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RESEARCH ARTICLE

A characterization of the groups $PSL_n(q)$ and $PSU_n(q)$ by their 2-fusion systems, q odd

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Abstract
Let q be a nontrivial odd prime power, and let $n \geq 2$ be a natural number with $(n, q) \neq (2, 3)$. We characterize the groups $PSL_n(q)$ and $PSU_n(q)$ by their 2-fusion systems. This contributes to a programme of Aschbacher aiming at a simplified proof of the classification of finite simple groups.

Contents

1. Introduction

The classification of finite simple groups (CFSG) is one of the greatest achievements in the history of mathematics. Its proof required around 15,000 pages and spreads out over many hundred articles in various journals. Many mathematicians from all over the world were involved in the proof, whose final steps were published in 2004 by Aschbacher and Smith, after it was prematurely announced as finished already in 1983. Because of its extreme length, a simplified and shortened proof of the CFSG would be very valuable. There are three programmes working towards this goal: the Gorenstein–Lyons–Solomon programme (see [26]), the Meierfrankenfeld–Stellmacher–Stroth programme (see [43]) and Aschbacher's programme.

The goal of Aschbacher's programme is to obtain a new proof of the CFSG by using *fusion systems*. The standard examples of fusion systems are the fusion categories of finite groups over p-subgroups (p a prime). If G is a finite group and S is a p-subgroup of G for some prime p, then the *fusion category* of G over S is defined to be the category $\mathcal{F}_S(G)$ given as follows: The objects of $\mathcal{F}_S(G)$ are precisely the subgroups of S, the morphisms in $\mathcal{F}_S(G)$ are precisely the group homomorphisms between subgroups of S induced by conjugation in G and the composition of morphisms in $\mathcal{F}_S(G)$ is the usual composition of group homomorphisms. Abstract fusion systems are a generalization of this concept. A fusion system over a finite p-group S, where p is a prime, is a category whose objects are the subgroups of S and whose morphisms behave as if they are induced by conjugation inside a finite group containing S as a p-subgroup. For the precise definition, we refer to [10, Part I, Definition 2.1]. A fusion system is called *saturated* if it satisfies certain axioms motivated by properties of fusion categories of finite groups over Sylow subgroups (see [10, Part I, Definition 2.2]). If G is a finite group and $S_1, S_2 \in \mathrm{Syl}_p(G)$ for some prime p, then $\mathcal{F}_{S_1}(G)$ and $\mathcal{F}_{S_2}(G)$ are easily seen to be isomorphic (in the sense of [11, p. 560]). Given a finite group G, a prime p and a Sylow p-subgroup S of G, we refer to $\mathcal{F}_S(G)$ as the *p-fusion system* of G.

Originally considered by the representation theorist Puig, fusion systems have become an object of active research in finite group theory, representation theory and algebraic topology. It has always been a problem of great interest in the theory of fusion systems to translate group-theoretic concepts into suitable concepts for fusion systems. For example, there is a notion of normalizers and centralizers of p-subgroups in fusion systems, a notion of the center of a fusion system, a notion of factor systems, a notion of normal subsystems of saturated fusion systems and a notion of simple saturated fusion systems (see [10, Parts I and II]). Roughly speaking, Aschbacher's programme consists of the following two steps.

1. Classify the simple saturated fusion systems on finite 2-groups. Use the original proof of the CFSG as a 'template'.
2. Use the first step to give a new and simplified proof of the CFSG.

There is the hope that several steps of the original proof of the CFSG become easier when working with fusion systems. For example, in the original proof of the CFSG, the study of centralizers of involutions plays an important role. The $2'$-cores of the involution centralizers, i.e., their largest normal odd order subgroups, cause serious difficulties and are obstructions to many arguments. Such difficulties are not present in fusion systems since cores do not exist in fusion systems. This is suggested by the well-known fact that the 2-fusion system of a finite group G is isomorphic to the 2-fusion system of

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$G/O(G)$, where $O(G)$ denotes the $2'$-core of G. For an outline of and recent progress on Aschbacher's programme, we refer to [7].

