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KERNELS OF ORTHODOX SEMIGROUP HOMOMORPHISMS

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1. Introduction

Any congruence on an orthodox semigroup S induces a partition of the set E of idempotents of S satisfying certain normality conditions. Meakin (1970) has characterized those partitions of E which are induced by congruences on S as well as the largest congruence ρ and the smallest congruence σ on S corresponding to such a partition of E. In this paper a more precise description of ρ and σ is given.

For an inverse semigroup S, Scheiblich (1974) has used the description of ρ and σ corresponding to a given normal partition of E to characterize the set of congruences on S which induce this partition of E. The aim of this paper is to present an analogue of these results for an orthodox semigroup.

2. Preliminary results and definitions

The reader is assumed to be familiar with the basic concepts, definitions, and terminology of semigroup theory (Clifford and Preston, 1961). Throughout, unless otherwise specified, S will denote an orthodox semigroup; that is, a regular semigroup in which the set of idempotents forms a subsemigroup. For any semigroup S, $E(S)$ will be used to denote the set of idempotents of S. When there is no danger of ambiguity, E will be used instead of $E(S)$. The set of inverses of an element a in S will be represented by $V(a)$.

The following lemma will be used frequently in this paper.

LEMMA 2.1 (Reilly and Scheiblich, 1967, Lemma 1.3 and Lemma 1.4). *Let S be an orthodox semigroup. Then*

(i) *for each $a, b \in S$, if $a' \in V(a)$, $b' \in V(b)$, then $b'a' \in V(ab)$;*
(ii) *for each $a \in S$, if $a' \in V(a)$, then $aEa' \subseteq E$;*
(iii) *for each $e \in E$, $V(e) \subseteq E$.*

A subsemigroup H of S will be called self-conjugate if $xHx' \subseteq H$ for each $x \in S$, $x' \in V(x)$. This is merely an extension of Howie's (1964) definition of self-conjugacy for subsemigroups of inverse S.

For any subset G of S, define the closure of G to be $G\omega = \{a \in S : ga \in G$ for some $g \in G\}$. G will be called *closed* whenever $G = G\omega$. In general, $G \subseteq G\omega$ does not hold; for example, consider $G = \{g\}$. However, if G is a subsemigroup of S then $G \subseteq G\omega$.

3. The lattice of idempotent-separating congruences

Meakin (1971, Theorem 4.4) characterizes μ, the maximum idempotent-separating congruence on orthodox S, as

$$\mu = \{(a, b) \in S \times S : \text{there are inverses } a' \text{ of } a \text{ and } b' \text{ of } b$$

$$\text{for which } aea' = beb' \text{ and } a'ea = b'eb \text{ for each } e \in E\}.$$

An alternate characterization of μ will be presented here.

Define the centralizer of E to be $C(E) = \{x \in S : x\mu \in E(S/\mu)\}$; that is, $C(E) = \{x \in S : (x, e) \in \mu$ for some $e \in E\}$ (Lallement, 1966, Lemma 2.2). One can readily verify that $C(E)$ is a self-conjugate, regular subsemigroup of S.

THEOREM 3.1. *Let* $\tau = \{(a, b) \in S \times S :$ *there are inverses* a' *of* a *and* b' *of* b *for which* $aa' = bb'$, $a'a = b'b$, *and* ab', $a'b \in C(E)\}$. *Then* $\mu = \tau$.

PROOF. Let $(a, b) \in \mu$. Then $aa' = bb'$ and $a'a = b'b$ where a', b' are the inverses of a, b respectively given in Meakin's characterization of μ (Meakin, 1971, proof of Theorem 4.4). In addition, (ab, bb'), $(a'a, a'b) \in \mu$ so that ab', $a'b \in C(E)$.

Conversely, let $(a, b) \in \tau$. Then there are inverses a' of a and b' of b for which $aa' = bb'$, $a'a = b'b$, and ab', $a'b \in C(E)$. So, $a\mathscr{H}b$ and $a'\mathscr{H}b'$ which gives $ab'\mathscr{H}bb'$. Since $ab' \in C(E)$, it follows that $(ab')\mu = (bb')\mu$. Therefore, $a\mu = a\mu(a'a)\mu = a\mu(b'b)\mu = (ab')\mu b\mu = (bb')\mu b\mu = b\mu$.

The characterization of μ just presented is the analogue for orthodox S of Howie's (1964, theorem 2.5) characterization of μ as $\{(a, b) \in S \times S : aa^{-1} = bb^{-1}$ and $a^{-1}b \in C(E)\}$ for inverse S.

LEMMA 3.2. *Let* $A = \{a \in S :$ *there is an inverse* a' *of* a *for which* $a'eae = aa'e$ *and* $eaea' = ea'a$ *for each* $e \in E\}$. *Then* $C(E) = A$.

PROOF. Let $a \in C(E)$ so that $(a, f) \in \mu$ for some $f \in E$. Since $\mu \subseteq \mathscr{H}$, there exists $a' \in V(a) \cap H_a$ such that $aa' = a'a = f$ so that $(a', f) \in \mu$. Choose $e \in E$. Then $(aea', fef) \in \mu$ and $(a'ea, fef) \in \mu$ which says that $aea' = fef$ and $a'ea = fef$. Therefore, $a'eae = fefe = fe = aa'e$ and $eaea' = efef = ef = ea'a$.

Conversely, if $a \in A$, then there is an inverse a' of a such that a, a' satisfy the equalities in the definition of A for each $e \in E$. Since $aa', a'a \in E$, $aa' = aa'(aa') = a'(aa')a(aa') = a'aaa' = (a'a)a(a'a)a' = (a'a)a'a = a'a$. So, given $e \in E$

$$aea' = a(a'a)ea' = a(aa'e)a' = a(a'eae)a' = aa'(eaea')$$
$$= aa'e(a'a) = aa'eaa'$$

and

$$a'ea = a'e(aa')a = a'(ea'a)a = a'(eaea')a = (a'eae)a'a$$
$$= aa'e(a'a) = aa'eaa'.$$

Hence, $(a,aa') \in \mu$.

