NORMAL RADICALS AND NORMAL CLASSES OF MODULES

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The study of special radicals was begun by Andrunakievič [1]. A class $\mathcal{P}$ of prime rings is called special if it is hereditary and closed under prime extensions. The upper radicals determined by special classes are called special. In later works Andrunakievič and Rjabuhin [2] and [3] defined the concept of a special class of modules.

A left R-module M is called *prime* if $RM \neq 0$ and every non-zero submodule has the same annihilator as M (equivalently, if $Im = 0$, where I is an ideal of R and $m \in M$, then either $m = 0$ or $IM = 0$). Let $\mathcal{S}(R)$ be a class of prime R-modules and $\mathcal{S} = \bigcup \mathcal{S}(R)$, the union being over all rings R. Then $\mathcal{S}$ is called *special* if it satisfies the following conditions:

(S.1) for every ring R, R-module M, and ideal I of R with $I \subseteq (0:M)$, $M \in \mathcal{S}(R)$ if and only if $M \in \mathcal{S}(R/I)$;

(S.2) if $M \in \mathcal{S}(R)$ and I is an ideal of R with $IM \neq 0$ then $M \in \mathcal{S}(I)$;

(S.3) if I is an ideal of R and $M \in \mathcal{S}(I)$ then $IM \in \mathcal{S}(R)$.

If $\mathcal{S}$ is a special class of modules then

$$\mathcal{P} = \{R : R \text{ has a faithful module in } \mathcal{S}(R)\}$$

is a special class of prime rings. Conversely, if $\mathcal{P}$ is a special class of rings and we set

$$\mathcal{S}(R) = \{{}_RM : M \text{ is a prime } R\text{-module and } R/(0:M) \in \mathcal{P}\}$$

then $\mathcal{S} = \bigcup \mathcal{S}(R)$ is a special class of modules.

The notion of a normal class of prime rings was defined in [5], where it was also shown that every such class is special and that a radical is normal and special if and only if it is the upper radical determined by a normal class. In this note we introduce the idea of a normal class of modules and prove that every normal class of prime rings is determined as above by a normal class of modules. It is also proved that such module classes are special and we note that the classes of prime modules, irreducible modules, and prime modules with non-zero socle are normal.

Normal classes of rings arise in studying rings connected in a Morita context. This is a four-tuple $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$, where R and S are rings and ${}_RV_S$ and ${}_SW_R$ are bimodules, together with bimodule homomorphisms $V \otimes_S W \to {}_RR_R$, $W \otimes_R V \to {}_SS_S$ satisfying associativity conditions which are equivalent to insisting that $C = \begin{bmatrix} R & V \\ W & S \end{bmatrix}$ be an associative ring under the usual matrix operations. We shall refer to C as the *context ring*. The context is called *S-faithful* if $S \neq 0$ and $VsW \neq 0$ for all non-zero $s \in S$. If P is an ideal of R then we denote $\{s \in S : VsW \subseteq P\}$ by S_P.

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PROPOSITION 1 [5]. *The following are equivalent for a class $\mathcal{P}$ of rings.*

(a) *If* $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ *is a Morita context and P is an ideal of R such that* $R/P \in \mathcal{P}$ *then either* $S_P = S$ *or* $S/S_P \in \mathcal{P}$.

(b) *If* $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ *is a Morita context and* $R \in \mathcal{P}$ *then either* $S_0 = S$ *or* $S/S_0 \in \mathcal{P}$.

(c) *If* $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ *is an S-faithful Morita context, then* $R \in \mathcal{P}$ *implies* $S \in \mathcal{P}$.

A class $\mathcal{P}$ of prime rings is called *normal* if it satisfies the conditions of Proposition 1.

DEFINITION 1. Let $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ be a Morita context. Then a *context module* is a pair of modules ${}_RM$, ${}_SN$ with module homomorphisms $\alpha: V \otimes_S N \to {}_RM$, $\beta: W \otimes_R M \to {}_SN$ satisfying associativity conditions so that $D = \begin{bmatrix} M \\ N \end{bmatrix}$ is a C-module for the context ring C under the usual matrix operations.

