FINITE NEAR-RINGS WITH TRIVIAL ANNIHILATORS

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1. Introduction

In [3] and [4], the near-rings R with no zero divisors are studied. In particular, a near-ring R is a near-field if it has a non-zero right distributive element ([4], Theorem 1.2.). Also, $(R, +)$ is a nilpotent group if not all non-zero elements of R are left identities of R ([3], Theorem 2). The purpose of the present paper is to extend the above results to a class of near-rings with zero divisors; that is, the set of annihilators of an element x in R, $T(x) = \{g/xg = 0\}$ is either $\{0\}$ or R. The examples of such near-rings are those R with $(R, +)$ simple groups and those R with no zero divisors as given in [1], [2], [3] and [4]. For this R, we can easily see that $R = A \cup S$ where $A = \{x/T(x) = R\}$ and $S = \{x/T(x) = \{0\}\}$. Then the second part of this paper will give a structural theorem on the semi-group $(S, \cdot)$, and more properties on R can be derived.

Throughout the present paper $(R, +, \cdot)$ is assumed a finite near-ring such that for each x in R, $T(x) = \{y/xy = 0\}$ is either R or $\{0\}$. If $T(0) = \{0\}$ then each $0a \neq 0$ for each $a \neq 0$ in R. But then $0R = R$; and so $0a$ is a left identity of R for each $a \neq 0$. From now on R is assumed not this kind just mentioned. So, R has the property that $T(0) = R$.

2

Section 1. Assume $T(x)$ is either R or $\{0\}$ for each x in R, we shall show that either the multiplication operation on R is trivial (that is, for $r \neq 0$ in R, $rg = 0$ for all g in R or $rg = g$ for all g in R), or the additive group $(R, +)$ is nilpotent. This extends Theorem 2 in [3].

LEMMA 1.1. *Let $S = \{x/T(x) = \{0\}\}$ and let $A = \{x/T(x) = R\}$; then* (1) $R = A \cup S$ *such that $A \cap S$ is a void set and* (2) $sS = S$ *and* $sA = A$ *for each s in S, where $sA = \{sa$ for a in $A\}$ and $sS = \{st$ for t in $S\}$.*

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PROOF. For each x in R $T(x)$ is either R or $\{0\}$, so part (1) is trivial. Next for each a in A $saR = s(aR) = s0 = 0$, so $sA \subset A$. Let x be an element in S and r in R such that $sxr = 0$. Then $s(xr) = 0$, $xr = 0$; and so $r = 0$. Thus $sS \subset S$. On the other hand, for any r' and r'' in R such that $sr' = sr''$, we have $s(r' - r'') = 0$. Hence $r' = r''$ because $T(s) = \{0\}$. Therefore $sA = A$ and $sS = S$.

LEMMA 1.2. *By keeping the notations of lemma* 1.1, *if $st = t$ for some elements s and t in S, then $sr = r$ for each r in R.*

PROOF. Since $tS = S$ and $tA = A$ by lemma 1.1, $tR = R$; and so for each r in R, $r = tr'$ for some r' in R. Hence $sr = s(tr') = (st)r' = tr' = r$.

Using similar idea to [3] we can show our main theorem in this section.

THEOREM 1.3. *By keeping the notations of Lemma* 1.1, *we have*

(1) *if $T(x) = R$ for each x in R then $R^2 = \{0\}$;*

(2) *if $T(x) = \{0\}$ with some $x \neq 0$ and if A contains no non-zero subgroups of $(R, +)$; then either x is a left identity of R or $(R, +)$ is a nilpotent group.*

PROOF. Part (1) is trivial by the definition of $T(x)$. Next, since $T(x) = \{0\}$ with some $x \neq 0$ in R, x is in S; and so $xR = R$ by lemma 1.1. Suppose x is not a left identity of R. Then $x^2 \neq x$. For otherwise $xr = r$ for each r in R by lemma 1.2. This contradicts that x is not a left identity. Hence we can have the identity x^n of the cyclic group generated by x under multiplication with $n > 1$; that is $x^n x = xx^n$ with a minimal integer n. Again since $xR = R$, $x^n(xr)$ is equal to xr for each xr in xR; so x^n is a left identity of R with $n > 1$. Futhermore, it is not hard to show that α_y defined by $\alpha_y(r) = yr$ for each r in R is a group automorphism of $(R, +)$ if y is in S. Since $n > 1$, $n = pm$ for some prime integer p and an integer m. Noting that the element x^m has order p, and that x^m is in S, we have that $\alpha_{(x^m)}$ is an automorphism of $(R, +)$ of order p. Also, $\alpha_{(x^m)}$ is a fixed point free automorphism. In fact, let $\alpha_{(x^m)}(r) = r$, that is, $x^m r = r$. Then there are two cases.