If S is an inverse semigroup, then $C(E) = \{a \in S : ea = ae \text{ for each } e \in E\}$ which is precisely Howie's (1964) definition of $C(E)$. To see this, let $a \in S$ and let a^{-1} denote the inverse of a in S. If $ea = ae$ for each $e \in E$, then $aa^{-1} = (aa^{-1}a)a^{-1} = (a^{-1}aa)a^{-1} = a^{-1}(aaa^{-1}) = a^{-1}(aa^{-1}a) = a^{-1}a$. Therefore, for each $e \in E$, $a^{-1}eae = a^{-1}eea = a^{-1}(ea) = a^{-1}ae = aa^{-1}e$ and $eaea^{-1} = aeea^{-1} = (ae)a^{-1} = eaa^{-1} = ea^{-1}a$. On the other hand, if $a^{-1}eae = aa^{-1}e$ and $eaea^{-1} = ea^{-1}a$ for each $e \in E$, then as in the proof of Lemma 3.2 $aa^{-1} = a^{-1}a$. Hence, $ea = e(aa^{-1})a = (ea^{-1}a)a = (eaea^{-1})a = ea(ea^{-1}a) = ea(a^{-1}ae) = (eaa^{-1})ae = (aa^{-1}e)ae = a(a^{-1}eae) = a(aa^{-1}e) = a(a^{-1}a)e = ae$.

The following theorem gives a description of the lattice of idempotent-separating congruences on orthodox S. Define $\mathscr{C} = \{K \subset S : E \subset K \subset C(E)$ and K is a self-conjugate, regular subsemigroup of $S\}$. Then, clearly, E and $C(E)$ belong to $\mathscr{C}$.

THEOREM 3.3. *The map $K \to (K) = \{(a, b) \in S \times S$: there are inverses a' of a and b' of b for which $aa' = bb'$, $a'a = b'b$, and $ab', a'b \in K\}$ is a $1:1$ order preserving map of $\mathscr{C}$ onto the set of idempotent-separating congruences on S.*

PROOF. First it will be shown that if $K \in \mathscr{C}$, then (K) is an idempotent-separating congruence. Since $E \subseteq K$, it is clear that (K) is a reflexive relation. Furthermore, $K \subseteq C(E)$ implies that $(K) \subset \mu$ [Theorem 3.1] so that (K) is an idempotent-separating relation. For $(a, b) \in (K)$, let a', b' be the inverses of a, b respectively given in the definition of (K). Then, since K is self-conjugate, $ba' = (bb')ba' = a(a'b)a' \in aKa' \subseteq K$ and

$$b'a = b'a(a'a) = b'(ab')b \in b'Kb \subset K$$

so that $(b, a) \in (K)$. Hence, (K) is symmetric. Before proceeding with the proof of this theorem, it is important to note that

(3.4) $(a, b) \in (K)$ *implies* $ab^*, a^*b \in K$ *for each* $a^* \in V(a), b^* \in V(b)$.

The verification of this result readily follows from the symmetry of (K) and the

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self-conjugacy of K. Suppose now that (a,b), $(b,c)\in(K)$. Then there are inverses a' of a, b' and b^* of b, c^* of c such that $aa'=bb'$, $a'a=b'b$, $bb^*=cc^*$, $b^*b=c^*c$. Thus $a\mathscr{H}b\mathscr{H}c$ so that there exists $c'\in V(c)$ such that $aa'=bb'=cc'$ and $a'a=b'b=c'c$. Furthermore, ab', $a'b$, bc', $b'c\in K$ [3.4]. Therefore,

$$ac'=aa'ac'=(ab')(bc')\in K \text{ and } a'c=a'aa'c=(a'b)(b'c)\in K.$$

So, $(a,c)\in(K)$. To see that (K) is compatible, let (a,b), $(c,d)\in(K)$. Since $K\subset C(E)$, $(K)\subset\mu$ so that $(a,b),(c,d)\in\mu$. Hence, there are inverses a' of a, b' of b, c' of c, d' of d such that the defining conditions of Meakin's characterization of μ are satisfied. It then follows that $aa'=bb'$, $a'a=b'b$, $cc'=dd'$, $c'c=d'd$ (Meakin, 1971, proof of Theorem 4.4). So, since $c'a'=(ac)'\in V(ac)$ and $d'b'=(bd)'\in V(bd)$ [Lemma 2.2],

$$(ac)(ac)'=a(cc')a'=a(dd')a'=b(dd')b'=(bd)(bd)',$$

$$(ac)'(ac)=c'(a'a)c=c'(b'b)c=d'(b'b)d=(bd)'(bd),$$

and

$$(ac)(bd)'=acd'b'=acd'(b'b)b'=(acd'a')ab'\in aKa'K\subset K,$$

$$(ac)'(bd)=c'a'bd=c'a'b(dd')d=(c'a'bc)c'd\in c'KcK\subset K.$$

Thus, $(ac,bd)\in(K)$.

Now, if τ is an idempotent-separating congruence on S, it will be shown that there exists an element K in $\mathscr{C}$ such that $(K)=\tau$. First, recall that the kernel of τ is defined to be Ker $\tau=\{a\in S: a\tau\in E(S/\tau)\}$ or equivalently Ker $\tau=\{a\in S:(a,e)\in\tau$ for some $e\in E\}$ (Lallement, 1966, Lemma 2.2). Note that this use of the word kernel differs from that of Clifford and Preston (1961) and that of Meakin (1970, 1971). Then Ker τ is a self-conjugate subsemigroup of S containing E. Moreover, Ker τ is regular. To see this, let $a\in$ Ker τ so that $(a,e)\in\tau$ for some $e\in E$. Choose $a'\in V(a)$. Then $a'\tau\in V(a\tau)=V(e\tau)$. Since S is orthodox, S/τ must be orthodox so that $a'\tau\in V(e\tau)$ implies that $a'\tau\in E(S/\tau)$ implies that $a'\tau\in E(S/\tau)$ [Lemma 2.1]. Finally, since τ is an idempotent-separating congruence on S, Ker $\tau\subset C(E)$. Therefore, Ker $\tau\in\mathscr{C}$. Furthermore, (Ker $\tau)=\tau$. For, if $(a,b)\in$ (Ker τ) then there are inverses a' of a and b' of b such that $aa'=bb'$, $a'a=b'b$, and ab', $a'b\in$ Ker τ. So, $a\mathscr{H}b$ and $a'\mathscr{H}b'$ which gives $ab'\mathscr{H}bb'$. Hence, $ab'\in$ Ker τ implies that $(ab')\tau=(bb')\tau$. Therefore, $a\tau=a\tau(a'a)\tau=a\tau(b'b)\tau=(ab')\tau b\tau=(bb')\tau b\tau=b\tau$. Conversely, if $(a,b)\in\tau$, then $(a,b)\in\mathscr{H}$ so that there are inverses a' of a and b' of b such that $aa'=bb'$ and $a'a=b'b$. In addition, (ab',bb'), $(a'a,a'b)\in\tau$ so that ab', $a'b\in$ Ker τ. Therefore, the given map is onto.