Given a context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ and an R-module M we can construct an S-module $M°$ so that $D = \begin{bmatrix} M \\ M° \end{bmatrix}$ is a context module. The construction appeared in [4].

For every $v \in V$ there is a $\mathbb{Z}$-morphism $v\cdot: W \otimes_R M \to M$ defined by $v\cdot(w \otimes m) = (vw)m$ for all $w \in W$, $m \in M$. Put $X = \bigcap_{v \in V} \ker(v\cdot)$ and $M° = (W \otimes M)/X$. Then $M°$ is an S-module. The map β of Definition 1 is given by $\beta: w \otimes m \to (w \otimes m) + X$ and we write this image as wm. The map α of Definition 1 is given by $\alpha: v \otimes n \to v\cdot t$, where $t \in W \otimes M$ and $n = t + X$. This image is written as vn. Thus $\begin{bmatrix} M \\ M° \end{bmatrix}$ is a C-module.

Some properties of this module are worth identifying here. A number of module properties are known to pass from M to $M°$ (see [4]). In particular if M is faithful then $M°$ is faithful. Also from the construction we have:

(a) if $n \in M°$ and $Vn = 0$ then $n = 0$;
(b) $M° = WM$;
(c) if M is faithful and $VSW \neq 0$ then $SM° \neq 0$.

For (a), note that if $n = t + X$, $t \in W \otimes M$ then $v\cdot t = 0$ for all $v \in V$; so $t \in X$ and $n = 0$; (b) is clear from the definition of $M°$ and wm, and (c) follows from $(VS)M° = (VSW)M$.

DEFINITION 2. Let $\mathcal{N}(R)$ be a class of prime R-modules and $\mathcal{N} = \bigcup \mathcal{N}(R)$, the union being over all rings R. Then $\mathcal{N}$ is called *normal* if it satisfies (S.1) and

(N) for every context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ and context module $D = \begin{bmatrix} M \\ N \end{bmatrix}$ such that

(i) for all $n \in N$, $Vn = 0$ implies $n = 0$, and
(ii) $N = WM$ and $SN \neq 0$, $M \in \mathcal{N}(R)$ implies $N \in \mathcal{N}(S)$.

THEOREM. *Let $\mathcal{N}$ be a normal class of modules. Then*

$$\mathcal{P} = \{R : R \text{ has a faithful module in } \mathcal{N}(R)\}$$

is a normal class of prime rings. Conversely, if $\mathcal{P}$ is a normal class of prime rings and we define, for every ring R,

$$\mathcal{N}(R) = \{{}_R M : M \text{ is a prime } R\text{-module and } R/(0:M) \in \mathcal{P}\}$$

then $\mathcal{N} = \cup \mathcal{N}(R)$ is a normal class of modules.

Proof. If $\mathcal{N}$ is a normal class of modules and $M \in \mathcal{N}(R)$ is a faithful R-module, then $(0:M) = 0$ is a prime ideal of R and $\mathcal{P}$, as defined, is a class of prime rings. From the comments after Definition 1, if $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ is an S-faithful Morita context then the context module $\begin{bmatrix} M \\ M^\circ \end{bmatrix}$ satisfies (i) and (ii) of (N) and M° is faithful. Hence $M^\circ \in \mathcal{N}(S)$, $S \in \mathcal{P}$, and $\mathcal{P}$ is a normal class of prime rings.

Now let $\mathcal{P}$ and $\mathcal{N}$ be as in the statement of the converse. For (S.1), suppose that M is an R-module and I is an ideal of R with $I \subseteq (0:M)$. Put $\bar{R} = R/I$. Note that $(0:M)_{\bar{R}} = (0:M)/I$ and if $r \in R$ and $\bar{r} = r + I$ then $\bar{r}m = rm$ for all $m \in M$. Thus $\bar{R}/(0:M)_{\bar{R}} \cong R/(0:M)$ and M is a prime R-module if and only if it is a prime $\bar{R}$-module. Therefore $M \in \mathcal{N}(R)$ if and only if $M \in \mathcal{N}(\bar{R})$.

