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ON THE SOLUTION SETS TO DIFFERENTIAL INCLUSIONS ON AN UNBOUNDED INTERVAL

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Dedicated to the memory of Aristide Halanay

Abstract We prove that for $F : [0,\infty) \times \mathbb{R}^n \to \mathcal{K}(\mathbb{R}^n)$ a Lipschitzian multifunction with compact values, the set of derivatives of solutions of the Cauchy problem

$$x' \in F(t,x), x(0) = \xi,$$

is a retract of $L^1_{\text{loc}}([0,\infty), \mathbb{R}^n)$.

Keywords: Lipschitzian differential inclusion; solution set; derivatives of solutions; absolute retract

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1. Introduction and the main result

Let $\mathcal{S}_F(\xi)$ be the set of solutions of the Cauchy problem

$$x' \in F(t,x), \quad x(0) = \xi,$$

where $F : [0,T] \times \mathbb{R}^n \to \mathcal{K}(\mathbb{R}^n)$ is a compact-valued multifunction, Lipschitzian with respect to x, $\xi \in \mathbb{R}^n$, and let

$$\mathcal{S}'_F(\xi) = \{x' : x \in \mathcal{S}_F(\xi)\}$$

be the set of derivatives of solutions.

Bressan *et al.* [**2**] proved that the set of fixed points of a multivalued contraction on $L^1([0,T], \mathbb{R}^n)$ is an absolute retract (for the case when the multivalued contraction has convex values, such a result was obtained by Ricceri [**12**]), and using this they established that $\mathcal{S}'_F(\xi)$ is a retract of the space $L^1([0,T], \mathbb{R}^n)$. As a consequence, one has that the solution set $\mathcal{S}_F(\xi)$ turns out to be an absolute retract [**7**].

A different approach based on the Baire category was used by De Blasi and Pianigiani in [**6**] to prove the contractibility of the set $\mathcal{S}_{\text{ext}\,F}(\xi)$, where ext F is the set of extreme points of a Lipschitzian, closed convex-valued multifunction F. Other topological properties of the solution sets were obtained by many authors, and we refer among others to [**2**, **4**, **8**, **10**, **11**, **14**].

Let consider the Cauchy problem

$$x' \in F(t,x), x(0) = \xi, \tag{P_ξ}$$

where $F : [0,\infty) \times \mathbb{R}^n \to \mathcal{K}(\mathbb{R}^n)$ is a compact-valued multifunction satisfying the following assumptions.

(H_1) F is $\mathcal{L} \otimes \mathcal{B}(\mathbb{R}^n)$-measurable.

(H_2) There exists $l \in L^1_{\mathrm{loc}}([0,\infty),(0,\infty))$ such that, for any x, $y \in \mathbb{R}^n$,

$$d_H(F(t,x),F(t,y)) \leqslant l(t)\|x-y\|, \quad \text{a.e. } t \in [0,\infty).$$

(H_3) There exists $\beta \in L^1_{\mathrm{loc}}([0,\infty),\mathbb{R})$ such that

$$d_H(\{0\},F(t,0)) \leqslant \beta(t), \quad \text{a.e. } t \in [0,\infty).$$

It was recently proved in [**13**] that under the assumptions (H_1)–(H_3) the set $\mathcal{S}_F(\xi)$ of all solutions of the Cauchy problem (P_ξ) is arcwise connected in the space of continuous functions $x : [0,\infty) \to \mathbb{R}^n$ with derivative $x' \in L^1_{\mathrm{loc}}([0,\infty),\mathbb{R}^n)$ endowed with the distance

$$d(x,y) = \|x(0)-y(0)\| + \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\displaystyle\int_0^n \|x'(t)-y'(t)\|\,\mathrm{d}t}{1+\displaystyle\int_0^n \|x'(t)-y'(t)\|\,\mathrm{d}t}.$$

Let

$$\mathcal{S}'_F(\xi) = \left\{u \in L^1_{\mathrm{loc}}([0,\infty),\mathbb{R}^n) : u(t) \in F\left(t, \xi + \int_0^t u(s)\,\mathrm{d}s\right), \text{ a.e. } t \in [0,\infty)\right\}$$

be the set of derivatives of solutions of the problem (P_ξ).

The aim of this paper is to establish a more general topological property of the solution set $\mathcal{S}_F(\xi)$, namely the following theorem.