Since the given map is clearly order preserving, it only remains to show that the map is $1:1$. So, let $K,L\in\mathscr{C}$ with $(K)=(L)$. Choose $k\in K$. Since

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$K \subset C(E)$, $k \in C(E)$ so that $(k, e) \in \mu$ for some $e \in E$. Since $\mu \subseteq \mathscr{H}$, there exists $k' \in V(k) \cap H_k$ such that $kk' = k'k = e$. Now K is regular, so there exists $k^* \in V(k) \cap K$. Then $k' = (k'k)k^*(kk') \in EKE \subset K$. Thus, $k\mathscr{H}k'k$, $k(k'k) = k \in K$, and $k'(k'k) = k'(kk') = k' \in K$ so that $(k, k'k) \in (K)$. Since $(K) = (L)$, $(k, k'k) \in (L)$; that is, $k(k'k)^* \in L$ for each $(k'k)^* \in V(k'k)$ [3.4]. In particular, $k(k'k) \in L$ so that $k \in L$. Similarly $L \subset K$ so that $K = L$.

4. Idempotent-equivalent congruences

Let $P = \{E_\alpha : \alpha \in J\}$ be a partition of E. Then P is a normal partition of E if

(i) for each $\alpha, \beta \in J$ there exists $\gamma \in J$ such that $E_\alpha E_\beta \subseteq E_\gamma$;

(ii) for each $\alpha \in J$, $a \in S$, $a' \in V(a)$ there exists $\beta \in J$ such that $aE_\alpha a' \subseteq E_\beta$.

Denote by π_P the equivalence relation on E induced by the normal partition P. Clearly, if τ is a congruence on S, then τ induces a normal partition of E. Meakin (1970, Theorem 2.3 and Theorem 3.3) has determined the smallest and largest congruences on S whose restriction to E is π_p. In this section, more precise characterizations of these congruences will be given.

It will be useful to introduce the following notation. If e, f are two idempotents of S, then define $e \sim f$ if e, f are in the same class E_α of the normal partition P.

THEOREM 4.1. *Let* $\sigma = \{(a, b) \in S \times S :$ *there exist* $a' \in V(a)$, $b' \in V(b)$; $\alpha, \beta, \gamma, \delta, \in J$; *and* $e \in E_\alpha$, $f \in E_\delta$, $g \in E_\beta$, $h \in E_\gamma$ *such that* aa', $bb'aa' \in E_\alpha$; $a'a$, $a'ab'b \in E_\beta$; bb', $aa'bb' \in E_\gamma$; $b'b$, $b'ba'a \in E_\delta$; *and* $ea = bf$, $ag = hb\}$. *Then* σ *is the smallest congruence on* S *whose restriction to* E *is* π_p.

PROOF. It is trivial to verify that σ is a reflexive, symmetric relation. To see that σ is transitive, let $(a, b), (b, c) \in \sigma$. Then there exist $a' \in V(a)$, $b' \in V(b)$; $e \sim aa'$, $f \sim b'b$, $g \sim a'a$, $h \sim bb'$ such that $aa' \sim bb'aa'$, $a'a \sim a'ab'b$, $bb' \sim aa'bb'$, $b'b \sim b'ba'a$, $ea = bf$ and $ag = hb$. And there exist $b^* \in V(b)$, $c^* \in V(c)$; $\bar{e} \sim bb^*$, $\bar{f} \sim c^*c$, $\bar{g} \sim b^*b$, $\bar{h} \sim cc^*$ such that $bb^* \sim cc^*bb^*$, $b^*b \sim b^*bc^*c$, $cc^* \sim bb^*cc^*$, $c^*c \sim c^*cb^*b$, $\bar{e}b = c\bar{f}$ and $b\bar{g} = \bar{h}c$. Thus,

$$aa' \sim bb'aa' = bb^*(bb^*)bb'aa' \sim bb^*(cc^*bb^*)bb'aa'$$
$$= (bb^*cc^*)(bb'aa') \sim cc^*aa'$$

and

$$c^*c \sim c^*cb^*b = c^*cb^*b(b'b)b'b \sim c^*cb^*b(b'ba'a)b'b$$
$$= (c^*cb^*b)(a'ab'b) \sim c^*ca'a.$$

Also, $(\bar{e}e)a = \bar{e}bf = c(\bar{f}f)$ where $\bar{e}e \sim bb^*(aa') \sim bb^*bb'aa' = bb'aa' \sim aa'$

and $\bar{f}f \sim (c^*c)b'b \sim c^*cb^*bb'b = c^*cb^*b \sim c^*c$. Symmetrically, $cc^* \sim aa'cc^*a'a \sim a'ac^*c$, $a(g\bar{g}) = (h\bar{h})c$ where $g\bar{g} \sim a'a$, $h\bar{h} \sim cc^*$. Therefore, $(a, c) \in \sigma$.

