ON THE COHOMOLOGY OF SOME HOPF ALGEBRAS

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To Professor Kiyoshi Noshiro on the occasion of his 60th birthday.

In this paper we study certain injective resolutions for some Hopf algebras. An injective resolution for a coalgebra A with augmentation η is defined to be such an exact sequence that

$$0 \to K \xrightarrow{\eta} A = X^0 \to X^1 \to \ldots\ldots \to X^n \xrightarrow{\delta^n} \ldots\ldots$$

where K is the basic field, X^n are injective A-comodules and δ^n are morphism of A-comodules. If A is connected and of finite type, then the cohomology $H^\cdot(A^*)$ is, by definition, given by $H^\cdot(K \square_A X) = \text{Cotor}_A^\cdot(K, K) = \text{Ext}_{A*}^\cdot(K, K)$, where A^* is the graded dual of A (Milnor-Moore [10]). Moreover if A is a Hopf algebra, an injective A-comodule M is known to be free, that is $M = A \otimes_k \overline{M}$ with comultiplication $\psi_M = \psi_A \otimes 1_{\overline{M}}$ (Cartan [4], th. 9).

To calculate $\text{Ext}_B^\cdot(K, K)$ for an algebra B, it is sometimes convenient to use an injective resolution for the coalgebra B^* (for example, cobar construction).

In this paper we will use the method of Brown's twisted tensor product construction so as to construct injective resolutions. As an application of this method we can calculate directly the algebra $\text{Ext}_{\mathscr{A}_2}^\cdot(Z_2, Z_2)$ for the subalgebra $\mathscr{A}_2 = Z_2\{s_q^1, s_q^2, \ldots\ldots, s_q^7\}$ of the Steenrod algebra $\mathscr{A}(2)$, which is determined by 13 generators and 54 relations between them.

In the following, multiplications (denoted by φ) and comultiplications (denoted by ψ) are to be associative, and differential operators (denoted by δ) are of degree 1. Let L be a graded vector space over the ground field K of characteristic p. Let L^+ and L^- be the parts of even and odd degree respectively. Let $s: L \to sL$ and $\pi: L \to \pi L$ be the suspension map and p-th power operation respectively ([11], p. 190).

1. Twisted tensor product construction

The following theorem is owing to the associativity of ψ_A and φ_B ([3] and

[11], §1).

THEOREM (*Brown*). *Let A be a differential augmented coalgebra over K, B be a differential augmented algebra over K with augmentation ε. Let $\theta\colon A \to B$ be a K-linear map of degree 1 satisfying the following condition*

$$\varepsilon\theta=0, \qquad \delta_B\theta+\theta\delta_A+\theta\cup\theta=0$$

where $\quad \theta\cup\theta=\varphi_B(\theta\otimes\theta)\psi_A$.
Then $A\otimes B$ is a differential K-module with the differential operator

$$\delta_\theta=\delta_A\otimes1+1\otimes\delta_B+(1\otimes\varphi_B)(1\otimes\theta\otimes1)(\psi_A\otimes1).$$

THEOREM. *Let A be a differential augmented coalgebra over K with augmentation η, L be a K-submodule of $J(A)=\mathrm{Coker}\ \eta$, $\iota\colon L \to A$ be the injection, $\theta\colon A \to L$ be a map such that $\theta\iota=1_L$. Let $\bar{\theta}=s\theta\colon A \to sL$, $\bar{\iota}=\theta s^{-1}\colon sL \to A$, and $\bar{X}=T(sL)/I$ the quotient of the tensor algebra where I is the ideal generated by $\mathrm{Im}(\bar{\theta}\cup\bar{\theta})(1-\bar{\iota}\bar{\theta})=(\bar{\theta}\cup\bar{\theta})(\mathrm{Ker}\ \bar{\theta})$. Then $\bar{X}$ is a differential K-algebra with the differential operator such that $\bar{\delta}=-(\bar{\theta}\cup\bar{\theta})\bar{\iota}$ on $sL=\bar{X}^1$. The twisted tensor product $X=A\otimes\bar{X}$ for $\bar{\theta}$ is defined. X is a differential A-comodule and a quotient of the cobar construction $C(A)$ of A. $\bar{X}$ is a quotient of $\bar{C}(A)=T(sJ(A))$.*
Especially if we take $\theta=1-\varepsilon\colon A \to J(A)$, then $X=C(A)$.

