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A CHARACTERIZATION OF CHAOS

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Consider the continuous mappings f from a compact real interval to itself. We show that when f has a positive topological entropy (or equivalently, when f has a cycle of order $\neq 2^n$, $n = 0, 1, 2, \ldots$) then f has a more complex behaviour than chaoticity in the sense of Li and Yorke: something like strong or uniform chaoticity, distinguishable on a certain level $\varepsilon > 0$. Recent results of the second author then imply that any continuous map has exactly one of the following properties: It is either strongly chaotic or every trajectory is approximable by cycles. Also some other conditions characterizing chaos are given.

Denote by $C^0(I,I)$ the class of continuous mappings $I \to I$, where I is a compact real interval. An $f \in C^0(I,I)$ is said to be chaotic in the sense of Li and Yorke [5], when there is an uncountable set $S \subseteq I$ such that for any $x, y \in S, x \neq y$, and any periodic point p of f,

(1) $$\limsup_{n \to \infty} |f^n(x) - f^n(y)| > 0$$

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(2) $$\liminf_{n \to \infty} |f^n(x) - f^n(y)| = 0$$

(3) $$\limsup_{n \to \infty} |f^n(x) - f^n(p)| > 0$$

Here f^n denotes the n-th iterate of f. Any set S whose points satisfy condition (1) - (3) is called a scrambled set for f.

In [10] is given the following stronger concept: given $\varepsilon > 0$, a set $S \subseteq I$ is an ε-scrambled set for some $f \in C^0(I,I)$ if for any $x, y \in S$, $x \neq y$, and any periodic point p of f,

(4) $$\limsup_{n \to \infty} |f^n(x) - f^n(y)| > \varepsilon$$

(5) $$\limsup_{n \to \infty} |f^n(x) - f^n(p)| > \varepsilon$$

and (2) is true.

Moreover, in [10] it is shown that for any $f \in C^0(I,I)$ with zero topological entropy (or equivalently, without cycles of order divisible by an odd prime, see [6]) the chaoticity in the sense of Li and Yorke is equivalent to the existence of a perfect non-empty ε-scrambled set, for some $\varepsilon > 0$. The following main result of this paper shows that this is also true for mappings with positive topological entropy.

THEOREM 1. *Let $f \in C^0(I,I)$ have a cycle of order divisible by an odd prime. Then for some $\varepsilon > 0$, f has a non-empty perfect ε-scrambled set S.*

In the proof we use methods of symbolic dynamics, see, for example, [2] or [7]. First we recall the following well-known result.

LEMMA 1. *(Block [1], see also [12]). If $f \in C^0(I,I)$ has a cycle of order $\neq 2^n$, $n = 0, 1, 2, \ldots,$ then there are closed disjoint intervals $J_0, J_1 \subseteq I$ and an integer $m > 0$ such that*

(6) $$f^m(J_0) \cap f^m(J_1) \supseteq J_0 \cup J_1 .$$

Next we give a generalization of this lemma.

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LEMMA 2. *Let* f, J_0, J_1 *and* m *be as in Lemma 1. Then there are closed intervals*

$$J_0 = J_0^0 \supseteq J_1^0 \supseteq J_2^0 \supseteq \dots \quad \text{and} \quad J_1 = J_0^1 \supseteq J_1^1 \supseteq J_2^1 \supseteq \dots$$

and a sequence $\{m(k)\}_{k=0}^{\infty}$ *of positive integers such that for every* $k = 0, 1, 2, \dots$ *and* $j = 0, 1,$