Suppose next that $(a, b) \in \sigma$ and $c \in S$. Then there exist $a' \in V(a)$, $b' \in V(b)$; $e \sim aa'$, $f \sim b'b$, $g \sim a'a$, $h \sim bb'$ such that the defining properties of σ are satisfied. Let $c' \in V(c)$. Then $c'a' = (ac)' \in V(ac)$ and $c'b' = (bc)' \in V(bc)$ [Lemma 2.1]. So,

$$\begin{aligned}(ac)(ac)' &= a(a'a)cc'a' \sim aa'a(b'bcc')a' = a(a'a)b'bcc'b'b(b'b)cc'a' \\ &\sim agb'bcc'b'bfcc'a' = hbb'bcc'b'eacc'a' \sim bb'bb'bcc'b'aa'acc'a' \\ &= bcc'b'acc'a' = (bc)(bc)'(ac)(ac)'\end{aligned}$$

and

$$\begin{aligned}(bc)'(bc) &= c'(b'b)cc'c \sim c'b'b(a'acc')c = c'(b'ba'a)cc'a'acc'c \\ &\sim c'b'bcc'a'acc'c = (bc)'(bc)(ac)'(ac).\end{aligned}$$

Also, $(bcc'b'eacc'a')ac = bc(c'b'bfcc'a'ac)$ where $bcc'b'eacc'a' \sim bcc'b'aa'acc'a' = bcc'b'acc'a' = (bc)(bc)'(ac)(ac)' \sim (ac)(ac)'$ and $c'b'bfcc'a'ac \sim c'b'bb'bcc'a'ac = c'b'bcc'a'ac = (bc)'(bc)(ac)'(ac) \sim (bc)'(bc)$. Symmetrically, $(bc)(bc)' \sim (ac)(ac)'(bc)(bc)'$, $(ac)'(ac) \sim (ac)'(ac)(bc)'(bc)$, and $acx = ybc$ where $x, y \in E$, $x \sim (ac)'(ac)$, $y \sim (bc)(bc)'$. Therefore, $(ac, bc) \in \sigma$. The left compatibility of σ is similarly established. Thus, σ is a congruence on S.

It will now be shown that $\sigma | E$ coincides with π_p. Suppose first that e, $f \in E_\alpha$. Then, since $e \in V(e)$, $f \in V(f)$, e and f clearly satisfy the defining properties of σ. Hence, $(e, f) \in \sigma$. Conversely, suppose that $e, f \in E$ for which $(e, f) \in \sigma$. Then there exist idempotents g, h such that $eg = hf$ where $g \sim e'e$, $h \sim ff'$ for some $e' \in V(e)$, $f' \in V(f)$. So, $e = e(e'e) \sim eg = hf \sim (ff')f = f$.

Lastly, let τ be a congruence on S for which $\tau | E = \pi_p$. For $(a, b) \in \sigma$ there exist $a' \in V(a)$, $b' \in V(b)$; $e \sim aa'$, $f \sim b'b$, $g \sim a'a$, $h \sim bb'$ such that the defining properties of σ are satisfied. Consequently, $a\tau = (aa')\tau a\tau = e\tau a\tau = (ea)\tau = (bf)\tau = b\tau f\tau = b\tau(b'b)\tau = b\tau$. Hence, $\sigma \subseteq \tau$ and the proof of the theorem is completed.

THEOREM 4.2. *Let* $\rho = \{(a, b) \in S \times S$: *there exist* $a' \in V(a)$, $b' \in V(b)$ *such that* $\varepsilon \in J$ *implies* $aE_\varepsilon a'$, $bE_\varepsilon b'aE_\varepsilon a' \subset E_\alpha$; $a'E_\varepsilon a$, $a'E_\varepsilon ab'E_\varepsilon b \subset E_\beta$; $bE_\varepsilon b'$, $aE_\varepsilon a'bE_\varepsilon b' \subset E_\gamma$; $b'E_\varepsilon b$, $b'E_\varepsilon ba'E_\varepsilon a \subset E_\delta$ *for some* $\alpha, \beta, \gamma, \delta, \in J$. *Then* ρ *is the largest congruence on* S *whose restriction to* E *is* π_p.

PROOF. It is obvious that ρ is a reflexive, symmetric relation. To see that ρ is transitive, let (a, b), $(b, c) \in \rho$. Then there are inverses a' of a, b' of b and b^* of b, c^* of c such that the defining properties of ρ are satisfied. Let $\varepsilon \in J$ and choose $e \in E_\varepsilon$. Note that $eb', eb^* \in V(be)$ [Lemma 2.1] so that $beb'\mathscr{R}beb^*$ and $b'e$, $b^*e \in V(eb)$ [Lemma 2.1] so that $b'eb\mathscr{L}b^*eb$. Thus,

$$aea' \sim beb'aea' = beb^*(beb^*)beb'aea' \sim beb^*cec^*(beb^*beb')aea'$$

$$= (beb^*cec^*)(beb'aea') \sim cec^*aea'$$

and

$$a'ea \sim a'eab'eb = a'eab'eb(b^*eb)b^*eb \sim a'ea(b'ebb^*eb)c^*ecb^*eb$$

$$= (a'eab'eb)(c^*ecb^*eb) \sim a'eac^*ec.$$

Symmetrically, $cec^* \sim aea'cec^*$ and $c^*ec \sim c^*eca'ea$ so that $(a, c) \in \rho$.

Suppose next that $(a, b) \in \rho$ and $c \in S$. Then there are inverses a' of a and b' of b such that the defining properties of ρ are satisfied. Let $c' \in V(c)$ so that $c'a' = (ac)' \in V(ac)$ and $c'b' = (bc)' \in V(bc)$ [Lemma 2.1]. Now let $\varepsilon \in J$ and choose $e \in E_\varepsilon$. Then

$$(ac)e(ac)' = a(cec')a' \sim b(cec')b'a(cec'a)a' = (bc)e(bc)'(ac)e(ac)'$$

and

$$(ac)'e(ac) = c'(a'ea)cc'c \sim c'a'ea(b'ebcc')c = c'(a'eab'eb)cc'b'ebcc'c$$

$$\sim c'a'eacc'b'ebcc'c = (ac)'e(ac)(bc)'e(bc).$$

Likewise $(bc)e(bc)' \sim (ac)e(ac)'(bc)e(bc)'$ and $(bc)'e(bc) \sim (bc)'e(bc)(ac)'e(ac)$ so that ρ is right compatible. In a similar fashion, one shows that ρ is left compatible. Thus, ρ is a congruence on S.

To see that $\rho \mid E = \pi_p$, first suppose that $e, f \in E_\alpha$. Then, since $e \in V(e)$, $f \in V(f)$, e and f clearly satisfy the defining properties of ρ so that $(e, f) \in \rho$. Now let $e, f \in E$ and suppose that $(e, f) \in \rho$. Then there are inverses e' of e and f' of f such that the defining properties of ρ are satisfied. So, $e = (ee'e')e \sim (fe'f'ee'e')e = fe'f'e = f(f'f'fe'f'e) \sim f(f'f'f) = f$.

