomorphic map of $\mathfrak{G}$ onto $\mathfrak{H}$.

Proof. Consider a direct decomposition

$$\mathfrak{G} = \mathfrak{K} + \psi\mathfrak{H}, \tag{6}$$

and define $\mathfrak{G}_1$ by

$$\mathfrak{G}_1 = \mathfrak{K} + \psi\phi\mathfrak{K} + \psi\phi\psi\phi\mathfrak{K} + \dots. \tag{7}$$

This makes sense, since the property of being a direct summand is transitive. $\psi\phi\mathfrak{K}$ is a direct summand of $\psi\mathfrak{H}$, so

$$\mathfrak{G} = \mathfrak{K} + \psi\phi\mathfrak{K} + \mathfrak{X},$$

and so on.

Since the sequence of partial sums in (7) are direct summands of $\mathfrak{G}$, the subgroup $\mathfrak{G}_1$ is a direct summand of $\mathfrak{G}$. $\phi\mathfrak{G}_1$ is a direct summand of $\phi\mathfrak{G}$, $\phi\mathfrak{G}$

[2]A group, decomposable into groups of rank 1, is called completely decomposable. A group is said to be of rank 1, if every finite subset of elements generates a cyclic subgroup (cf. **7**).

[3]This theorem admits considerable generalizations; its proper formulation will be in terms of lattices or semi-lattices.

a direct summand of H, so $\phi\mathfrak{G}_1$ is a direct summand of $\mathfrak{H}$. Applying the map ψ, we see that

$$\psi\phi\mathfrak{G}_1 = \psi\phi\mathfrak{K} + \psi\phi\psi\phi\mathfrak{K} + \dots \tag{8}$$

is a direct summand of $\psi\mathfrak{H}$. Thus there exists a decomposition

$$\psi\mathfrak{H} = \psi\phi\mathfrak{G}_1 + \mathfrak{G}_2, \tag{9}$$

which defines $\mathfrak{G}_2$ uniquely up to isomorphism. Using (6), we see

$$\mathfrak{G} = \mathfrak{K} + \psi\phi\mathfrak{G}_1 + \mathfrak{G}_2.$$

It follows from (7) and (8) that

$$\mathfrak{G}_1 = \mathfrak{K} + \psi\phi\mathfrak{G}_1,$$

so we are in accordance with (2).

Defining $\mathfrak{H}_1$ and $\mathfrak{H}_2$ by (4) and (5), we find, applying ψ^{-1} to (9)

$$\begin{aligned}\mathfrak{H} &= \phi\mathfrak{G}_1 + \psi^{-1}\mathfrak{G}_2\\ &= \mathfrak{H}_1 + \mathfrak{H}_2,\end{aligned}$$

in accordance with (3).

Thus (4) and (5) define the required isomorphic map of $\mathfrak{G}$ on $\mathfrak{H}$, q.e.d.

We give the following simple application.

COROLLARY. *If $\mathfrak{G}$ and $\mathfrak{H}$ are equivalent complete groups or equivalent additive groups of a division-ring, then $\mathfrak{G}$ and $\mathfrak{H}$ are isomorphic.*

Proof. Each subgroup $\mathfrak{U}$ of an additive group of a division-ring with characteristic $p \neq 0$ is a direct summand, so we can apply Theorem I, if the characteristic equals $p \neq 0$. In case $p = 0$, the additive group of a division-ring is complete, so we have only to consider equivalent complete groups $\mathfrak{G}$ and $\mathfrak{H}$. Since a subgroup of a complete group is a direct summand if and only if it is complete and since the sum of an arbitrary number of complete subgroups is again complete, we can apply Theorem I to obtain the required result.

It has to be noted that this result can also be obtained easily by applying the set-theoretical Schröder-Bernstein theorem directly in view of the (up to isomorphisms) unique decomposition of a complete group into a direct sum of groups of rational numbers and groups of type p^∞ for various prime numbers p.

This corollary can be generalized to more general classes of completely decomposable groups, but these are not very interesting since strong conditions must be imposed to avoid counter-examples like Example (i).

The condition in Theorem I requiring that the sum of an ascending sequence of direct summands is again a direct summand is also a very strong one. It is satisfied, for example, for the class of groups **(4)** in which every serving subgroup is a direct summand, but one can prove that there are already

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serving subgroups in a free abelian group of infinite rank which are the sums of an ascending sequence of direct summands, but nevertheless not direct summands themselves.