Proof. A map $\bar{\delta}=-(\bar{\theta}\cup\bar{\theta})\bar{\iota}\colon sL \to T(sL)$ is uniquely extended to $\bar{\delta}\colon T(sL) \to T(sL)$ satisfying $\bar{\delta}\varphi=\varphi(\bar{\delta}\otimes1+1\otimes\bar{\delta})$. It follows that $\bar{\delta}I\subset I$ and which makes $\bar{X}$ a differential algebra.

By the fact that X is a quotient of $C(A)$, if X is acyclic and $\delta_A=0$, the product in $\bar{X}$ induces the cup product in $\mathrm{Ext}_{A*}(K,K)$.

2. Exterior algebra

Let $E(L)$ be the exterior algebra generated by L such that $L=L^-$ if $p\neq2$. It is a Hopf algebra with a comultiplication $\psi(a)=a\otimes1+1\otimes a$ for $a\in L$. Let $\theta\colon E(L) \to L$ be the canonical projection. Then by §1 we have a differential $E(L)$-comodule

$$X=E(L)\otimes P(sL)$$

where P denotes the polynomial algebra. The Explicit formula for δ_X is given by

$$\delta(a_1\ldots\ldots a_n x)=\sum_{i=1}^{n}(-1)^{i+1}a_1\ldots\ldots\hat{a}_i\ldots\ldots a_n(sa_i)x, \qquad a_i\in L,\ x\in P.$$

Acyclicity is easily verified. X is a minimal resolution.

3. Divided polynomial algebra

Let $\Gamma(L)$ be the divided polynomial algebra generated by L such that $L=L^+$. It is a Hopf algebra with a comultiplication $\psi(\gamma_k(a)) = \sum_{i+j=k} \gamma_i(a) \otimes \gamma_j(a)$ for $a \in L$. Let $\theta: \Gamma(L) \to L$ be the canonical projection. Then we have a differential $\Gamma(L)$-comodule

$$X = \Gamma(L) \otimes E(sL).$$

The Explicit formula for δ_X is given by

$$\delta(\gamma_{r_1}(a_1)\ldots\ldots\gamma_{r_n}(a_n)x) = \sum \gamma_{r_1}(a_1)\ldots\ldots\gamma_{r_i-1}(a_i)\ldots\ldots\gamma_{r_n}(a_n)(sa_i)x, \qquad a_i \in L, \, x \in E.$$

Acyclicity is easily verified. X is a minimal resolution.

4. Truncated polynomial algebra

Let $Q(L)$ be the truncated polynomial algebra generated by L such that $L = L^+$ and $p > 0$. It is a Hopf algebra with a comultiplication $\phi(a) = a \otimes 1 + 1 \otimes a$ for $a \in L$. If $p = 2$, then $Q(L) = E(L)$ which is studied in §2. Suppose $p > 2$. Let $\theta: Q(L) \to L$ be the canonical projection. Then we have a differential $Q(L)$-comodule

$$X = Q(L) \otimes E(sL).$$

It is not acyclic. However, X is also a Hopf algebra, we can again use the twisted tensor product construction. Take a homogeneous basis $\{a_\alpha\}$ of L, then $\{\Pi_\alpha a_\alpha^{r_\alpha} \Pi_\beta sa_\beta^{\varepsilon_\beta} | r_\alpha \geqq 0, \ \varepsilon_\beta = 0, 1\}$ is a basis of X. Let $M = \sum_\alpha Z_2 a_\alpha^{p-1} sa_\alpha$, $\omega: X \to M$ be the projection with respect to the above basis of X. Then we have a differential K-module

$$Y = Q(L) \otimes E(sL) \otimes P(s^2\pi L).$$

To prove Y being acyclic, we may suppose that L has the only one generator, when a contracting homotopy is easily constructed. Y is a minimal resolution. Y is also a quotient of the cobar construction $C(Q(L))$. Let L have the only one generator a. Define a map $\bar{f}: \bar{C}(Q(L)) \to \bar{Y}$ as follows

$$\bar{f}([a]^\varepsilon [a^{p-1} | a]^n) = (sa)^\varepsilon (s^2\pi a)^n, \quad \varepsilon = 0, 1, \quad n \geqq 0$$