(7) $\qquad m(k)$ *is divisible by* $k!$ *and* m ,

(8) $\qquad f^{m(k)}(J_k^j) \supseteq J_0 \cup J_1$,

(9) $\qquad \mu(J_{k+1}^j) < \frac{1}{2}\mu(J_k^j)$,

where μ *is the Lebesgue measure.*

Proof. Put $m(0) = m$ and assume by induction that $m(k)$, J_k^0 and J_k^1 are defined for $k \le n$. Choose a closed interval $U_j \subseteq J_n^0$ such that $f^{m(n)}(U_j) = J_j$, for $j = 0, 1$. Then at least one of the sets U_0, U_1 has Lebesgue measure less than $\frac{1}{2}\mu(J_n^0)$. Denote this set by J_{n+1}^0 and put $m(n + 1) = m(n) + p$, where $p \ge m$ is choosen such that (7) is true for $k = n + 1$. Then by (6)

$$f^{m(n+1)}(J_{n+1}^0) = f^{m(n)+p}(J_{n+1}^0) = f^p(J_0) \supseteq J_0 \cup J_1$$

since p is divisible by m . Similarly we find J_{n+1}^1 . □

In the sequel the following notation will be useful. Let $X(k)$ be the set $\{0, 1\}^k$ of all ordered k-tuples and $X = \{0, 1\}^N$ the set of all sequences of two symbols $0, 1.$ If $\alpha \in X(k)$, $\beta \in X(s)$ then $\alpha\beta \in X(k + s)$ is the concatenation of α and β . For $\alpha \in X(k)$ or $\alpha \in X$, $\alpha(j)$ will denote the j-th coordinate of α . Assume X is equipped with the topology of pointwise convergence (given for example by the metric $\rho(\alpha, \beta) = \sum_n 2^{-n} |\alpha(n) - \beta(n)|)$.

LEMMA 3. *There is a perfect, non-empty set* $Y \subseteq X$ *such that any*

$\alpha \in Y$ has infinitely many 0's and 1's, and for any two $\alpha, \beta \in Y$, $\alpha \neq \beta$ implies $\alpha(n) \neq \beta(n)$ for infinitely many n.

Proof. Let ξ be an irrational number. Define $\tau : [0, 1] \to X$ in the following way: For $t \in [0, 1]$, $\tau(t) = \{\alpha(k)\}_{k=1}^{\infty}$, where

$$\alpha(k) = \begin{cases} 0 & \text{if} \quad N(t + \xi k) \in [0, 1/2) \\ 1 & \text{if} \quad N(t + \xi k) \in [1/2, 1) \end{cases}$$

Here $N(x) \in [0, 1)$ is the fractional part of x. Considering the well-known fact that $\{N(\xi k)\}_{k=1}^{\infty}$ is uniformly distributed and hence dense in $[0, 1]$, we can easily verify that $\tau(t)(n) \neq \tau(s)(n)$ for infinitely many n, whenever $t, s \in [0, 1]$, $t \neq s$.

Next observe that τ has at most a countable set of discontinuity points: for each k there is exactly one $t \in [0, 1]$ so that $N(t + \xi k) = 1/2$. Denote this t by $t(k)$. Clearly τ is continuous on $B = [0, 1] \setminus \{t(k)\}_{k=1}^{\infty}$. Since B is a Borel set we have that $\tau(B) \subseteq X$ is analytic and uncountable and by [4] it contains a non-empty perfect set P.

For any $\alpha \in P$ write

$$\alpha^* = \alpha(1)\ 0\ \alpha(2)\ 1\ \alpha(3)\ 0\ \alpha(4)\ 1\ \alpha(5)\ 0 \ldots$$

and let $Y = \{\alpha^* ; \alpha \in P\}$. It is easy to see that Y is closed (as the intersection of closed sets) and has no isolated points, that is Y is perfect. □

LEMMA 4. *Let f have a cycle of order divisible by an odd prime. Then there is a set $\{I_\alpha ; \alpha \in X(k)\}_{k=1}^{\infty}$ of closed intervals and a sequence $\{n(k)\}_{k=1}^{\infty}$ of positive integers such that, for every k, s, $k > s$,*

(10) $\quad I_{\alpha\beta} \subseteq I_\alpha \quad$ *for* $\quad \alpha \in X(k - s), \beta \in X(s)$,

(11) $\quad f^{n(k)}(I_\alpha) = J_k^{\alpha(k)} \quad$ *whenever* $\quad \alpha \in X(k)$,

(12) $\quad n(k) - n(s)$ *is divisible by* $s!$;

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here J^i_k *are the intervals from Lemma 2.*

Proof. Keep the notation from Lemma 2. Since $f^{m(0)}(J^j_0) \supseteq J_0 \cup J_1$, there is a closed interval $I_j \subseteq J^j_0 = J_j$ such that $f^{m(0)}(I_j) = J_j$, $j = 0, 1$. Put $n(1) = m(0)$.