Lastly, suppose τ is a congruence on S for which $\tau \mid E = \pi_p$. Let $(a, b) \in \tau$. Choose $a' \in V(a)$, $b' \in V(b)$. Suppose $\varepsilon \in J$ and let $e \in E_\varepsilon$. Then $(ae, be) \in \tau$ and $(ea')\tau \in V((ae)\tau)$, $(eb')\tau \in V((be)\tau)$ [Lemma 2.1] so that $(aea')\tau\mathscr{R}(ae)\tau = (be)\tau\mathscr{R}(beb')\tau$. Thus, $(aea')\tau = (beb')\tau(aea')\tau = (beb'aea')\tau$ implying that $aea' \sim beb'aea'$ and $(beb')\tau = (aea')\tau(beb')\tau = (aea'beb')\tau$ implying that $beb' \sim aea'beb'$. Similarly, $a'ea \sim a'eab'eb$ and $b'eb \sim b'eba'ea$. Therefore, $\tau \subseteq \rho$ and the theorem is proved.

Given the characterizations of σ, the smallest congruence on S such that $\sigma \mid E = \pi_p$, and ρ, the largest congruence on S such that $\rho \mid E = \pi_p$, it will be worthwhile to examine σ and ρ from another viewpoint.

For each $\alpha \in J$, $[E_\alpha]\sigma \in E(S/\sigma)$. Denote this by $[E_\alpha]\sigma = \alpha$. Hence, for each $\alpha \in J$, H_α is a subgroup of S/σ. Let $G_\alpha = (H_\alpha)\sigma^{-1}$. Then G_α is a subsemigroup of S with E_α as its set of idempotents.

LEMMA 4.3. *For each $\alpha \in J$, $G_\alpha = \{a \in S$: there exist $e \in E_\alpha$ and $a' \in V(a)$ such that $aa'e$, $ea'a \in E_\alpha$ and $eaa' \sim aa'$, $a'ae \sim a'a\}$.*

PROOF. Let $a \in S$ and suppose that there exist $e \in E_\alpha$, $a' \in V(a)$ such that $aa'e$, $ea'a \in E_\alpha$; eaa', $aa' \in E_\beta$ and $a'ae$, $a'a \in E_\gamma$ for some $\beta, \gamma \in J$. Now $e(aa') \in E_\alpha E_\beta$ and $eaa' \in E_\beta$ imply that $\alpha\beta = \beta$ and $(aa')e \in E_\beta E_\alpha$ and $aa'e \in E_\alpha$ imply that $\beta\alpha = \alpha$ which says $\beta\mathcal{R}\alpha$. So $a\sigma\mathcal{R}(aa')\sigma = \beta\mathcal{R}\alpha$. Similarly, $a\sigma\mathcal{L}\alpha$. Hence, $a\sigma \in H_\alpha$ or equivalently $a \in G_\alpha$.

For $a \in G_\alpha$, $a\sigma \in H_\alpha$. Let $e \in E_\alpha$, $a' \in V(a)$. Then $e\sigma = \alpha\mathcal{R}a\sigma\mathcal{R}(aa')\sigma$ and $e\sigma = \alpha\mathcal{L}a\sigma\mathcal{L}(a'a)\sigma$. It thus follows that $(aa'e)\sigma = e\sigma$, $(ea'a)\sigma = e\sigma$, $(eaa')\sigma = (aa')\sigma$, $(a'ae)\sigma = (a'a)\sigma$ so that $aa'e \sim e$, $ea'a \sim e$, $eaa' \sim aa'$, $a'ae \sim a'a$.

Before proceeding, it is important to consider G_α as it relates to any congruence τ on S for which $\tau \mid E = \pi_p$.

LEMMA 4.4. *Let τ be a congruence on S for which $\tau \mid E = \pi_p$. Then*
(i) *a, $b \in G_\alpha$ imply $a\tau\mathcal{H}b\tau$;*
(ii) *$a\tau\mathcal{H}e\tau$ for some $e \in E_\alpha$ implies $a \in G_\alpha$;*
(iii) *$a \in G_\alpha$ implies $\alpha(a'\tau)\alpha \in V(a\tau) \cap H_\alpha$ for each $a' \in V(a)$.*

PROOF. Suppose a, $b \in G_\alpha$. Then there exist $a' \in V(a)$, $b' \in V(b)$; e, $f \in E_\alpha$; and $\beta, \gamma, \bar{\beta}, \bar{\gamma} \in J$ such that $aa'e$, $ea'a$, $bb'f$, $fb'b \in E_\alpha$; eaa', $aa' \in E_\beta$; $a'ae$, $a'a \in E_\gamma$; fbb', $bb' \in E_{\bar{\beta}}$; $b'bf$, $b'b \in E_{\bar{\gamma}}$. As in the proof of Lemma 4.3, it follows that $\beta\mathcal{R}\alpha$ and $\bar{\beta}\mathcal{R}\alpha$ so that $\beta\mathcal{R}\bar{\beta}$. Hence, $a\tau\mathcal{R}(aa')\tau = \beta\mathcal{R}\bar{\beta} = (bb')\tau\mathcal{R}b\tau$. Similarly, $a\tau\mathcal{L}(a'a)\tau = \gamma\mathcal{L}\bar{\gamma} = (b'b)\tau\mathcal{L}b\tau$. Therefore, $a\tau\mathcal{H}b\tau$.

If $a\tau\mathcal{H}e\tau$ for some $e \in E_\alpha$, then for any $a' \in V(a)$ $(aa')\tau\mathcal{R}a\tau\mathcal{R}e\tau$ so that $(aa'e)\tau = e\tau$ and $(eaa')\tau = (aa')\tau$ implying $aa'e \sim e$ and $eaa' \sim aa'$. Also, $(a'a)\tau\mathcal{L}a\tau\mathcal{L}e\tau$ so that $(ea'a)\tau = e\tau$ and $(a'ae)\tau = (a'a)\tau$ implying $ea'a \sim e$ and $a'ae \sim a'a$. Thus, $a \in G_\alpha$ [Lemma 4.3].