So far, Aschbacher's programme has focused mainly on Step 1, while not much has been done on Step 2. An important part of Step 2 is to identify finite simple groups from their 2-fusion systems. The present paper contributes to Step 2 of Aschbacher's programme by characterizing the finite simple groups $PSL_n(q)$ and $PSU_n(q)$ in terms of their 2-fusion systems, where $n \geq 2$ and where q is a nontrivial odd prime power with $(n,q) \neq (2,3)$.

In order to state our results, we introduce some notation and recall some definitions. Let G be a finite group. A *component* of G is a quasisimple subnormal subgroup of G, and a 2-*component* of G is a perfect subnormal subgroup L of G such that $L/O(L)$ is quasisimple. The natural homomorphism $G \to G/O(G)$ induces a one-to-one correspondence between the set of 2-components of G and the set of components of $G/O(G)$ (see [27, Proposition 4.7]). We use $Z^*(G)$ to denote the full preimage of the center $Z(G/O(G))$ in G. In Step 2 of Aschbacher's programme, one may assume that a finite group G is a minimal counterexample to the CFSG. Such a group G has the following property.

$$\text{Whenever } x \in G \text{ is an involution and } J \text{is a 2-component of } C_G(x), \text{ then } J/Z^*(J) \text{ is a known finite simple group.} \tag{$\mathcal{CK}$}$$

By a known finite simple group, we mean a finite simple group appearing in the statement of the CFSG.

For each integer $n \neq 0$, we use n_2 to denote the 2-part of n, i.e., the largest power of 2 dividing n. Given odd integers a, b with $|a|, |b| > 1$, we write $a \sim b$ provided that $(a-1)_2 = (b-1)_2$ and $(a+1)_2 = (b+1)_2$. If q is a nontrivial prime power and if n is a positive integer, then we write $PSL_n^+(q)$ for $PSL_n(q)$ and $PSL_n^-(q)$ for $PSU_n(q)$. With this notation, we can now state our main results.

Theorem A. *Let q be a nontrivial odd prime power, and let $n \geq 2$ be a natural number. Let G be a finite simple group. Suppose that G satisfies ($\mathcal{CK}$) if $n \geq 6$. Then the 2-fusion system of G is isomorphic to the 2-fusion system of $PSL_n(q)$ if and only if one of the following holds:*

(i) *$G \cong PSL_n^{\varepsilon}(q^*)$ for some nontrivial odd prime power q^* and some $\varepsilon \in \{+,-\}$ with $\varepsilon q^* \sim q$;*
(ii) *$n = 2$, $|PSL_2(q)|_2 = 8$, and $G \cong A_7$;*
(iii) *$n = 3$, $(q+1)_2 = 4$, and $G \cong M_{11}$.*

Our second main result is an extension of Theorem A. In order to state it, we briefly mention some concepts from the local theory of fusion systems. Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S for some prime p, and let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$. In [6, Chapter 6], Aschbacher introduced a subgroup $C_S(\mathcal{E})$ of S, which plays the role of the centralizer of $\mathcal{E}$ in S. In [6, Chapter 9], he defined a normal subsystem $F^*(\mathcal{F})$ of $\mathcal{F}$, called the *generalized Fitting subsystem* of $\mathcal{F}$, and proved that $C_S(F^*(\mathcal{F})) = Z(F^*(\mathcal{F}))$, where the latter denotes the center of $F^*(\mathcal{F})$.

Theorem B. *Let q be a nontrivial odd prime power, and let $n \geq 2$ be a natural number. If $n = 2$, suppose that $q \equiv 1$ or $7 \bmod 8$. Let G be a finite simple group, and let S be a Sylow 2-subgroup of G. Suppose that $\mathcal{F}_S(G)$ has a normal subsystem $\mathcal{E}$ on a subgroup T of S such that $\mathcal{E}$ is isomorphic to the 2-fusion system of $PSL_n(q)$ and such that $C_S(\mathcal{E}) = 1$. Then $\mathcal{F}_S(G)$ is isomorphic to the 2-fusion system of $PSL_n(q)$. In particular, if $n \leq 5$ or if G satisfies ($\mathcal{CK}$), then one of the properties (i)–(iii) from Theorem A holds.*

Corollary C. *Let q be a nontrivial odd prime power, and let $n \geq 2$ be a natural number. If $n = 2$, suppose that $q \equiv 1$ or $7 \bmod 8$. Let G be a finite simple group, and let S be a Sylow 2-subgroup of G. Suppose that $F^*(\mathcal{F}_S(G))$ is isomorphic to the 2-fusion system of $PSL_n(q)$. Then $\mathcal{F}_S(G)$ is isomorphic*

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to the 2-fusion system of $PSL_n(q)$. In particular, if $n \leq 5$ or if G satisfies ($\mathcal{CK}$), then one of the properties (i)–(iii) from Theorem A holds.