Case 1. r is in S. Then $rR = R$ by Lemma 1.2; and so x^m is a left identity of R. Thus $x^m x = x$, a contradiction to the minimal property of n such that $x^n x = x$. This implies that $\alpha_{(x^m)}$ is a fixed point free automorphism of order p. Therefore $(R, +)$ is a nilpotent group by [6].

Case 2. r is in A. Let $C = \{h/h \text{ in } R \text{ and } x^m h = h\}$. Then 0 and r are in C. For each h' and h'' is C, $x^m(h' - h'') = x^m h' - x^m h'' = h' - h''$. Hence $(C, +)$ is a subgroup of $(R, +)$. Noting that C can be assumed a subset of A. For otherwise there exists h in S such that $x^m h = h$; and so this leads to case 1. But by hypothesis the set A has no non-zero subgroup of $(R, +)$, so $C = \{0\}$. Hence $r = 0$. Thus $\alpha_{(x^m)}$ is a fixed point free automorphism of order p; and so $(R, +)$ is nilpotent.

A near-ring R is a near-integral domain if and only if $A = \{0\}$ and not all elements of R are left identities ([3], Def. Section 2). Then our theorem extends the Theorem 2 in [3].

COROLLARY 1.4. (Ligh) *Let R be a finite near-integral domain. Then $(R, +)$ is nilpotent.*

From $R = A \cup S$ with $A \cap S$ a void set, it can be shown that a trivial multiplication on A implies a trivial multiplication on R.

PROPOSITION 1.5. *If $(S \cup \{0\}, +)$ is a proper subgroup of $(R, +)$ and if $sa = a$ for each a in A and some s in R (and so in S), then $sr = r$ for each r in R.*

PROOF. For any element t in S, $a + t$ is not in S because $(S \cup \{0\}, +)$ is a subgroup of $(R, +)$ and because a is not in $S \cup \{0\}$. Hence $s(a + t) = a + t$. But then $sa + st = a + st = a + t$, $st = t$ for each t in S. Thus $sr = r$ for each r in R.

Since for each s in S, α_s defined by $\alpha_s(r) = sr$ is a group automorphism of $(R, +)$. Hence from Proposition 1.5 we have:

COROLLARY 1.6. *If $(S \cup \{0\}, +)$ is a subgroup of $(R, +)$ and if the automorphism α_s has at least two non-zero fixed points in a same coset of $(S \cup \{0\}, +)$; then α_s is an identity automorphism.*

PROOF. Let a' and a'' be two non-zero fixed points of α_s in a same coset of $(S \cup \{0\}, +)$ in R. Then $a' = a'' + r$ for some r in S such that $\alpha_s(a') = a'$ and $\alpha_s(a'') = a''$; and so

$$\alpha_s(a') = \alpha_s(a'' + r) = a'' + r.$$

But $\alpha_s(a'' + r) = \alpha_s(a'') + \alpha_s(r)$ then $\alpha_s(a'' + r) = a'' + \alpha_s(r) = a'' + r$. Hence $\alpha_s(r) = r$; that is, $sr = r$. This implies that $st = t$ for all t in R by Lemma 1.2. Thus α_s is the identity automorphism of $(R, +)$.

(3) *Section* 2. By Lemma 1.1, $R = A \cup S$, so, in case S is a void set, we have $R^2 = \{0\}$, and in case $A = \{0\}$, we have a near-integral domain. In this section, S is always assumed non-void. We shall give the following structural theorem on the semi-group $(S, \cdot)$: S is partitioned as isomorphism multiplicative groups. Consequently, some of the results of [4] can be extended.

LEMMA 2.1. *For each element s in S, it has a unique right identity s' which is also a left identity of R.*

PROOF. Since $T(s) = \{0\}$ and since R is finite, there is a multiplicative group generated by s of order n,

$$\{s = s^{n+1}, s^2, \cdots, s^n\}.$$

Hence s^n is a right identity of s. Suppose t is also a right identity of s. Then $st = ss^n$; and so $s(t - s^n) = 0$. Thus $t = s^n = s'$ because $T(s) = \{0\}$ again. This implies that s' is unique. Moreover, noting that $s's = s^ns = s$ and that $sR = R$ we conclude that $s' = s^n$ is a left identity of R by Lemma 1.2.