Theorem 1.1. *If $F : [0,\infty) \times \mathbb{R}^n \to \mathcal{K}(\mathbb{R}^n)$ is a compact-valued multifunction satisfying* (H_1)–(H_3) *and $\xi \in \mathbb{R}^n$, then there exists a continuous map $H : L^1_{\mathrm{loc}}([0,\infty),\mathbb{R}^n) \to L^1_{\mathrm{loc}}([0,\infty),\mathbb{R}^n)$, such that*

(i) *$H(u) \in \mathcal{S}'_F(\xi)$, for all $u \in L^1_{\mathrm{loc}}([0,\infty),\mathbb{R}^n)$;*

(ii) *$H(u) = u$, whenever $u \in \mathcal{S}'_F(\xi)$.*

2. Preliminaries

Let $\mathbb{R}^n$ be a real n-dimensional Euclidean space with norm $\|\cdot\|$. Denote by $\mathcal{K}(\mathbb{R}^n)$ the family of all compact non-empty subsets of $\mathbb{R}^n$ with the Hausdorff–Pompeiu distance $d_H(\cdot,\cdot)$ defined by

$$d_H(A,B) = \max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\ \sup_{b\in B}\inf_{a\in A}\|a-b\|\}.$$

Let $\mathcal{B}(\mathbb{R}^n)$ be the family of Borel subsets of $\mathbb{R}^n$ and $\mathcal{L}$ be the σ-algebra of Lebesgue measurable subsets of $[0,\infty)$. We denote by $\mathcal{L}\otimes\mathcal{B}(\mathbb{R}^n)$ the product σ-algebra on $[0,\infty)\times\mathbb{R}^n$, generated by the sets $A\times B$, where $A\in\mathcal{L}$ and $B\in\mathcal{B}(\mathbb{R}^n)$.

For every $k\geqslant 1$ we denote by I_k the interval $[0,k]$ and by $L^1(I_k,\mathbb{R}^n)$ the space of integrable functions $u: I_k\to\mathbb{R}^n$ with the norm

$$\|u\|_{1,k} = \int_0^k \|u(t)\|\,\mathrm{d}t. \tag{2.1}$$

As usual, $L^1_{\text{loc}}([0,\infty),\mathbb{R}^n)$ denotes the space of locally integrable functions $u:[0,\infty)\to\mathbb{R}^n$, whose topology is generated by the family of seminorms $\{p_k : k\geqslant 1\}$, where

$$p_k(u) = \|u|_{I_k}\|_{1,k} = \int_0^k \|u(t)\|\,\mathrm{d}t.$$

A subset $K\subset L^1(I_k,\mathbb{R}^n)$ is called decomposable (see [**9**]) if for any $u,v\in K$ and any Lebesgue measurable subset $A\subset I_k$,

$$u\chi_A + v\chi_{I_k\setminus A}\in K,$$

where χ_A is the characteristic function of A. Denote by $\mathcal{D}(L^1(I_k,\mathbb{R}^n))$ the family of all closed and decomposable subsets of $L^1(I_k,\mathbb{R}^n)$.

Let S be a separable metric space, X be a separable Banach space and let $\mathcal{C}(X)$ be the family of all closed non-empty subsets of X. Let $\mathcal{A}$ be a σ-algebra of subsets of S.

A multifunction $\Phi : S\to\mathcal{C}(X)$ is said to be lower semicontinuous if the set $\{s\in S : \Phi(s)\subset C\}$ is closed in S for any closed subset $C\subset X$.

We say that $\Phi : S\to\mathcal{C}(X)$ is $\mathcal{A}$-measurable if $\{s\in S : \Phi(s)\cap C\neq\emptyset\}\in\mathcal{A}$ for any closed subset $C\subset X$.

By selection from the multifunction $\Phi : S\to\mathcal{C}(X)$ we mean any function $\varphi : S\to X$ such that $\varphi(s)\in\Phi(s)$ for all $s\in S$.

The following lemma follows from Proposition 2.1 in [**5**].

Lemma 2.1. *Let S be a separable metric space and $F^* : I_k\times S\to\mathcal{C}(\mathbb{R}^n)$ be a $\mathcal{L}\otimes\mathcal{B}(S)$-measurable multifunction such that $s\mapsto F^*(t,s)$ is lower semicontinuous. Then the multifunction $s\mapsto G_{F^*}(s)$, defined by*

$$G_F(s) = \{v\in L^1(I_k,\mathbb{R}^n) : v(t)\in F^*(t,s),\ \text{a.e. } t\in I_k\},$$

is lower semicontinuous from S into $\mathcal{D}(L^1(I,\mathbb{R}^n))$ if and only if there exists a continuous map $\beta : S\to L^1(I_k,\mathbb{R})$ such that

$$d(0,F^*(t,s))\leqslant\beta(s)(t),\ \text{a.e. in } I_k.$$

Theorem 3 and Proposition 4 in [3] imply the following lemma.