Now assume by induction that we have defined intervals $\{I_\alpha ; \alpha \in X(r)\}$ and $n(r)$. Let $\alpha \in X(r+1)$. Then $\alpha = \beta 0$ or $\alpha = \beta 1$ where $\beta \in X(r)$. By the hypothesis, $f^{n(r)}(I_\beta) = J^{\beta(r)}_r$ and Lemma 2 gives

$$f^{n(r)+m(r)}(I_\beta) \supseteq I_0 \cup I_1 .$$

Hence there is a closed interval $I_\alpha \subseteq I_\beta$ such that $f^{n(r)+m(r)}(I_\alpha) = J^{\alpha(r+1)}_{r+1}$. If we take $n(r+1) = n(r) + m(r)$, then by (7) and hypothesis, (12) if true for $k = r + 1$. The other conditions are clearly satisfied.□

Now we are ready to give

Proof of Theorem 1. Keep the notation from Lemmas 1 - 4. Write $F_k = \cup\{I_\alpha ; \alpha \in X(k)\}$ and $A = \bigcap_{k=1}^{\infty} F_k$. Define a mapping $\phi : A \to X$ in the following way:

For any $x \in A$ let $\phi(x) = \alpha \in X$ be such that $x \in M_\alpha$, where

$$M_\alpha = I_{\alpha(1)} \cap I_{\alpha(1)\alpha(2)} \cap I_{\alpha(1)\alpha(2)\alpha(3)} \cap \ldots$$

(it is easy to see that for every x there is exactly one α with $x \in M_\alpha$), Since $M_\alpha \neq \emptyset$ for every α, ϕ is surjective.

The mapping ϕ is also continuous. Indeed, let $O(\alpha)$ be a neighbourhood of $\alpha \in X$. Then there is an n such that

$$O(\alpha) \supseteq O^n(\alpha) = \{\beta \in X; \beta(k) = \alpha(k) \text{ for } k = 1, \ldots, n\} .$$

Write $G = I_{\alpha(1)\ldots\alpha(n)}$. Let $x \in A$ with $\phi(x) = \alpha$. Then G is a relatively open neighbourhood of x in A, and clearly $\phi(G) \subseteq O(\alpha)$.

Note that for every α, M_α is closed and connected, and ϕ is constant on M_α. Let x_α be the left-end point of M_α. Then clearly

$B = \{x_\alpha ; \alpha \in X\} \subseteq A$ is an uncountable Borel set and ϕ restricted to B is a bijection $B \to A$. Therefore $\phi^{-1}(Y) \cap B$ is an uncountable Borel set. Hence there is a non-empty perfect set $S \subseteq \phi^{-1}(Y) \cap B$ (see [4]; here Y is the set from Lemma 3).

It remains to verify that S is the desired ε-scrambled set for f, where

$$\varepsilon = \frac{1}{3} \text{dist } (J_0, J_1) > 0 .$$

Let $x, y \in S, x \neq y$. Then $\phi(x) = \alpha$, $\phi(y) = \beta$, where $\alpha \neq \beta$, $\alpha, \beta \in Y$. Hence by Lemma 3 and (11), for infinitely many k either

$$f^{n(k)}(x) \in J_0 \quad \text{and} \quad f^{n(k)}(y) \in J_1$$

or

$$f^{n(k)}(x) \in J_1 \quad \text{and} \quad f^{n(k)}(y) \in J_0$$

since $J_j^i \subseteq J_i$ for every i, j. Thus (4) is true.