Plan of the Paper

In Sections 2 and 3, we collect several results needed for the proofs of our main results. Preliminary results on abstract finite groups and abstract fusion systems are proved in Section 2. Section 3 presents some results on linear and unitary groups over finite fields, mainly focusing on 2-local properties and on the automorphisms of these groups.

In Section 4, we will verify Theorem A for the case $n \leq 5$. Our proofs strongly depend on work of Gorenstein and Walter [30] (for $n = 2$), on work of Alperin, Brauer and Gorenstein [1], [2] (for $n = 3$) and on work of Mason [40], [41], [42] (for $n = 4$ and $n = 5$).

For $n \geq 6$, we will prove Theorem A by induction over n. In order to do so, we will consider a finite group G realizing the 2-fusion system of $PSL_n(q)$, where q is a nontrivial odd prime power and where $n \geq 6$ is a natural number such that Theorem A is true with m instead of n for any natural number m with $6 \leq m < n$. We will also assume that $O(G) = 1$ and that G satisfies ($\mathcal{CK}$). To prove that Theorem A is satisfied for the natural number n, we will prove the existence of a normal subgroup G_0 of G such that G_0 is isomorphic to a nontrivial quotient of $SL_n^{\varepsilon}(q^*)$ for some nontrivial odd prime power q^* and some $\varepsilon \in \{+, -\}$ with $\varepsilon q^* \sim q$. This will happen in Sections 5-8.

In Section 5, we will introduce some notation and prove some preliminary lemmas. Section 6 describes the 2-components of the centralizers of involutions of G. In Section 7, we will use signalizer functor methods to describe the components of the centralizers of certain involutions of G. This will be used in Section 8 to construct the subgroup G_0 of G. One of the main tools here will be a version of the Curtis–Tits theorem [29, Chapter 13, Theorem 1.4] and a related theorem of Phan reproved by Bennett and Shpectorov in [13].

Finally, in Section 9, we will give a full proof of Theorem A (basically summarizing Sections 4–8), and we will prove Theorem B and Corollary C.

Notation and Terminology

Our notation and terminology are fairly standard. The reader is referred to [23], [27], [37] for unfamiliar definitions on groups and to [10], [18] for unfamiliar definitions on fusion systems.

However, we shall now explain some particularly important notation and definitions (before stating our main results, we already introduced some other important definitions).

Given a map $\alpha : A \longrightarrow B$ and an element or a subset X of A, we write X^{α} for the image of X under α. Also, if $C \subseteq A$ and $D \subseteq B$ such that $C^{\alpha} \subseteq D$, we use $\alpha|_{C,D}$ to denote the map $C \longrightarrow D, c \mapsto c^{\alpha}$. Given two maps $\alpha : A \longrightarrow B$ and $\beta : B \longrightarrow C$, we write $\alpha\beta$ for the map $A \longrightarrow C, a \mapsto (a^{\alpha})^{\beta}$.

Sometimes, we will interpret the symbols + and − as the integers 1 and −1, respectively. For example, if n is an integer and if ε is assumed to be an element of $\{+, -\}$, then $n \equiv \varepsilon \mod 4$ shall express that $n \equiv 1 \mod 4$ if $\varepsilon = +$ and that $n \equiv -1 \mod 4$ if $\varepsilon = -$.

Let G be a finite group. We write $G^{\#}$ for the set of nonidentity elements of G. Given an element g of G and an element or a subset X of G, we write X^g for $g^{-1}Xg$. The inner automorphism $G \longrightarrow G, x \mapsto x^g$ is denoted by c_g. For subgroups Q and H of G, we write $\mathrm{Aut}_H(Q)$ for the subgroup of $\mathrm{Aut}(Q)$ consisting of all automorphisms of Q of the form $c_h|_{Q,Q}$, where $h \in N_H(Q)$.