$$\bar{f} = 0 \quad \text{for the other monomials of } \bar{C}(Q(L)),$$

then $\bar{f}$ is a morphism of differential K-module. Cocycles $[a]$,

$\sum_{i=1}^{p-1} \left[\dfrac{a^i}{i!} \Big| \dfrac{a^{p-1-i}}{(p-1-i)!} \right]$ are mapped to sa, $s^2\pi a$ respectively. $f=1\otimes\bar{f}: C(Q(L)) \to Y$ is a morphism of differential $Q(L)$-comodules. Therefore we have an isomorphism Cotor $^{Q(L)}(K, K) \cong E(sL)\otimes P(s^2\pi L)$ as algebras.

5. Polynomial algebra

Let $P(L)$ be a polynomial algebra generated by L such that $L=L^+$. If $p>0$, $P(L) \cong Q(\sum_{n\geq 0}\pi^n L)$ as coalgebras.

By §2 for $p=2$, §3 for $p=0$ and §4 for $p>2$, we have the following minimal resolutions

$$P(L)\otimes E(sL) \quad \text{for} \quad p=0$$

$$P(L)\otimes E(\textstyle\sum_{n\geq 0} s\pi^n L) \quad \text{for} \quad p=2,$$

$$P(L)\otimes E(\textstyle\sum_{n\geq 0} s\pi^n L)\otimes P(\textstyle\sum_{n\geq 0} s^2\pi^{n+1} L) \quad \text{for} \quad p>2.$$

6. Let A be a connected Hopf algebra of characteristic 2 such that the primitive elements are indecomposable. Such an algebra is isomorphic to $E(L)$ as algebras where $L=I(A)/I(A)^2$ (Milnor-Moore [10] 5.18 and 6.11). Let $\theta: A \to L$ be the canonical projection and $\iota: L \to E(L)=A$. Then we have a differential A-comodule

$$X=A\otimes P(sL).$$

Taking the augmentation filtrations, we observe that the associated graded module of A is isomorphic to $E(L)$ as Hopf algebras, and the associated graded module of X is the minimal resolution for $E(L)$ (§2).

Especially if A is the associated graded algebra of the dual of the Steenrod algebra $\mathscr{A}(2)$ with respects to the following filtration, the above consideration is available. For an integer $n=\sum n_r 2^r$, $n_r=0, 1$, set $|n|=\sum n_r$. By $|m+n|\leq |m|+|n|$, a filtration given by

$$F_r \mathscr{A}(2)^* = Z_2\{\xi_1^{r_1}\xi_2^{r_2}\ldots\ldots | \quad |r_1|+2|r_2|+3|r_3|+\ldots\ldots \leq n\}$$

is closed under the multiplication and comultiplication. Denoting a_i^j by the class of $\xi_i^{2^j}$ in $E^\circ \mathscr{A}(2)^*$, we have

$$E^\circ \mathscr{A}(2)^* = E(L), \quad L = Z_2\{a_i^j \,|\, i>0, j\geq 0\}.$$

Comultiplication ψ_A is given by $\psi(a_n^k)=\sum a_{n-i}^{i+k}\otimes a_i^k$. These are discussed by May [8].

7. Let $\mathscr{A}_n$ be the subalgebra of the Steenrod algebra $\mathscr{A}(2)$ generated by $s_q^1, \ldots\ldots, s_q^{2^n}$. It is a Hopf subalgebra of $\mathscr{A}(2)$. The dual Hopf algebra A_n of $\mathscr{A}_n$ is a quotient Hopf algebra

$$A_n = Z_2[\xi_1, \ldots\ldots, \xi_{n+1}]/(\xi_1^{2^{n+1}}, \xi_2^{2^n}, \ldots\ldots, \xi_{n+1}^2)$$

of $\mathscr{A}(2)^*$ (Steenrod [12]).

A minimal injective resolution for A_0 is given by $A_0 \otimes P(s\xi_1)$, and $H^\cdot(\mathscr{A}_0) = P(s\xi_1)$ (§2).