Again by Lemma 3 and (11), for infinitely many k we have $\alpha(k) = \beta(k)$, and thus $f^{n(k)}(x), f^{n(k)}(y) \in J_k^{\alpha(k)} = J_k^{\beta(k)}$, hence by (9)

$$|f^{n(k)}(x) - f^{n(k)}(y)| \leq \mu(J_k^{\alpha(k)}) \leq 2^{-k} \mu(J_0^{\alpha(k)})$$

for every such k and this implies (2).

Finally, let $x \in S$ and let $p \in I$ be a periodic point of f. Let s be the period of p. For $k > s$ we have

$$f^{n(k)}(p) = f^{n(k)-n(s)}(f^{n(s)}(p)) = f^{n(k)-n(s)}(q) = q ,$$

since q has period s and s divides $n(k) - n(s)$ (see (12)). Let $r \in \{0, 1\}$ be such that dist $(J_r, \{q\}) > \varepsilon$. Choose $k > s$ so that for $\alpha = \phi(x)$, $\alpha(k) = r$. Then by (11),

$$|f^{n(k)}(x) - f^{n(k)}(p)| \geq \text{dist } (J_r, \{q\}) > \varepsilon .$$

Since k can be choosen arbitrarily large we obtain (5) and our theorem is proved. □

Before we state the next result, we recall some terminology (see

[10]). Let $f \in C^0(I, I)$. We say that an interval $J \subseteq I$ is an f-periodic interval of order k if $f^k(J) = J$ and $f^i(J) \cap f^j(J) = \emptyset$ for $i \neq j$, $i, j = 1, \ldots, k$. Two points $u, v \in I$ are f-separable if there are disjoint periodic intervals $J_u, J_v \subseteq I$ with $u \in J_u$, $v \in J_v$. Otherwise u, v are f-nonseparable. The set of all limit points of a trajectory $\{f^k(x)\}_{k=1}^{\infty}$ is called the attractor of f and x , and is denoted by $L_f(x)$.

The following theorem generalizes a result from [10].

THEOREM 2. *A function $f \in C^0(I, I)$ is chaotic in the sense of Li and Yorke if and only if there is an infinite attractor $L_f(x)$ containing two f-nonseparable points u, v.*

Proof. In [10] the theorem is proved for functions with zero topological entropy. Thus in view of Theorem 1 it suffices to show that any $f \in C^0(I, I)$ with positive topological entropy has an infinite attractor $L_f(x)$ containing two f-nonseparable points u, v.

Hence assume f has a cycle of order divisible by an odd prime (see [6]). By [11] or [12] there is an uncountable attractor $L_f(x)$ containing a cycle of f . Let the order of this cycle be $m \geq 1$. Clearly $L_f(x)$ contains two accumulation points u, v of $L_f(x)$. Assume that there are disjoint periodic intervals J_u, J_v, $u \in J_u$, $v \in J_v$, with periods $m(u), m(v) \geq 1$ (otherwise u and v would be f-nonseparable).

Then there is a k such that $f^k(x) \in J_u$, and hence $L_f(x) \subseteq \mathrm{Orb}_f(J_u) = \bigcup_{i=1}^{m(u)} f^i(J_u)$, and similarly $L_f(x) \subseteq \mathrm{Orb}_f(J_v)$. Since J_u, J_v are disjoint, we have $m(u) > 1$, $m(v) > 1$. Consider the mapping $f^{m(n)}$ restricted to J_u ; denote it f_1 . By the periodicity of J_u , the set $L_f(x) \cap J_u$ is uncountable (since $f(L_f(x)) = L_f(x))$. Choose two accumulation points $u_1, v_1 \in L_f(x) \cap J_u$ of $L_f(x)$. Assume there

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are disjoint f_1-periodic intervals $J_u^1, J_v^1 \subset J_u$ with periods $m(u_1)$, $m(v_1) \geq 1$ such that $u_1 \in J_u^1$ and $v_1 \in J_v^1$ (otherwise u_1, v_1 are f-nonseparable). Similarly as in the preceding step we can see that $m(u_1) > 1$ and $m(v_1) > 1$. Hence J_u^1 is an f-periodic interval of period $m(u).m(u_1) > m(u)$, and such that $L_f(x) \subseteq \mathrm{Orb}_f(J_u^1)$.