Lemma 2.2. *If $\Phi : S \to \mathcal{D}(L^1(I_k, \mathbb{R}^n))$ is a lower continuous multifunction with closed, decomposable and non-empty values, $\varphi : S \to L^1(I_k, \mathbb{R}^n)$ and $\psi : S \to L^1(I_k, \mathbb{R})$ are continuous maps, and if, for every $s \in S$, the set*

$$H(s) = \mathrm{cl}\{v \in \Phi(s) : \|v(t) - \varphi(s)(t)\| < \psi(s)(t), \text{ a.e. } t \in I_k\}$$

is non-empty, then the multifunction $s \mapsto H(s)$ is lower semicontinuous, and, consequently, it admits a continuous selection (cl stands for closure).

Now let $F : [0, \infty) \times \mathbb{R}^n \to \mathcal{K}(\mathbb{R}^n)$ satisfy (H_1)–(H_3) and $\xi \in \mathbb{R}^n$ be given. For every $k \geqslant 1$ and for $u \in L^1(I_k, \mathbb{R}^n)$ define

$$\hat{u}(t) = \xi + \int_0^t u(s)\,\mathrm{d}s, \quad t \in I_k, \tag{2.2}$$

and

$$\beta_0(u)(t) = \|u(t)\| + \beta(t) + l(t)\|\hat{u}(t)\|, \quad t \in I_k, \tag{2.3}$$

where the functions $l, \beta \in L^1_{\mathrm{loc}}([0, \infty), \mathbb{R}^n)$ are given by (H_2) and (H_3). Since, for any $u_1, u_2 \in L^1(I_k, \mathbb{R}^n)$,

$$\|\beta_0(u_1) - \beta_0(u_2)\|_{1,k} \leqslant (1 + \|l|_{I_k}\|_{1,k})\|u_1 - u_2\|_{1,k},$$

it follows that $\beta_0 : L^1(I_k, \mathbb{R}^n) \to L^1(I_k, \mathbb{R})$ is continuous, for any $k \geqslant 1$.

Moreover, by (H_2) and (H_3) we obtain that for any $k \in \mathbb{N}$ and any $u \in L^1(I_k, \mathbb{R}^n)$:

$$d(u(t), F(t, \hat{u}(t))) \leqslant \beta_0(u)(t), \quad \text{a.e. } t \in I_k. \tag{2.4}$$

Denote

$$\mathcal{S}'_{F,I_k}(\xi) = \{u \in L^1(I_k, \mathbb{R}^n) : u(t) \in F(t, \hat{u}(t)), \text{ a.e. } t \in I_k\}.$$

Then we have the following proposition.

Proposition 2.3. *If $\varphi : L^1(I_k, \mathbb{R}^n) \to L^1(I_k, \mathbb{R}^n)$ is a continuous map such that $\varphi(u) = u$ for any $u \in \mathcal{S}'_{F,I_k}(\xi)$, then the multifunction $u \mapsto \Phi^k(u)$ defined by*

$$\Phi^k(u) = \begin{cases} \Psi^k(u), & \text{if } u \notin \mathcal{S}'_{F,I_k}(\xi), \\ \{u\}, & \text{if } u \in \mathcal{S}'_{F,I_k}(\xi), \end{cases}$$

where

$$\Psi^k(u) = \{v \in L^1(I_1, \mathbb{R}^n) : v(t) \in F(t, \widehat{\varphi(u)}(t)), \text{ a.e. } t \in I_k\}$$

is lower semicontinuous with closed decomposable and non-empty values.

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Proof. Let $C \subset L^1(I_k, \mathbb{R}^n)$ be a closed subset and let $(u_n)_{n\in\mathbb{N}}$ converge to some u_0 in $L^1(I_k, \mathbb{R}^n)$ and $\Phi^k(u_n) \subset C$ for any $n \in \mathbb{N}$. Let $v_0 \in \Phi^k(u_0)$ and for every $n \in \mathbb{N}$ consider a measurable selection v_n from the measurable multifunction $t \mapsto F(t, \widehat{\varphi(u_n)}(t))$ such that: $v_n = u_n$ if $u_n \in \mathcal{S}'_{F,I_k}(\xi)$, and

$$\|v_n(t) - v_0(t)\| = d(v_0(t), F(t, \widehat{\varphi(u_n)}(t))), \quad \text{a.e. } t \in I_k$$

if $u_n \notin \mathcal{S}'_{F,I_k}(\xi)$. In both cases,

$$\begin{aligned}\|v_n(t) - v_0(t)\| &\leqslant d_H(F(t, \widehat{\varphi(u_n)}(t)), F(t, \widehat{\varphi(u_0)}(t))) \\ &\leqslant l(t)\|\widehat{\varphi(u_n)}(t) - \widehat{\varphi(u_0)}(t)\|,\end{aligned}$$

which implies

$$\|v_n - v_0\|_{1,k} \leqslant \|l|_{I_k}\|_{1,k}\|\widehat{\varphi(u_n)} - \widehat{\varphi(u_0)}\|_{1,k}.$$