In A_1, let L be a submodule generated by the classes of ξ_1, ξ_1^2 and ξ_2, θ: $A_1 \to L$ be the projection with respect to monomial basis of A_1. Then we have a differential A_1-comodule $X = A_1 \otimes \bar{X}$, where $\bar{X}$ is a differential algebra with generators $a_1 = \bar{\theta}\xi_1$, $a_2 = \bar{\theta}\xi_1^2$ and $b = \bar{\theta}\xi_2$ satisfying the following relations

$$[a_1, a_2] = 0, \ [a_1, b] = a_2^2, \ [a_2, b] = 0$$
$$\delta a_1 = 0, \ \delta a_2 = 0, \ \delta b = a_2 a_1.$$

A basis of X is given by $\{\xi_1^{\varepsilon_1 + 2\varepsilon_2}\xi_2^{\varepsilon_3} a_1^{r_1} a_2^{r_2} b^{r_3} \mid \varepsilon_i = 0, 1, r_i \geqq 0\}$. Defining a filtration of X such that $F_n X = \{\xi_1^{\varepsilon_1 + 2\varepsilon_2}\xi_2^{\varepsilon_3} a_1^{r_1} a_2^{r_2} b^{r_3} \mid \varepsilon_1 + \varepsilon_2 + 2\varepsilon_3 + r_1 + r_2 + 2r_3 \leqq n\}$, we observe that the associated graded module of X is such a resolution as one in §6. This implies that X is acyclic. As b^2 belongs to the center of $\bar{X}$, and $\delta b^4 = 0$, explicit formula for $\delta_{\bar{X}}$ is given by

$$\delta(f_0 + f_1 b + f_2 b^2 + f_3 b^3) = a_2(a_1 f_1 + a_2^2 f_2) + a_2 f_3(a_2^2 b + a_1 b^2)$$

where f_0, f_1, f_2, f_3 are polynomials of a_1, a_2, b^4. Hence we have

$$\text{Ext}_{\mathscr{A}_1}(Z_2, Z_2) = Z_2[h_0, h_1, u_0, \omega_0]/(h_0 h_1, h_1^3, h_1 u_0, u_0^2 + a_1^2 \omega_0)$$

where h_0, h_1, u_0, ω_0 are cohomology classes of $a_1, a_2, a_2^2 b + a_1 b^2, b^4$ respectively ([6]).

8. Let L be a subspace of $A_2 = Z_2[\xi_1, \xi_2, \xi_3]/(\xi_1^8, \xi_2^4, \xi_3^2)$ generated by the classes of $\xi_1, \xi_1^2, \xi_1^2, \xi_1^3, \xi_1^4, ; \xi_2, \xi_2^2; \xi_3$.

Let $\theta : A_2 \to L$ be the projection with respect to the monomial basis. Then we have a differential A_2-comodule $X = A_3 \otimes \bar{X}$.

$\bar{X}$ is a differential Z_2-algebra generated by $a_1, a_2, a_3, a_4; b_1, b_2; c$ which are $\bar{\theta}$-images of the basic elements of L.

The following relations hold in $\bar{X}$.

$$[a_1, a_4]=0, \quad [a_1, b_1]=a_2^2+a_3a_1, \quad [a_1, b_2]=a_4a_3, \quad [a_1, c]=b_2a_2$$

$$[a_2, a_4]=0, \quad [a_2, b_1]=a_4a_1+a_2a_3, \quad [a_2, b_2]=a_4^2, \quad [a_2, c]=b_2a_3,$$

$$[a_3, a_4]=0, \quad [a_3, b_1]=a_3^2, \quad\quad [a_3, b_2]=0, \quad [a_3, c]=b_2a_4,$$

$$[a_4, b_1]=0, \quad\quad\quad\quad [a_4, b_2]=0, \quad [a_4, c]=0,$$

$$[b_1, b_2]=0, \quad [b_1, c]=a_4b_2,$$

$$[b_2, c]=0,$$

$$\delta a_1 = 0, \quad\quad\quad \delta b_1 = a_2 a_1, \quad\quad\quad \delta c = b_2 a_1 + a_4 b_1,$$

$$\delta a_2 = 0, \quad\quad\quad \delta b_2 = a_4 a_2,$$

$$\delta a_3 = [a_1, a_2],$$

$$\delta a_4 = 0.$$

A basis of $\bar{X}$ is given by

$$\{f a_4^k b_1^l b_2^m c^n\}$$

where f is a word consisting of the letters a_1, a_2, a_3, and k, l, m, n are non-negative integers. Taking a filtration of X such that

$$F_r X = Z_2 \{\xi_1^h \xi_2^{\varepsilon_1 + 2\varepsilon_2} \xi_3^{\varepsilon_3} f a_4^k b_1^l b_2^m c^n \mid \varepsilon_1 + \varepsilon_2 + 2\varepsilon_3 + l + m + 2n \leqq r\},$$

we see that the asociated graded module $E^\circ X$ is isomorphic to the tensor product of the cobar construction of an exterior algebra with two generators and the minimal injective resolution for an exterior algebra with four generators. This implies the acyclicity of X.