Finally, if $a \in G_\alpha$ then by (i) $a\tau\mathcal{H}\alpha$. Choose $a' \in V(a)$. One can readily show that $\alpha(a'\tau)\alpha \in V(a\tau)$. To see that $\alpha(a'\tau)\alpha \in \mathcal{H}_\alpha$, first note that $(a'a)\tau \in R_{a'\tau} \cap L_{a\tau} = R_{a'\tau} \cap L_\alpha$ implies that $\alpha(a'\tau) \in R_\alpha \cap L_{a'\tau} = R_{a\tau} \cap L_{a'\tau}$ (Clifford and Preston, 1961, Lemma 2.17). So, $\alpha(a'\tau) \in H_{(aa')\tau}$. Therefore, $(aa')\tau \in R_{a\tau} \cap L_{a'\tau} = R_\alpha \cap L_{(aa')\tau}$ implies that $\alpha(a'\tau)\alpha \in R_{(aa')\tau} \cap L_\alpha = R_{a\tau} \cap L_\alpha = R_\alpha \cap L_\alpha$ (Clifford and Preston, 1961, Lemma 2.17). So, $\alpha(a'\tau)\alpha \in H_\alpha$.

For any congruence τ on S, recall that the kernel of τ is defined to be Ker $\tau = \{a \in S : a\tau \in E(S/\tau)\}$ or equivalently Ker $\tau = \{a \in S : (a, e) \in \tau$ for some $e \in E\}$ (Lallement, 1966, Lemma 2.2). Moreover, Ker τ is a self-conjugate, regular subsemigroup of S containing E [Proof of Theorem 3.3]. Let $M =$ Ker σ and $N =$ Ker ρ. Then M and N are each self-conjugate, regular subsemigroups of S containing E. For each $\alpha \in J$, define $M_\alpha = \{a \in S : a\sigma = \alpha\}$ and $N_\alpha = \{a \in S : a\rho = \alpha\}$. It follows that both M_α and N_α are subsemigroups of S containing E_α. In addition, $M = \cup M_\alpha$ and $N = \cup N_\alpha$. More precise descriptions of M_α and N_α are given in the following proposition.

PROPOSITION 4.5. *For each $\alpha \in J$, $M_\alpha = \{a \in G_\alpha : eae = e$ for some $e \in E_\alpha\}$. Moreover, M_α is closed in G_α; that is, $M_\alpha = M_\alpha\omega \cap G_\alpha$.*

PROOF. If $a \in G_\alpha$, then $a\sigma \in H_\alpha$ so that $eae = e$ for some $e \in E_\alpha$ implies that $a\sigma = \alpha a\sigma\alpha = e\sigma a\sigma e\sigma = (eae)\sigma = e\sigma = \alpha$. Hence, $a \in M_\alpha$.

Conversely, for $a \in M_\alpha$ $a\sigma = \alpha$ so that $a \in G_\alpha$. Furthermore, $a\sigma = \alpha$ implies that $a\sigma = e\sigma$ for some $e \in E_\alpha$. So, $(ea, e) \in \sigma$. From the definition of σ, there exist $(ea)' \in V(ea)$, $e' \in V(e)$; $f, g \in E$ with $f \sim (ea)(ea)'$, $g \sim e'e$ such that $ee' \sim (ea)(ea)'ee'$ and $f(ea) = (e)g$. Thus $(egfe)a(egfe) = egfe$ with $egfe \sim e(e'e)(ea)(ea)'e = (ea)(ea)'e = [(ea)(ea)'ee']e \sim ee'e = e$.

The proof that M_α is closed in G_α is omitted as this readily follows from the definition of M_α and $M_\alpha\omega$.

It should be noted here that, in fact, $M_\alpha = E_\alpha\omega \cap G_\alpha$. To see this, first note that $E_\alpha \subset M_\alpha$ so that $E_\alpha\omega \cap G_\alpha \subset M_\alpha\omega \cap G_\alpha = M_\alpha$. On the other hand, for $a \in M_\alpha$, $a \in G_\alpha$ and $eae = e$ for some $e \in E_\alpha$. Thus, $eaea = ea$ so that $ea \in E$. Since $e, a \in G_\alpha$, it follows that $ea \in E_\alpha$. Hence, $a \in E_\alpha\omega \cap G_\alpha$.

PROPOSITION 4.6. *For each $\alpha \in J$, $N_\alpha = \{a \in G_\alpha : E_\alpha E_\beta E_\alpha \subset E_\gamma$ implies $aE_\beta E_\alpha a'E_\alpha$, $E_\alpha a'E_\alpha E_\beta a \subset E_\gamma$ for each $a' \in V(a)\}$. Moreover, N_α is closed in G_α; that is, $N_\alpha = N_\alpha\omega \cap G_\alpha$.*

PROOF. First note that $N_\alpha \subset G_\alpha$; for if $a \in N_\alpha$ then $a\rho = \alpha$ so that $a\rho\mathscr{H}\alpha$ which implies $a \in G_\alpha$ [Lemma 4.4 (ii)]. Now suppose that $a \in N_\alpha$, $E_\alpha E_\beta E_\alpha \subset E_\gamma$, and $a' \in V(a)$. Then $\alpha(a'\rho)\alpha = (a\rho)(a'\rho)(a\rho) = a\rho = \alpha$. Thus, $(aE_\beta E_\alpha a'E_\alpha)\rho = (a\rho)\beta[\alpha(a'\rho)\alpha] = \alpha\beta\alpha = \gamma$ and $(E_\alpha a'E_\alpha E_\beta a)\rho = [\alpha(a'\rho)\alpha]\beta(a\rho) = \alpha\beta\alpha = \gamma$. Consequently, $aE_\beta E_\alpha a'E_\alpha$, $E_\alpha a'E_\alpha E_\beta a \subset E_\gamma$.