Then, by the continuity of $\varphi : L^1(I_k, \mathbb{R}^n) \to L^1(I_k, \mathbb{R}^n)$, we obtain that $(v_n)_{n\in\mathbb{N}}$ converges to v_0 in $L^1(I_k, \mathbb{R}^n)$. Since $v_n \in \Phi^k(u_n) \subset C$, $\forall n \in \mathbb{N}$, and since C is closed we get $v_0 \in C$. Therefore, $\Phi^k(u_0) \subset C$ and the lower semicontinuity of Φ^k is proved.

On the other hand, the inequality (2.4), the continuity of β_0, and Lemma 2.1 imply that Ψ^k has closed, decomposable and non-empty values, and the same holds for the multifunction Φ^k. □

3. Proof of the main result

We shall prove that for every integer $k \geqslant 1$, there is a continuous map $h^k : L^1(I_k, \mathbb{R}^n) \to L^1(I_k, \mathbb{R}^n)$ with the following properties:

(P_1) $h^k(u) = u$ whenever $u \in \mathcal{S}'_{F,I_k}(\xi)$;

(P_2) $h^k(u) \in \mathcal{S}'_{F,I_k}(\xi)$ for every $u \in L^1(I_k, \mathbb{R}^n)$;

(P_3) $h^k(u)(t) = h^{k-1}(u|_{I_{k-1}})(t)$, for $t \in I_{k-1}$.

Fix $\varepsilon > 0$ and for $n \geqslant 0$ set

$$\varepsilon_n = (n + 1/n + 2)\varepsilon.$$

For $u \in L^1(I_1, \mathbb{R}^n)$ and $n \geqslant 0$ define

$$\beta_0^1(u)(t) = \|u(t)\| + \beta(t) + l(t)\|\hat{u}(t)\|, \quad t \in I_1,$$

and

$$\delta_{n+1}^1(u)(t) = \int_0^t \beta_0^1(u)(s)\frac{[m(t) - m(s)]^n}{n!}\,\mathrm{d}s + \frac{[m(t)]^n}{n!}\varepsilon_{n+1}, \tag{3.1}$$

where $m(t) = \int_0^t l(s)\,\mathrm{d}s$ and l is given by (H_2).

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By the continuity of the map $\beta_0^1 = \beta_0$ proved in the previous section we obtain easily that $\delta_n^1 : L^1(I_1, \mathbb{R}^n) \to L^1(I_1, \mathbb{R})$ is continuous. Moreover, by a similar computation to the one provided in [**1**, p. 122] we get

$$\int_0^t l(s)\delta_n^1(u)(s)\,\mathrm{d}s = \int_0^t \beta_0^1(u)(s)\frac{[m(t)-m(s)]^n}{n!}\,\mathrm{d}s + \frac{[m(t)]^n}{n!}\varepsilon_n < \delta_{n+1}^1(u)(t). \tag{3.2}$$

Set $h_0^1(u) = u$. We claim that for any $n \geqslant 1$ there exists a continuous map $h_n^1 : L^1(I_1, \mathbb{R}^n) \to L^1(I_1, \mathbb{R}^n)$ satisfying the following conditions:

(i) $h_n^1(u) = u$ whenever $u \in \mathcal{S}'_{F,I_1}(\xi)$;

(ii) $h_n^1(u)(t) \in F(t, \widehat{h_{n-1}^1(u)}(t))$, a.e. $t \in I_1$;

(iii) $\|h_n^1(u)(t) - h_{n-1}^1(u)(t)\| \leqslant l(t)\delta_{n-1}^1(u)(t)$, a.e. $t \in I_1$;

where, for simplicity, $l(t)\delta_0^1(u)(t)$ is understood as $\beta_0^1(u)(t) + \varepsilon_0$.