We come to calculate $\mathrm{Ext}\,\dot{\mathscr{A}}_2(Z_2, Z_2)$. Let Y be the subalgebra of $\bar{X}$ generated by $a_1, a_2, a_3, a_4, b_1, b_2$ and let I be the ideal of Y generated by $[a_1, a_2]$ and a_3. Both Y and I are differentiable, hence $Y'=Y/I$ is a differential algebra. $X'=Y'\otimes_Y X$ is isomorphic to $Y'\otimes Z_2[c]$ as differential Y'-modules. It is not differential algebra but it contains a differential algebra $Y'\otimes Z_2[c^2]$. X' is differential module over this algebra. Moreover the submodule $I\otimes\bar{X}=I\cdot\bar{X}$ of $\bar{X}$ is acyclic. A contracting homotopy is easily constructed. Hence we have $H^\cdot(\bar{X}')=H^\cdot(\bar{X})=\mathrm{Ext}\,\dot{\mathscr{A}}_2(Z_2, Z_2)$.

Let Z' be a subalgebra of Y' generated by a_1, a_2, a_4, b_2. Then we have $H^\cdot(Z')=Z_2[h_0, h_1, h_2, u_1, \omega_1]/(h_1h_2, h_2^3, h_2u_1, u_1^2+h_1^2\omega_1)$ by §7 where $h_0, h_1, h_2, u_1, \omega_1$

are the classes of $a_1, a_2, a_4, a_2b_2^2+a_4^2b_2, b_2^4$ respectively.

Considering Y' as a free (left or right) Z'-module, we can determine $H^\cdot(Y')$. It is a quotient of $Z_2[h_0, h_1, h_2, u_1, v, \omega_0, \omega_1]$ by the ideal generated by $h_0h_1, h_1h_2,$ $h_0^2h_2+h_1^3, h_2^3, h_1v+h_0h_2^2, h_2^2v+h_0u_1, h_2v^2+h_1^2u_1, v^4+h_0^4\omega_1, u_1^2+h_1^2\omega_1, h_2u_1, vu_1$ where v, ω_0 are the classes of $a_1b_2+a_4b_1, b_1^4+a_2^3b_2$ respectively. Considering X' as a free Y'-module, we can determine $H^\cdot(X')$. The cup products in $H^\cdot(X')$ are induced by the multiplication of $\bar{X}$.

Ext $\mathscr{A}_2(Z_2, Z_2)$ is generated by the following cohomology classes.

Class in Ext	Bidegree	Representation in X
h_0	$(1, 1)$	a_1
h_1	$(1, 2)$	a_2
h_2	$(1, 4)$	a_4
ω_0	$(4, 12)$	$b_1^4+a_2^3b_2+[a_1, a_3]a_4^2+(a_2a_3^2+a_3a_2a_3+a_3^2a_2)a_4$
ω_1	$(4, 24)$	b_2^4
α_0	$(8, 56)$	c^8
α_1	$(3, 11)$	$a_2^2c+a_2b_1b_2+a_4^2b_1$
α_2	$(3, 15)$	$a_1c^2+a_4^2c+(b_1+a_3)b_2^2$
α_3	$(3, 18)$	$a_4c^2+b_2^3$
α_4	$(4, 18)$	$a_2^2c^2+a_4(b_2a_1+a_4b_1)c+b_1^2b_2^2$
α_5	$(4, 21)$	$(a_2b_2^2+a_4^2b_2)c+b_1b_2^3$
α_6	$(5, 30)$	$a_2c^4+b_2^5$
α_7	$(7, 39)$	$a_2^2c^5+(a_2b_1b_2+a_4^2b_1)c^4+a_2b_2^5c+b_1b_2^6+b_2^4\alpha_2$

The following relations hold, and all the relations in Ext $\dot{\mathscr{A}_2}(Z_2, Z_2)$ are induced by them.