THEOREM II. *Groups $\mathfrak{G}$ and $\mathfrak{H}$, equivalent by serving subgroups, of which at least one has a base, are isomorphic.*

Proof. Since subgroups of a group with a base have themselves a base, $\mathfrak{G}$ and $\mathfrak{H}$ are both expressible (up to isomorphisms uniquely) as a direct sum of infinite cyclic groups and finite primary cyclic groups. Take decompositions of this kind in $\mathfrak{G}$ and $\mathfrak{H}$.

$\mathfrak{G}$ and $\mathfrak{H}$ have the same rank, since they are equivalent. Hence the number of infinite cyclic summands is the same in both group decompositions. Now take the direct sum $\mathfrak{S}$ of those cyclic summands in $\mathfrak{G}$ which are of order p^k (p and k fixed). We shall show that the number of cyclic summands of this type is the same (in the given decompositions of $\mathfrak{G}$ and $\mathfrak{H}$). $\mathfrak{S}$ is a direct summand of $\mathfrak{G}$, $\phi\mathfrak{G}$ a serving subgroup of $\mathfrak{H}$, $\phi\mathfrak{S}$ therefore a serving subgroup of $\mathfrak{H}$. Using the well-known fact that a periodic serving subgroup with all elements of bounded order is a direct summand, we see that $\phi\mathfrak{S}$ is a direct summand of $\mathfrak{H}$. However, since each decomposition of $\mathfrak{H}$ can be refined to a decomposition into indecomposable cyclic direct summands, and since any two such decompositions are isomorphic, the number of cyclic summands in $\phi\mathfrak{S}$, all of order p^k, is less than or equal to the number of cyclic summands of this type in the decomposition of $\mathfrak{H}$, given above. So the number of cyclic summands of this type in $\mathfrak{G}$ is less than or equal to the corresponding number in $\mathfrak{H}$. Since the converse is equally true, the number is the same.

Now the theorem follows by taking direct sums in $\mathfrak{G}$ and $\mathfrak{H}$ corresponding to different numbers p^k.

3. Indecomposable groups of finite rank. Let α be a transcendental number and put

$$a_i = \alpha^i \qquad (i = 1, 2, \ldots, n).$$

We define the group $\mathfrak{G}$ as the additive group of real numbers

$$\mathfrak{G} = \left\{\frac{a_1}{p_1^{\,k}}, \frac{a_2}{p_2^{\,k}}, \ldots, \frac{a_n}{p_n^{\,k}}, \frac{a_1 + a_2 + \ldots + a_n}{p^k}\right\},$$

that is, $\mathfrak{G}$ is generated by the elements in the right hand side, where the k are variable integers and $p, p_1, p_2, \ldots, p_n$ are distinct prime numbers.

$\mathfrak{G}$ is an indecomposable group[4] *of rank n* ($\mathfrak{G}$ is clearly countable and torsion-free).

[4]The first indecomposable groups of rank two were constructed by Levi (**10**) and Pontrjagin (**12**, p. 384). Baer (cf. **7**, p. 217) proved that every serving subgroup of the additive group of p-adic integers is indecomposable. Erdös (**3**) gave an example of an indecomposable group of rank two, which is essentially the same as our example for the case $n = 2$, though following a different line of thought.

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Proof. Using the transcendency of α we see that $\mathfrak{G}$ only contains elements of $n+2$ different types, that is, the type 0 (of the infinite cyclic group) and the types of $a_1, a_2, \ldots, a_n$ and

$$b = \sum_{i=1}^{n} a_i .$$

Moreover, we see in the same way that only the elements of the serving subgroups

$$\mathfrak{A}_i = \left\{\frac{a_i}{p_i^k}\right\} \ (i = 1, 2, \ldots, n) \text{ and } \mathfrak{B} = \left\{\frac{b}{p^k}\right\}$$

have the types of $a_i (i = 1, 2, \ldots, n)$ and b, respectively.