Indeed, define

$$\Phi_1^1(u) = \begin{cases} \Psi_1^1(u), & \text{if } u \notin \mathcal{S}'_{F,I_1}(\xi), \\ \{u\}, & \text{if } u \in \mathcal{S}'_{F,I_1}(\xi), \end{cases}$$

where

$$\Psi_1^1(u) = \{v \in L^1(I_1, \mathbb{R}^n) : v(t) \in F(t, \hat{u}(t)),\ \text{a.e. } t \in I_1\},$$

and, by Proposition 2.3 (for $\varphi(u) = u$ and $k = 1$), we obtain that $\Phi_1^1 : L^1(I_1, \mathbb{R}^n) \to \mathcal{D}(L^1(I_1, \mathbb{R}^n))$ is lower semicontinuous. Moreover, due to (2.4), the set

$$H_1^1(u) = \mathrm{cl}\{v \in \Phi_1^1(u) : \|v(t) - u(t)\| < \beta_0^1(u)(t) + \varepsilon_0,\ \text{a.e. } t \in I_1\}$$

is non-empty for any $u \in L^1(I_1, \mathbb{R}^n)$. Then a continuous selection h_1^1 from $u \mapsto H_1^1(u)$ exists by Lemma 2.2 and it satisfies (i)–(iii).

Assume we have constructed $h_0^1, \dots, h_n^1$ satisfying (i)–(iii). Then by (ii), (iii) and (3.2) we get

$$d(h_n^1(u)(t), F(t, \widehat{h_n^1(u)})) \leqslant l(t)\int_0^t l(s)\delta_{n-1}^1(u)(s)\,\mathrm{d}s < l(t)\delta_n^1(u)(t), \quad \text{a.e. } t \in I_1. \tag{3.3}$$

Define the multifunction $\Phi_{n+1}^1 : L^1(I_1, \mathbb{R}^n) \to \mathcal{C}(L^1(I_1, \mathbb{R}^n))$ by

$$\Phi_{n+1}^1(u) = \begin{cases} \Psi_{n+1}^1(u), & \text{if } u \notin \mathcal{S}'_{F,I_1}(\xi), \\ \{u\}, & \text{if } u \in \mathcal{S}'_{F,I_1}(\xi), \end{cases}$$

where

$$\Psi_{n+1}^1(u) = \{v \in L^1(I_1, \mathbb{R}^n) : v(t) \in F(t, \widehat{h_n^1(u)}(t)),\ \text{a.e. } t \in I_1\}.$$

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Apply Proposition 2.3 (for $\varphi(u) = h^1_n(u)$) and obtain that Φ^1_{n+1} is a lower semicontinuous multifunction with closed decomposable and non-empty values. Moreover, by (3.3), the set

$$H^1_{n+1}(u) = \mathrm{cl}\{v \in \Phi^1_{n+1}(u) : \|v(t) - h^1_n(u)(t)\| < l(t)\delta^1_n(u)(t), \text{ a.e. } t \in I_1\}$$

is non-empty for any $u \in L^1(I_1, \mathbb{R}^n)$. Then we can apply Lemma 2.2 and obtain the existence of a continuous selection h^1_{n+1} from $u \mapsto H^1_{n+1}(u)$, hence satisfying (i)–(iii), proving the claim.

Now, by (iii) and (3.2) one obtains that

$$\|h^1_{n+1}(u) - h^1_n(u)\|_{1,1} \leqslant \frac{[m(1)]^n}{n!}[\|\beta^1_0(u)\|_{1,1} + \varepsilon],$$

and this implies that $(h^1_n(u))_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $L^1(I_1, \mathbb{R}^n)$, hence it converges to some $h^1(u) \in L^1(I_1, \mathbb{R}^n)$. Moreover, since the map $u \mapsto \|\beta^1_0(u)\|_{1,1}$ is continuous, it is locally bounded and the Cauchy condition is satisfied by $(h^1_n(u))_{n\in\mathbb{N}}$ locally uniformly with respect to u, so the map $u \mapsto h^1(u)$ is continuous from $L^1(I_1, \mathbb{R}^n)$ into $L^1(I_1, \mathbb{R}^n)$.

By (i) it follows that $h^1(u) = u$ if $u \in \mathcal{S}'_{F,I_1}(\xi)$ and, by (ii) and the closure of the values of F, we obtain that, for any $u \in L^1(I_1, \mathbb{R}^n)$,

$$h^1(u)(t) \in F(t, \widehat{h^1(u)}(t)),$$

hence $h^1(u) \in \mathcal{S}'_{F,I_1}(\xi)$. Therefore, $h^1 : L^1(I_1, \mathbb{R}^n) \to L^1(I_1, \mathbb{R}^n)$ is continuous and satisfies (P_1) and (P_2).