For (N), suppose that the context and context module are as described in Definition 2 and that $M \in \mathcal{N}(R)$. To see that N is a prime S-module, let J be an ideal of S and $n \in N$ with $Jn = 0$. Then $(VJW)(Vn) = 0$; so either $(VJW)M = 0$ or $Vn = 0$, since M is a prime R-module. If $Vn = 0$ then $n = 0$ from (i). If $(VJW)M = 0$ then $VJN = 0$ from (ii) and $JN = 0$ from (i). Thus N is a prime S-module. To prove that $S/(0:N) \in \mathcal{P}$, observe that $(0:N) = \{s \in S : VsW \subseteq (0:M)\}$ from (i) and (ii) in (N). From Proposition 1 (a), either $(0:N) = S$ or $S/(0:N) \in \mathcal{P}$. Since $SN \neq 0$, by hypothesis, it follows that $S/(0:N) \in \mathcal{P}$.

PROPOSITION 2. *Every normal class of modules is special.*

Proof. Let $\mathcal{N}$ be a normal class of modules. Let $M \in \mathcal{N}(R)$ and I be an ideal of R with $IM \neq 0$. Consider the context $\begin{bmatrix} R & I \\ R^1 & I \end{bmatrix}$ and context module $D = \begin{bmatrix} M \\ M \end{bmatrix}$. If $Im = 0$, $m \in M$, then $m = 0$ since M is a prime R-module and $IM \neq 0$. Therefore the conditions (N) (i) and (N) (ii) from Definition 2 are satisfied and so $M \in \mathcal{N}(I)$. This establishes (S.2).

For (S.3), let I be an ideal of a ring R and $M \in \mathcal{N}(I)$. Consider the context $\begin{bmatrix} I & R^1 \\ I & R \end{bmatrix}$ and context module $\begin{bmatrix} M \\ IM \end{bmatrix}$. Since M is a prime I-module, $IM \neq 0$ and

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$I(IM)\neq 0$. Hence $R(IM)\neq 0$ and the conditions (N) (i) and (N) (ii) of Definition 2 are satisfied; so that $IM \in \mathcal{N}(R)$.

EXAMPLES. The three classes we shall consider here were shown to be special in [**3**]. Thus (S.1) is satisfied. We shall use the notation of (N).

1. The class of all prime modules is normal. Let M be a prime R-module, J an ideal of S, and $n \in N$. As in the proof of the Theorem, $Jn = 0$ implies $JN = 0$ or $n = 0$; so N is a prime S-module.

2. The class of all irreducible modules is normal. Let M be an irreducible R-module and $n \in N$, $n \neq 0$. Then $0 \neq Vn$; so $Vn = M$ and $Sn \supset WVn = WM = N$. Thus N is an irreducible S-module.

3. The class of all prime modules with non-zero socle is normal. Let M be a prime R-module with minimal submodule K. If $(VW)K = 0$ then $(VW)M = VN = 0$, which implies that $N = 0$. But $SN \neq 0$; so $(VW)K \neq 0$. Hence $WK \neq 0$. Let $n \neq 0$, $n \in WK$. Then $0 \neq Vn \subseteq K$ and, by the minimality of K, $Vn = K$. Therefore $Sn \supseteq (WV)n = WK$ and WK is a minimal submodule of N. Along with Example 1 this proves the normality of this class.

REMARK. It was shown in [**3**] that the class $\mathcal{P}$ determined by the module classes in Examples 2 and 3 is the class of primitive rings.

REFERENCES

1. V. A. Andrunakievič, Radicals of associative rings. II. Examples of special radicals, *Mat. Sb.* (*N.S.*) **55** (**97**) (1961), 329–346; *Amer. Math. Soc. Transl.* (2) **52** (1966), 129–150.

2. V. A. Andrunakievič and Ju. M. Rjabuhin, Special modules and special radicals, *Dokl. Akad. Nauk SSSR* **147** (1962), 1274–1277.

3. V. A. Andrunakievič and Ju. M. Rjabuhin, Special modules and special radicals, *In Memoriam*: *N. G. Čebotarev*, 7–17, *Izdat. Kazan. Univ.*, *Kazan*, 1964.

4. G. Desale and W. K. Nicholson, Endoprimitive rings, *J. Algebra* **70** (1981), 548–560.

5. W. K. Nicholson and J. F. Watters, Normal radicals and normal classes of rings, *J. Algebra* **59** (1979), 5–15.

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