DEFINITION. For any x and y in S, we call x *equivalent* to y if and only if the identity of x = the identity of y.

THEOREM 2.2. (a) *The relation "$\sim$" defined above on S is an equivalence relation*;

(b) *Each equivalence class of "$\sim$", $R_x = \{y/$the identity of $y =$ the identity of $x\}$, is a multiplicative group*;

(c) *Any two equivalence classes, R_x and R_y, for x and y in S, are isomorphic.*

PROOF. Part (a) is obvious. For part (b), let a and b be in R_x with the right identity x^n. Then $(ab)x^n = a(bx^n) = ab$. Also, $aa^{k-1} = a^k = x^n$, where k is the order of a, so $a^{k-1} = a^{-1}$. Hence R_x is a multiplicative group with the identity x^n. Finally for part (c), let R_x and R_y be any two equivalence classes with the right identities x' and y' respectively. Define a map β from R_x to R_y by $\beta(rx') = (rx')y'$ for each rx' in R_x (for $R_x = S_x'$). Since x' is also a left identity of R by Lemma 2.1,

$$\beta(rx') = (rx')y' = r(x'y') = ry'.$$

We claim that β is a group isomorphism from R_x onto R_y. In fact, for any ax' and bx' in R_x,

$$\beta(ax'bx') = \beta(abx') = (aby') = (ab)y' = (ay')(by')) =$$

$$(ax'y')(bx'y') = \beta(ax')\beta(bx')$$

by Lemma 2.1. again. Next let $\beta(ax') = y'$ for an element ax' in R_x, then $(ax')y' = y'$. Since $y'R = R$, ax' is a left identity of R; and so ax' is the identity of the multiplicative group R_x. Hence $ax' = x'$. Thus β is one to one. Furthermore, let ay' be an element in R_y, then ax' is in R_x such that

$$\beta(ax') = (ax')y' = ay'.$$

This implies that β is onto and therefore the theorem is proved.

The following consequences are immediate.

COROLLARY 2.3. *The number of elements in S = (order of R_x) times (the number of equivalence classes of "$\sim$").*

COROLLARY 2.4. *The following statements are equivalent*:

(a) $R_x \cup \{0\}$ *is a subnear-ring of* $(R, +, \cdot)$;

(b) $R_x \cup \{0\}$ *is a near-field*;

(c) $(R_x \cup \{0\}, +)$ *is a subgroup of* $(R, +)$.

PROOF. Since R_x is a multiplicative group, the proof is trivial.

In theorem 1.3, we assumed that the set A contains no non-zero subgroups of R under addition ([1], 2–3, example 6). But if R has a non-zero right distributive element, then this assumption does not hold.

THEOREM 2.5. *If R has a non-zero right distributive element, then* $(A, +)$ *is a normal subgroup of* $(R, +)$.

PROOF. Let x be a non-zero right distributive element in R. Then for any elements a and b in A $(a + b)x = ax + bx = 0$; and so $a + b$ is in A. Hence $(A, +)$ is a subgroup of $(R, +)$. Moreover, for each c in R, $(-c + a + c)x = (-c)x + ax + cx = -(cx) + 0 + cx = 0$ because x in a right distributive element. Thus $(A, +)$ is normal in $(R, +)$.

REMARK 1. The near-ring R under consideration is $S \cup A$ by Lemma 1.1. From the definitions of A and S, we know that $A - \{0\}$ is the set of left zero divisors of R and that S is the set of elements without right zero divisors. Hence if R has a right distributive non-zero elements, then the number of elements of R is less than n^2 where $n + 1$ is the order of the normal subgroup $(A, +)$ in Theorem 2.5 ([4], Th. 2.3).

REMARK 2. If R has a right distributive element in S with $S \cup 0$ a group under $+$ then S has only one equivalence class in the sense of Theorem 2.2. S is equal to the class, so it is a multiplicative group. $S \cup \{0\}$ is a near-ring, for $(S \cup \{0\}, +)$ is a subgroup of $(R, +)$. This implies that $R = S \cup \{0\}$ because. the complement of S, $(A, +)$, is also a subgroup of $(R, +)$ by Theorem 2.5. Thus this leads to Theorem 1.2. of [4]; and so R is a near-field. After this paper had been submitted, the author learned that Theorem 2.2 for planar integral domains had been proved by J. Clay.

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