We shall now construct a continuous map $h^2 : L^1(I_2, \mathbb{R}^n) \to L^1(I_2, \mathbb{R}^n)$ from h^1, satisfying (P_1)–(P_3).

For this, define $h^2_0 : L^1(I_2, \mathbb{R}^n) \to L^1(I_2, \mathbb{R}^n)$ by

$$h^2_0(u)(t) = h^1(u|_{I_1})(t)\chi_{I_1} + u(t)\chi_{I_2\setminus I_1}(t) \tag{3.4}$$

and state that it is continuous.

Indeed, fix any $u_0 \in L^1(I_2, \mathbb{R}^n)$. Since h^1 is continuous at $u_0|_{I_1}$ for any $\sigma > 0$, there exists $\zeta_\sigma > 0$ such that $\zeta_\sigma < \frac{1}{2}\sigma$ and, for every $v \in L^1(I_1, \mathbb{R}^n)$:

$$\|v - u_0|_{I_1}\|_{1,1} < \zeta_\sigma \Rightarrow \|h^1(v) - h^1(u_0|_{I_1})\|_{1,1} < \tfrac{1}{2}\sigma.$$

Then, for any $u \in L^1(I_2, \mathbb{R}^n)$ with $\|u - u_0\|_{1,2} < \zeta_\sigma$, one has that

$$\|h^2_0(u) - h^2_0(u_0)\|_{1,2} = \|h^1(u|_{I_1}) - h^1(u_0|_{I_1})\|_{1,1} + \int_1^2 \|u(t) - u_0(t)\|\,\mathrm{d}t < \sigma,$$

which implies the continuity of h^2_0.

Moreover, since $h^1(u) = u$ for $u \in \mathcal{S}'_{F,I_1}(\xi)$, by (3.4) we obtain that

$$h^2_0(u) = u, \quad \text{whenever } u \in \mathcal{S}'_{F,I_2}(\xi).$$

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For any $u \in L^1(I_2, \mathbb{R}^n)$, define

$$\Phi_1^2(u) = \begin{cases} \Psi_1^2(u), & \text{if } u \notin \mathcal{S}'_{F,I_2}(\xi), \\ \{u\}, & \text{if } u \in \mathcal{S}'_{F,I_2}(\xi), \end{cases}$$

where

$$\Psi_1^2(u) = \{w \in L^1(I_2, \mathbb{R}^n) : w(t) = h^1(u|_{I_1})(t)\chi_{I_1}(t) + v(t)\chi_{I_2 \setminus I_1}(t),$$
$$v(t) \in F(t, \widehat{h_0^2(u)}(t)), \text{ a.e. } t \in I_2 \setminus I_1\}.$$

We can apply Proposition 2.3, for $k = 2$, $\varphi(u) = h_0^2(u)$, and obtain that Φ_1^2 is lower semicontinuous from $L^1(I_2, \mathbb{R}^n)$ into $\mathcal{D}(L^1(I_2, \mathbb{R}^n))$. Moreover, for any $u \in L^1(I_2, \mathbb{R}^n)$,

$$\begin{aligned} d(h_0^2(u)(t), F(t, \widehat{h_0^2(u)}(t))) &= d(u(t), F(t, \widehat{h_0^2(u)}(t)))\chi_{I_2 \setminus I_1}(t) \\ &\leqslant \beta_0^2(u)(t), \quad \text{a.e. } t \in I_2, \end{aligned} \tag{3.5}$$

where

$$\beta_0^2(u)(t) = [\|u(t)\| + \beta(t) + l(t)\|\widehat{h_0^2(u)}(t)\|], \quad t \in I_2.$$

Since

$$\beta_0^2(u)(t) = \beta_0(u)(t) + l(t)\|h^1(u|_{I_1}) - u\|_{1,1}\chi_{I_2 \setminus I_1}(t),$$

by the continuity of β_0 and h^1 we obtain that $\beta_0^2 : L^1(I_2, \mathbb{R}^n) \to L^1(I_2, \mathbb{R})$ is continuous.