We write $E(G)$ for the subgroup of G generated by the components of G and $L_{2'}(G)$ for the subgroup of G generated by the 2-components of G. We say that G is *core-free* if $O(G) = 1$. If G is core-free and if L is a subnormal subgroup of G, then L is said to be a *solvable* 2*-component* of G if $L \cong SL_2(3)$ or $PSL_2(3)$.

Let n be a natural number. Then we use E_{2^n} to denote an elementary abelian 2-group of order 2^n, and we say that n is the *rank* of E_{2^n}. The maximal rank of an elementary abelian 2-subgroup of a finite 2-group S is said to be the *rank* of S. It is denoted by $m(S)$.

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Now let p be a prime, and let $\mathcal{F}$ be a fusion system on a finite p-group S. Then S is said to be the *Sylow group* of $\mathcal{F}$, and $\mathcal{F}$ is said to be *nilpotent* if $\mathcal{F} = \mathcal{F}_S(S)$. Given a fusion system $\mathcal{F}_1$ on a finite p-group S_1, we say that $\mathcal{F}$ and $\mathcal{F}_1$ are *isomorphic* if there is a group isomorphism $\varphi : S \to S_1$ such that

$$\mathrm{Hom}_{\mathcal{F}_1}(Q^\varphi, R^\varphi) = \{(\varphi^{-1}|_{Q^\varphi,Q})\psi(\varphi|_{R,R^\varphi}) \mid \psi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)\}$$

for all $Q, R \leq S$. In this case, we say that φ *induces an isomorphism* from $\mathcal{F}$ to $\mathcal{F}_1$. Let T be a strongly $\mathcal{F}$-closed subgroup of S, i.e., for any subgroup P of T and any $\alpha \in \mathrm{Hom}_{\mathcal{F}}(P,S)$, we have $P^\alpha \leq T$. If Q and R are subgroups of S containing T and if $\alpha : Q \to R$ is a morphism in $\mathcal{F}$, we write α/T for the group homomorphism $Q/T \to R/T$ induced by α. The fusion system $\mathcal{F}/T$ on S/T with $\mathrm{Hom}_{\mathcal{F}/T}(Q/T, R/T) = \{\alpha/T \mid \alpha \in \mathrm{Hom}_{\mathcal{F}}(Q,R)\}$ for all $Q, R \leq S$ containing T is said to be the *factor system* of $\mathcal{F}$ modulo T.

Suppose now that $\mathcal{F}$ is saturated. We write $\mathfrak{foc}(\mathcal{F})$ for the focal subgroup of $\mathcal{F}$ and $\mathfrak{hnp}(\mathcal{F})$ for the hyperfocal subgroup of $\mathcal{F}$. We say that $\mathcal{F}$ is *quasisimple* if $\mathcal{F}/Z(\mathcal{F})$ is simple and $\mathfrak{foc}(\mathcal{F}) = S$. A *component* of $\mathcal{F}$ is a subnormal quasisimple subsystem of $\mathcal{F}$. Given a normal subsystem $\mathcal{E}$ of S and a subgroup R of S, we write $\mathcal{E}R$ for the product of $\mathcal{E}$ and R, as defined in [6, Chapter 8].

2. Preliminaries on finite groups and fusion systems

In this section, we present some general results on finite groups and fusion systems.

2.1. Preliminaries on finite groups

Lemma 2.1 [37, 3.2.8]. *Let G be a finite group, and let N be a normal p'-subgroup of G for some prime p. Set $\overline{G} := G/N$. If R is a p-subgroup of G, then we have $N_{\overline{G}}(\overline{R}) = \overline{N_G(R)}$ and $C_{\overline{G}}(\overline{R}) = \overline{C_G(R)}$.*

Corollary 2.2. *Let G be a finite group, and let N be a normal p'-subgroup of G for some prime p. Set $\overline{G} := G/N$. If $x \in G$ has order p, then we have $C_{\overline{G}}(\bar{x}) = \overline{C_G(x)}$.*