Conversely, let $a \in G_\alpha$ and suppose that $E_\alpha E_\beta E_\alpha \subset E_\gamma$ implies that $aE_\beta E_\alpha a'E_\alpha$, $E_\alpha a'E_\alpha E_\beta a \subset E_\gamma$ for each $a' \in V(a)$. Let $\mu_{S/\sigma}$ be the maximum idempotent-separating congruence on S/σ. Recall that $\mu_{S/\sigma} = \rho/\sigma = \{(a\sigma, b\sigma) \in S/\sigma \times S/\sigma : (a, b) \in \rho\}$ (Reilly and Scheiblich, 1967, Theorem 3.4). Also, since S is orthodox, S/σ is orthodox so that $\mu_{S/\sigma} = \{(a\sigma, b\sigma) \in S/\sigma \times S/\sigma :$ there are inverses x of $a\sigma$, y of $b\sigma$ for which $a\sigma(\delta)x = b\sigma(\delta)y$ and $x(\delta)a\sigma = y(\delta)b\sigma$ for each $\delta \in E(S/\sigma)\}$. It will be shown that if $e \in E_\alpha$ then $(a\sigma, e\sigma) \in \mu_{S/\sigma}$. Choose $a^* \in V(a)$. Since $a \in G_\alpha$, $\alpha(a^*\sigma)\alpha \in V(a\sigma)$ [Lemma 4.4 (iii)]. Then for any $\delta \in E(S/\sigma)$, $\alpha\delta\alpha \in E(S/\sigma)$, say $\alpha\delta\alpha = \gamma$. So, since $\alpha \in V(\alpha)$, $a\sigma\delta[\alpha(a^*\sigma)\alpha] = a\sigma\delta\alpha(a^*\sigma)\alpha = \gamma = \alpha\delta\alpha$ and $[\alpha(a^*\sigma)\alpha]\delta a\sigma = \alpha(a^*\sigma)\alpha\delta a\sigma = \gamma = \alpha\delta\alpha$. Therefore, $(a\sigma, \alpha) \in \mu_{S/\sigma}$. If $e \in E_\alpha$, then $(a\sigma, e\sigma) \in \mu_{S/\sigma} = \rho/\sigma$ or equivalently $(a, e) \in \rho$ so that $a\rho = e\rho = \alpha$ and $a \in N_\alpha$.

It is easy to show that N_α is closed in G_α and thus the proof is omitted.

5. Kernels of homomorphisms

If $P = \{E_\alpha : \alpha \in J\}$ is a normal partition of E, define $\theta(P)$ to be the set of congruences on S which induce P. Theorem 4.1 and Theorem 4.2 characterize σ and ρ, the smallest and largest members of $\theta(P)$, respectively. Scheiblich (1974) completely describes $\theta(P)$ for inverse S. An analogue of these results for orthodox S follows.

First define $\mathcal{K} = \{K \subset S : M \subset K \subset N,\ K$ is a self-conjugate, regular subsemigroup of S, and for all $\alpha \in J$ $K_\alpha = K \cap G_\alpha$ is closed in $G_\alpha(K_\alpha = K_\alpha\omega \cap G_\alpha)\}$. Note that both $M, N \in \mathcal{K}$.

THEOREM 5.1. *The map* $K \to (K) = \{(a, b) \in S \times S :$ *there exist* $a' \in V(a)$, $b' \in V(b)$ *and* $\alpha, \beta, \gamma, \delta \in J$ *such that* $aa', bb'aa' \in E_\alpha$; $a'a, a'ab'b \in E_\beta$; $bb', aa'bb' \in E_\gamma$; $b'b, b'ba'a \in E_\gamma$; *and* $ab', a'b \in K\}$ *is a* $1:1$ *order preserving map of* $\mathcal{K}$ *onto* $\theta(P)$.

The proof of the theorem will proceed as follows. First, it will be shown that the map

$$K \to K\sigma^{\square} \tag{5.2}$$

is a $1:1$ order preserving map of $\mathcal{K}$ onto $\mathcal{C}_{S/\sigma}$, the set of self-conjugate, regular subsemigroups of S/σ between $E(S/\sigma)$ and $C(E(S/\sigma))$. Then, by Theorem 3.3, the map

$$K\sigma^{\square} \to (K\sigma^{\square}) \tag{5.3}$$

will be a $1:1$ order preserving map of $\mathcal{C}_{S/\sigma}$ onto the lattice of idempotent-separating congruences on S/σ. Next it will be shown that $(K\sigma^{\square}) = (K)/\sigma$ so that the map

$$K \to (K)/\sigma \tag{5.4}$$

is a $1:1$ order preserving map of $\mathcal{K}$ onto the lattice of idempotent-separating congruences on S/σ [5.2 and 5.3]. Since the map

$$(K) \to (K)/\sigma \tag{5.5}$$

is also a $1:1$ order preserving map of $\theta(P)$ onto the lattice of idempotent-separating congruences on S/σ (Reilly and Scheiblich, 1967, proof of Theorem 3.4), it follows that $K \to (K)$ is a $1:1$ order preserving map of $\mathcal{K}$ onto $\theta(P)$ [5.4 and 5.5].

PROOF. First note that $M = (E(S/\sigma))\sigma^{\square-1}$. In addition, $N = C(E(S/\sigma))\sigma^{\square-1}$ for $a \in N$ iff $a\rho = \alpha$ for some $\alpha \in J$ iff $(a, e) \in \rho$ for some $e \in E_\alpha$ iff $(a\sigma, e\sigma) \in \rho/\sigma = \mu_{S/\sigma}$ for some $e \in E_\alpha$ iff $a \in C(E(S/\sigma))$.

Let $K \in \mathscr{K}$. Since K is a regular subsemigroup of S such that $M \subset K \subset N$, $K\sigma^\square$ must be a regular subsemigroup of S/σ for which $E(S/\sigma) \subset K\sigma^\square \subset C(E(S/\sigma))$. To see that $K\sigma^\square$ is self-conjugate in S/σ, let $a\sigma \in S/\sigma$ and choose $x\sigma \in V(a\sigma)$. Then $(ax)\sigma$, $(xa)\sigma \in E(S/\sigma)$ so that ax, $xa \in M$. Hence, for any $k \in K$, $a' \in V(a)$, $a(k)(xa)a'(ax) \in aKMa'M \subset aKa'K \subset KK \subset K$ and thus $a\sigma(k\sigma)x\sigma = a\sigma(k\sigma)x\sigma(a\sigma a'\sigma a\sigma)x\sigma = (akxaa'ax)\sigma \in K\sigma^\square$. Likewise $x\sigma(k\sigma)a\sigma \in K\sigma^\square$ so that $K\sigma^\square$ is self-conjugate in S/σ. Therefore, $K\sigma^\square \in \mathscr{C}_{S/\sigma}$.