Set

$$\delta_{n+1}^2(u)(t) = \int_0^t \beta_0^2(u)(s)\frac{[m(t) - m(s)]^n}{n!}\,ds + \frac{[m(t)]^n}{n!}\varepsilon_{n+1},$$

and, by the continuity of the map β_0^2, we easily obtain that $\delta_n^2 : L^1(I_2, \mathbb{R}^n) \to L^1(I_2, \mathbb{R})$ is continuous. Moreover, as in (3.2) with $\beta_0^2(u)$ instead of $\beta_0^1(u)$, we have

$$\begin{aligned} \int_0^t l(s)\delta_n^2(u)(s)\,ds &= \int_0^t \beta_0^2(u)(s)\frac{[m(t) - m(s)]^n}{n!}\,ds + \frac{[m(t)]^n}{n!}\varepsilon_n \\ &< \delta_{n+1}^2(u)(t). \end{aligned} \tag{3.6}$$

We shall prove that for any $n \geqslant 1$ there exists a continuous map $h_n^2 : L^1(I_2, \mathbb{R}^n) \to L^1(I_2, \mathbb{R}^n)$ satisfying

(i) $h_n^2(u)(t) = h^1(u|_{I_1})(t)$, for $t \in I_1$;

(ii) $h_n^2(u) = u$ whenever $u \in \mathcal{S}'_{F,I_2}(\xi)$;

(iii) $h_n^2(u)(t) \in F(t, \widehat{h_{n-1}^2(u)})$, a.e. $t \in I_2$;

(iv) $\|h_n^2(u)(t) - h_{n-1}^2(u)(t)\| \leqslant l(t)\delta_{n-1}^2(u)(t)$, a.e. $t \in I_1$;

where $l(t)\delta_0^2(u)(t)$ is understood as $\beta_0^2(u)(t) + \varepsilon_0$.

Define

$$H_1^2(u) = \text{cl}\{v \in \Phi_1^2(u) : \|v(t) - h_0^2(u)(t)\| < \beta_0^2(u)(t) + \varepsilon_0, \text{ a.e. } t \in I_2\},$$

and, by (3.5), the set $H_1^2(u)$ is non-empty for any $u \in L^1(I_2, \mathbb{R}^n)$. Since Φ_1^2 is lower semicontinuous, and the functions h_0^2 and β_0^2 are continuous, Lemma 2.2 can be applied and obtain the existence of a continuous selection h_1^1 from $u \mapsto H_1^2(u)$, which satisfies (i)–(iv).

Assume we have constructed $h_0^2, \dots, h_n^2$ satisfying (i)–(iv). Then, by (H_2), (iv) and (3.6), one obtains

$$d(h_n^2(u)(t), F(t, \widehat{h_n^2(u)})) \leqslant l(t) \int_0^t l(s)\delta_{n-1}^2(u)(s)\,\mathrm{d}s < l(t)\delta_n^2(u)(t), \quad \text{a.e. } t \in I_2. \tag{3.7}$$

Define the multifunction $\Phi_{n+1}^2 : L^1(I_1, \mathbb{R}^n) \to \mathcal{C}(L^1(I_1, \mathbb{R}^n))$ by

$$\Phi_{n+1}^2(u) = \begin{cases} \Psi_{n+1}^2(u), & \text{if } u \notin \mathcal{S}'_{F,I_2}(\xi), \\ \{u\}, & \text{if } u \in \mathcal{S}'_{F,I_2}(\xi), \end{cases}$$

where

$$\Psi_{n+1}^2(u) = \{w \in L^1(I_2, \mathbb{R}^n) : w(t) = h^1(u|_{I_1})(t)\chi_{I_1}(t) + v(t)\chi_{I_2 \setminus I_1}(t), \; v(t) \in F(t, \widehat{h_n^2(u)}(t)), \text{ a.e. } t \in I_2 \setminus I_1\},$$

and, by Proposition 2.3, we obtain that it is lower semicontinuous with closed decomposable and non-empty values. Moreover, by (3.7), the set

$$H_{n+1}^2(u) = \mathrm{cl}\{v \in \Phi_{n+1}^1(u) : \|v(t) - h_n^2(u)(t)\| < l(t)\delta_n^2(u)(t), \text{ a.e. } t \in I_1\}$$

is non-empty for any $u \in L^1(I_2, \mathbb{R}^n)$. By applying Lemma 2.2 we obtain the existence of a continuous selection h_{n+1}^2 from $u \mapsto H_{n+1}^1(u)$, satisfying (i)–(iv). We need to prove that the sequence $(h_n^2(u))_{n \in \mathbb{N}}$ is a Cauchy sequence in the Banach space $L^1(I_2, \mathbb{R}^n)$ with norm $\|\cdot\|_{1,2}$, locally uniformly with respect to u. But this follows by a similar reasoning to the one made for $(h_n^1(u))_{n \in \mathbb{N}}$ and the remark that (iv) and (3.6) imply

$$\|h_{n+1}^2(u) - h_n^2(u)\|_{1,2} \leqslant \frac{[\|l|_{I_2}\|_{1,2}]^n}{n!}[\|\beta_0^2(u)\|_{1,2} + \varepsilon].$$