$h_0h_1=0, h_1h_2=0, h_0^2h_2=h_1^3, h_0h_2^2=0, h_2^3=0, h_2\omega_1=0,$

$h_0\alpha_1=0, h_1^2\alpha_1=0, h_2\alpha_1=0, \omega_1\alpha_1=0,$

$h_1\alpha_2 = 0,$

$h_0\alpha_3 = h_2\alpha_2,\ h_1\alpha_3 = 0,\ h_2^2\alpha_3 = h_1\omega_1,$

$h_0^2\alpha_4 = h_2^2\omega_0,\ h_1\alpha_4 = h_0h_2\alpha_2,$

$h_0\alpha_5 = h_2\alpha_4,\ h_1\alpha_5 = h_2^2\alpha_2,\ h_2\alpha_5 = h_0\omega_1,$

$h_0\alpha_6 = 0,\ h_2\alpha_6 = 0,$

$h_0\alpha_7 = h_0\omega_1\alpha_2,\ h_1^2\alpha_7 = h_0\omega_1\alpha_4,\ h_2\alpha_7 = 0,\ \omega_1\alpha_7 = 0,$

$\alpha_1^2 = 0,\ \alpha_1\alpha_2 = h_0^2\omega_1,\ \alpha_1\alpha_3 = 0,\ \alpha_1\alpha_4 = 0,\ \alpha_1\alpha_5 = 0,\ \alpha_1\alpha_6 = h_1\alpha_7,\ \alpha_1\alpha_7 = 0,$

$\alpha_2\alpha_6 = \omega_1\alpha_5,\ \alpha_2\alpha_7 = 0,$

$\alpha_3\alpha_4 = \alpha_2\alpha_5,\ \alpha_3\alpha_5 = \omega_1\alpha_2,\ \alpha_3\alpha_6 = \omega_1^2,\ \alpha_3\alpha_7 = 0,$

$\alpha_4^2 = \omega_0\omega_1,\ \alpha_4\alpha_7 = 0,$

$\alpha_5^2 = \omega_1\alpha_4,\ \alpha_5\alpha_7 = 0,$

$\alpha_6^2 = h_1^2\alpha_0 + \omega_1\alpha_3^2,\ \alpha_6\alpha_7 = h_1\alpha_0\alpha_1,$

$\alpha_7^2 = 0$

$\alpha_2^2\alpha_3 = \alpha_4\alpha_6,\ \alpha_2\alpha_3^2 = \alpha_5\alpha_6,\ \alpha_2^2\alpha_4 = \omega_0\alpha_3^2,\ \alpha_3^3 = \omega_1\alpha_6,$

$\alpha_2^4 = h_0^4\alpha_0 + \omega_0\omega_1^2,$

$h_2\alpha_2^2 = h_1^2\alpha_6,\ h_2\alpha_2\alpha_3 = 0,\ h_0\alpha_2\alpha_4 = h_2\omega_0\alpha_3,\ h_2\alpha_3^2 = 0.$

These relations imply that $\mathrm{Ext}\,\dot{\mathscr{A}}_2(Z_2, Z_2)$ is a free $Z_2[\omega_0, \alpha_0]$-module.

Remark. If we denote the Massey product by $<\ ,\ ,\ >, <\ ,\ ,\ ,\ >$, we have the following expression (Massey [7]).

$$\omega_1 = <h_2, h_2^2, h_2^2, h_2>,$$

$$\alpha_0 = <h_2, \omega_1, h_2, \omega_1>,$$

$$\alpha_1 = <h_1, h_0, h_2^2>,$$

$$\alpha_2 = <h_0, h_1, h_2, h_2^2>,$$

$$\alpha_3 = <h_1, h_2, h_2^2, h_2>,$$

$$\alpha_4 = <h_0, h_1, \alpha_2> = <h_0, h_2^2, h_2^2, h_1>,$$

$$\alpha_5 = < h_2, h_1, \alpha_2 > = < h_0, h_1, \alpha_3 > = < h_0, h_2^2, h_2, h_2^2 >,$$

$$\alpha_6 = < h_1, h_2, \omega_1 >$$
$$\alpha_7 = < \omega_1, h_2, \alpha_1 >.$$

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