By repeating this construction we obtain f-periodic intervals $J_u \supseteq J_u^1 \supseteq J_u^2 \supseteq \ldots \supseteq J_u^n$, where $n \geq 1$ is the first index such that $L_f(x) \subseteq \mathrm{Orb}_f(J_u^n)$ and $m(u_n)$, the period of J_u^n, is greater than m. But this is a contradiction with the fact that $L_f(x)$ contains a cycle of order m. Hence u_{n-1}, v_{n-1} are f-nonseparable. □

Now we can prove the following survey theorem summarizing conditions equivalent to the chaoticity of mappings. Recall that for $f \in C^0(I, I)$ we say that the trajectory $\{f^k(x)\}_{k=1}^{\infty}$ of x is approximable by cycles if for any $\varepsilon > 0$ there is a periodic point p of f such that

$$\limsup_{n \to \infty} |f^n(x) - f^n(p)| < \varepsilon .$$

THEOREM 3. *Let* $f \in C^0(I, I)$. *The following conditions are equivalent:*

(a) f *is chaotic in the sense of Li and Yorke;*

(b) f *has an infinite attractor containing two* f*-nonseparable points;*

(c) for some $\varepsilon > 0$, f *has a nonempty perfect* ε*-scrambled set;*

(d) f *has a trajectory which is not approximable by cycles;*

(e) f *is topologically conjugate to a function which has a scrambled set of positive Lebesgue measure;*

(f) for some $\varepsilon > 0$, f *has a nonempty* ε*-scrambled set.*

Remark 1. We emphasize that, rather surprisingly, positive topological entropy (or the existence of a cycle of order divisible by an odd prime) is not equivalent to the chaoticity of a function f in

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the sense of Li and Yorke (an example is given in [10]). However, in view of Theorem 1, positive topological entropy of f implies that chaoticity of f.

On the other hand, existence of an infinite attractor does not imply (but is clearly implied by) the chaoticity of f, see [10].

Remark 2. The implication (a) → (e) in Theorem 3 generalizes recent results [3], [7], [8], in which particular functions with large (from a measure-theoretical point of view) scrambled sets are constructed. However, this implication does not generalize the result from [9], in which map g with a perfect scrambled set of positive Lebesgue measure is given. This is because g can easily be modified to be of class C^1 (this possibility is not mentioned in [9]).

Proof of Theorem 3. (a)<—>(b): This follows from Theorem 2.

(b) → (c): This was proved in [10] for functions having no cycles of order divisible by an odd prime; for other functions use Theorem 1.

(c) → (d): This follows immediately from (5).

(d) → (a ∨ b): For functions with zero topological entropy the implication (d) → (b) is proved in [10], otherwise Theorem 1 gives the validity of (a).

(c) → (e): Let $S \neq \emptyset$ be a perfect scrambled set for f. Let $h: I \to I$ be a homeomorphism such that $\mu(h(S)) > 0$. Then $h(S)$ is clearly a scrambled set for $g = h \circ f \circ h^{-1}$ (first apply h^{-1}).

(e) → (a) is trivial and since (f) is an another formulation of (d), also (d)<—>(f) is true. □

Problem. It is possible to show that for $f \in C^0(I, I)$ the following condition also is equivalent to the chaoticity of f:

(g) f has a scrambled set containing two points.

However, our proof is rather complicated. But this result should be probably provable in a simpler way. (Clearly, in view of Theorem 1 it suffices to consider only mappings with zero topological entropy satisfying the condition (g).)

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