Therefore, $(h_n^2(u))_{n \in \mathbb{N}}$ converges in $L^1(I_2, \mathbb{R}^n)$ to some $h^2(u) \in L^1(I_2, \mathbb{R}^n)$ and the map $u \mapsto h^2(u)$ is continuous from $L^1(I_2, \mathbb{R}^n)$ into $L^1(I_2, \mathbb{R}^n)$. Moreover, by (i) it follows that

$$h^2(u)(t) = h^1(u|_{I_1})(t), \quad \text{for } t \in I_1;$$

by (ii),

$$h^1(u) = u, \quad \text{if } u \in \mathcal{S}'_{F,I_1}(\xi);$$

and by (iii) and the closure of the values of F we obtain that for any $u \in L^1(I_2, \mathbb{R}^n)$

$$h^2(u)(t) \in F(t, \widehat{h^2(u)}(t)), \quad \text{a.e. } t \in I_2.$$

Therefore, h^2 satisfies properties (P_1)–(P_3).

Similarly, for any $k > 2$, we obtain a continuous map $h^k : L^1(I_k, \mathbb{R}^n) \to L^1(I_k, \mathbb{R}^n)$ from $h^{k-1} : L^1(I_{k-1}, \mathbb{R}^n) \to L^1(I_{k-1}, \mathbb{R}^n)$, satisfying properties (P_1)–(P_3).

Define $H : L^1_{\text{loc}}([0,\infty), \mathbb{R}^n) \to L^1_{\text{loc}}([0,\infty), \mathbb{R}^n)$ by

$$H(u)(t) = h^k(u|_{I_k})(t), \quad k = 1, 2, \dots.$$

By using (P_3) and the continuity of each h^k it is easy to see that H is well defined and continuous. Moreover, for each $u \in L^1_{\text{loc}}([0,\infty), \mathbb{R}^n)$, by ($P_2$) we have

$$H(u)|_{I_k}(t) = h^k(u|_{I_k})(t) \in \mathcal{S}'_{F,I_k}(\xi), \quad \text{for each } k = 1, 2, \dots,$$

hence

$$H(u) \in \mathcal{S}'_F(\xi).$$

□

References

1. J. P. Aubin and A. Cellina, *Differential inclusions* (Springer, Berlin, 1984).
2. A. Bressan, A. Cellina and A. Fryszkowski, A class of absolute retracts in space of integrable functions, *Proc. Am. Math. Soc.* **112** (1991), 413–418.
3. A. Bressan and G. Colombo, Extensions and selections of maps with decomposable values, *Studia Math.* **102** (1992), 209–216.
4. A. Bressan and G. Crasta, Extremal selections of multifunctions generating a continuous flow, *Ann. Polon. Math.* **40** (1994), 101–117.
5. R. Colombo, A. Fryszkowski, T. Rzezukowski and V. Staicu, Continuous selections of solution sets of lipschitzian differential inclusions. *Funkcial. Ekvac.* **34** (1991), 321–330.
6. F. S. De Blasi and G. Pianigiani, On the solution sets of nonconvex differential inclusions, *J. Diff. Eqns* **128** (1996), 541–555.
7. F. S. De Blasi, G. Pianigiani and V. Staicu, Topological properties of nonconvex differential inclusions of evolution type, *Nonlinear Analysis* **24** (1995), 711–720.
8. L. Gorniewicz, On the solution sets of differential inclusions, *J. Math. Analysis Appl.* **113** (1986), 235–244.
9. F. Hiai and H. Umegaki, Integrals, conditional expectations and martingales of multivalued maps, *J. Multivariate Analysis* **7** (1977), 149–182.
10. O. Naselli Ricceri and B. Ricceri, Differential inclusions depending on a parameter, *Bull. Pol. Acad. Sci.* **37** (1989), 665–671.
11. N. S. Papagiorgiou, A property of the solution set of differential inclusions in Banach spaces with Carathéodory orientor, *Applic. Analysis* **27** (1988), 279–286.
12. B. Ricceri, Une proprieté topologique de l'ensemble des points fixes d'une contraction multivoque à valeurs convexes, *Rend. Accad. Naz. Lincei* **81** (1987), 283–286.
13. V. Staicu, Arcwise connectedness of solution sets to differential inclusions, technical report CM/I 41 (Aveiro University, 1998).
14. A. A. Tolstonogov, On the structure of the solution sets for differential inclusions in Banach spaces, *Mat. USSR Sb.* **46** (1983), 1–15.