Suppose now that $\mathfrak{G}$ is decomposable

$$\mathfrak{G} = \mathfrak{P} + \mathfrak{Q}.$$

Then each of the groups $\mathfrak{A}_i$ and $\mathfrak{B}$ must be contained in either $\mathfrak{P}$ or $\mathfrak{Q}$ since' in the opposite case, as the reader may easily verify, both $\mathfrak{P}$ and $\mathfrak{Q}$ would contain elements of, say, the type of a_i in contradiction of the fact that only the serving subgroup $\mathfrak{A}_i$ of rank 1 contains such elements. Say

$$b \in \mathfrak{P}.$$

The a_i are spread over $\mathfrak{P}$ and $\mathfrak{Q}$, but then they are already contained in $\mathfrak{P}$, since otherwise

$$b = p + q, \qquad q \neq 0,$$

in contradiction to $b \in \mathfrak{P}$.

Hence $\mathfrak{Q} = 0$ and $\mathfrak{G}$ is indecomposable, q.e.d.

4. Example (ii). Define "disjoint" groups $\mathfrak{G}_i (i = 1, 2, \ldots)$

$$\mathfrak{G}_i \simeq \mathfrak{G},$$

where $\mathfrak{G}$ is the group of the preceding section with $n = 2$. Define

$$\mathfrak{G}_0 = \left\{\frac{1}{p_1^n}\right\},$$

where n is a variable integer. We form the (restricted) direct sums

$$\mathfrak{K} = \sum_{i=0}^{\infty} \mathfrak{G}_i, \quad \mathfrak{K}_1 = \sum_{i=1}^{\infty} \mathfrak{G}_i.$$

It is easy to see, using different transcendental numbers, that $\mathfrak{K}$ can be isomorphically embedded in the additive real group. So $\mathfrak{K}$ is torsion-free and countable.

$\mathfrak{K}$ and its subgroup $\mathfrak{K}_1$ are countable torsion-free groups, which are equivalent by serving subgroups, but $\mathfrak{K}$ and $\mathfrak{K}_1$ are not isomorphic.

Proof. $\mathfrak{K}_1$ is a direct summand, so certainly a serving subgroup of $\mathfrak{K}$. Conversely, we can map $\mathfrak{K}$ isomorphically in $\mathfrak{K}_1$:

$$\mathfrak{K} \rightarrow \phi\mathfrak{K} \subset \mathfrak{K}_1,$$

by mapping $\mathfrak{G}_0$ in the natural way on $\mathfrak{H}_1 \subset \mathfrak{G}_1$ (where $\mathfrak{H}_1$ corresponds to the subgroup

$$\left\{\frac{a_1}{p_1^{\,n}}\right\}$$

of $\mathfrak{G}$, defined in §**3**), and $\mathfrak{G}_i$ in the natural way on $\mathfrak{G}_{i+1}$ ($i = 1, 2, \ldots$) . $\phi\mathfrak{K}$ is a serving subgroup of $\mathfrak{K}_1$, since $\mathfrak{K}_1/\phi\mathfrak{K}$ is torsion-free. Thus $\mathfrak{K}$a nd $\mathfrak{K}_1$ are equivalent by serving subgroups.

To prove that $\mathfrak{K}$ and $\mathfrak{K}_1$ are not isomorphic, we prove that $\mathfrak{K}_1$ contains no direct summand isomorphic to the direct summand $\mathfrak{G}_0$ of $\mathfrak{K}$.

Suppose, on the contrary, that there is a subgroup

$$\mathfrak{G}_0^* \simeq \left\{\frac{1}{p_1^{\,n}}\right\}$$

with

$$\mathfrak{K}_1 = \mathfrak{G}_0^* + \mathfrak{M}. \tag{10}$$

Since (putting $\mathfrak{G}_j = \mathfrak{G}$) the elements $a_2/p_2^{\,n}$ are contained in $\mathfrak{G}_j$ for all natural numbers n, and since "unlimited division" in $\mathfrak{G}_0^*$ is only possible by powers of p_1, it follows easily from the decomposition (10) that all $a_2/p_1^{\,n}$ are contained in $\mathfrak{M}$. The b/p^n are equally contained in $\mathfrak{M}$. So $b - a_2 = a_1$ and therefore the serving subgroup $\{a_1/p_1^{\,n}\}$ is contained in the direct summand $\mathfrak{M}$. Hence

$$\mathfrak{G}_j \subset \mathfrak{M}.$$

This is true for all $j = 1, 2, \ldots$. Therefore

$$\mathfrak{M} = \mathfrak{K}_1$$

in contradiction to (10).

Remark. Actually, we proved the existence of two non-isomorphic groups, the first being isomorphic to a direct summand of the second, and the second being isomorphic to a serving subgroup of the first.

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