Conversely, if $H \in \mathscr{C}_{S/\sigma}$, let $K = H\sigma^{\square-1}$. Clearly, K is a self-conjugate subsemigroup of S such that $M \subset K \subset N$. To see that K is regular, let $k \in K$ and choose $k' \in V(k)$. Then $k\sigma \in H$ which is regular, so there exists $x\sigma \in V(k\sigma) \cap H$. Now $k'\sigma = (k'k)\sigma x\sigma(kk')\sigma \in E(S/\sigma)HE(S/\sigma) \subset H$ and therefore $k' \in K$. Finally, for K to be in $\mathscr{K}$, it is necessary to verify that K_α is closed in G_α. Since $K_\alpha = K \cap G_\alpha$ is a subsemigroup of K, $K_\alpha \subset K_\alpha\omega$ so that $K_\alpha \subset K_\alpha\omega \cap G_\alpha$. On the other hand, if $x \in K_\alpha\omega \cap G_\alpha$ then $kx \in K_\alpha$ for some $k \in K_\alpha$. Now $k \in K_\alpha = K \cap G_\alpha$ implies that there exists $k' \in V(k) \cap K$ and $\alpha(k'\sigma)\alpha \in V(k\sigma) \cap H_\alpha$. Thus, since $G_\alpha\sigma = H_\alpha$,

$$x\sigma = \alpha(x\sigma) = [\alpha(k'\sigma)\alpha(k\alpha)]x\sigma = \alpha(k'\alpha)\alpha(kx)\sigma \in E(S/\sigma)HE(S/\sigma)H \subset H$$

so that $x \in K$. But $x \in G_\alpha$ which gives that $x \in K_\alpha$.

So far it has been shown that $K \to K\sigma^\square$ is a map of $\mathscr{K}$ onto $\mathscr{C}_{S/\sigma}$. Since this map is clearly order preserving, it only remains to show that the map is 1 : 1. So, let $K, L \in \mathscr{K}$ with $K\sigma^\square = L\sigma^\square$. Choose $k \in K$. Then $k \in K_\alpha = K \cap G_\alpha$ for some $\alpha \in J$. So, $k\sigma \in K\sigma^\square \cap H_\alpha$. Since $K\sigma^\square = L\sigma^\square$, $k\sigma = l\sigma$ for some $l \in L$. Then $l\sigma = k\sigma \in H_\alpha$ so that $l \in L \cap G_\alpha = L_\alpha$ and thus for each $l' \in V(l)\, \alpha(l'\sigma)\alpha \in V(l\sigma) \cap H_\alpha$ [Lemma 4.4 (iii)]. Let $e \in E_\alpha$. Then $(el'ek)\sigma = \alpha(l'\sigma)\alpha k\sigma = \alpha(l'\sigma)\alpha l\sigma = \alpha$ so that $el'ek \in M_\alpha$. Therefore, $lel'ek \in L_\alpha M_\alpha \subset L_\alpha$ and $(lel'e)\sigma = l\sigma(\alpha l'\sigma\alpha) = \alpha$ so that $lel'e \in M_\alpha \subset L_\alpha$ giving that $k \in L_\alpha\omega$. Since $k \in K_\alpha = K \cap G_\alpha$, $k \in L_\alpha\omega \cap G_\alpha = L_\alpha \subset L$. In a similar fashion, $L \subset K$. Thus $K = L$.

By virtue of the comments on the proof of this theorem, it only remains to show that $(K\sigma^\square) = (K)/\sigma$ for each $K \in \mathscr{K}$ in order to complete the proof of this theorem. So, let $K \in \mathscr{K}$. For $(a\sigma, b\sigma) \in (K\sigma^\square)$, $(a\sigma, b\sigma) \in \mathscr{H}_{S/\sigma}$ and $a\sigma(b\sigma)'$, $(a\sigma)'b\sigma \in K\sigma^\square$ for each $(b\sigma)' \in V(b\sigma)$, $(a\sigma)' \in V(a\sigma)$ [Theorem 3.3]. Let $a' \in V(a)$, $b' \in V(b)$. Then $(aa')\sigma\mathscr{R}a\sigma\mathscr{R}b\sigma\mathscr{R}(bb')\sigma$ so that $(aa')\sigma = (bb')\sigma(aa')\sigma = (bb'aa')\sigma$ implying that $aa' \sim bb'aa'$ and $(bb')\sigma = (aa')\sigma(bb')\sigma = (aa'bb')\sigma$ implying that $bb' \sim aa'bb'$. Likewise

$(a'a)\sigma\mathscr{L}a\sigma\mathscr{L}b\sigma\mathscr{L}(b'b)\sigma$ so that $a'a \sim a'ab'b$ and $b'b \sim b'ba'a$. Also $(ab')\sigma = a\sigma b'\sigma$, $(a'b)\sigma = a'\sigma b\sigma \in K\sigma^{\square}$ so that ab', $a'b \in K$. Therefore, $(a, b) \in (K)$ or equivalently $(a\sigma, b\sigma) \in (K)/\sigma$.

Conversely, if $(a\sigma, b\sigma) \in (K)/\sigma$, then $(a, b) \in (K)$ so that there are inverses a' of a and b' of b such that $aa' \sim bb'aa'$, $a'a \sim a'ab'b$, $bb' \sim aa'bb'$, $b'b \sim b'ba'a$, and ab', $a'b \in K$. One can easily verify that $aa' \sim bb'aa'$, $bb' \sim aa'bb'$ imply $a\sigma\mathscr{R}b\sigma$ and $a'a \sim a'ab'b$, $b'b \sim b'ba'a$ imply $a\sigma\mathscr{L}b\sigma$. Thus, $a\sigma\mathscr{H}b\sigma$. Let $a' \in V(a)$. Then $a'\sigma \in V(a\sigma)$ so that $a\sigma\mathscr{H}b\sigma$ implies the existence of $y\sigma \in V(b\sigma) \cap H_{a'\sigma}$. So, $a\sigma a'\sigma = b\sigma y\sigma$ and $a'\sigma a\sigma = y\sigma b\sigma$. Furthermore, $a'\sigma b\sigma = (a'b)\sigma \in K\sigma^{\square}$ and $a\sigma y\sigma = a\sigma(a'a)\sigma t\sigma = a\sigma(a'ab'b)\sigma y\sigma = (aa')\sigma(ab')\sigma(by)\sigma \in E(S/\sigma)K\sigma^{\square}E(S/\sigma) \subsetneq K\sigma^{\square}$. Therefore, $(a\sigma, b\sigma) \in (K\sigma^{\square})$.

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