Lemma 2.3. *Let G be a finite group, and let Z be a cyclic central subgroup of G. Then each E_8-subgroup of G/Z has an involution which is the image of an involution of G.*

Proof. Let $Z \leq E \leq G$ such that $E/Z \cong E_8$. Let R be a Sylow 2-subgroup of E. Then $E = RZ$. It suffices to show that R has an involution not lying in $R \cap Z$. Assume that any involution of R is an element of $R \cap Z$. Then R has a unique involution since Z is cyclic. We have $R/(R \cap Z) \cong RZ/Z = E/Z \cong E_8$, and so R is not cyclic. Applying [37, 5.3.7], we conclude that R is generalized quaternion. In particular, $Z(R)$ has order 2, and so we have $R \cap Z = Z(R)$. Since R is a generalized quaternion group, $R/Z(R)$ is dihedral. In particular, $E/Z \cong R/(R \cap Z) = R/Z(R) \ncong E_8$. This contradiction shows that R has an involution not lying in $R \cap Z$, as required. □

The following proposition is well-known. We include a proof since we could not find a reference in which it appears in the form given here.

Proposition 2.4. *Let G be a finite group, and let N be a normal subgroup of G with odd order. If L is a 2-component of G, then LN/N is a 2-component of G/N. The map from the set of 2-components of G to the set of 2-components of G/N sending each 2-component L of G to LN/N is a bijection. Moreover, if $N \leq K \leq G$ and K/N is a 2-component of G/N, then $O^{2'}(K)$ is the associated 2-component of G.*

Proof. Let L be a 2-component of G. Hence, L is a perfect subnormal subgroup of G such that $L/O(L)$ is quasisimple. Clearly, LN/N is perfect and subnormal in G/N. Also, we have $(LN/N)/O(LN/N) \cong L/O(L)$, and so $(LN/N)/O(LN/N)$ is quasisimple. It follows that LN/N is a 2-component of G/N.

Let $N \leq K \leq G$ such that K/N is a 2-component of G/N. In order to prove the second statement of the proposition, it is enough to show that there is precisely one 2-component L of G such that $LN/N = K/N$.

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Since K/N is subnormal in G/N, we have that K is subnormal in G. Therefore, $L := O^{2'}(K)$ is subnormal in G. Since $O^{2'}(K/N) = K/N$, we have that $K/N = LN/N$. Clearly, $O^{2'}(L) = L$. We have $L/O(L) \cong (LN/N)/O(LN/N) = (K/N)/O(K/N)$, and so $L/O(L)$ is quasisimple. Applying [27, Lemma 4.8], we conclude that L is a 2-component of G.

Now let L_0 be a 2-component of G such that $K/N = L_0N/N$. Then $K = L_0N$. In particular, L_0 is a subgroup of K with odd index in K. Since L_0 is subnormal in G, we have that L_0 is subnormal in K. Applying [12, Lemma 1.1.11], we conclude that $L_0 = O^{2'}(L_0) = O^{2'}(K) = L$. The proof of the second statement of the proposition is now complete. The third statement also follows from the above arguments. $\square$

Lemma 2.5. *Let G be a finite group, and let n be a positive integer. Assume that $L_1, \ldots, L_n$ are the distinct 2-components of G, and assume that $L_i \trianglelefteq G$ for all $1 \leq i \leq n$. Let x be a 2-element of G, and let L be a 2-component of $C_G(x)$. Then L is a 2-component of $C_{L_i}(x)$ for some $1 \leq i \leq n$.*

Proof. For each $1 \leq i \leq n$, let $\mathfrak{L}_i$ denote the set of 2-components of $C_{L_i}(x)$, and let $\mathfrak{L} := \mathfrak{L}_1 \cup \ldots \mathfrak{L}_n$. It suffices to show that $L \in \mathfrak{L}$.

By [31, Corollary 3.2], we have $L_{2'}(C_G(x)) = L_{2'}(C_{L_{2'}(G)}(x))$, and by [31, Lemma 2.18 (iii)], we have $L_{2'}(C_{L_{2'}(G)}(x)) = \prod_{i=1}^n L_{2'}(C_{L_i}(x))$. Thus $L_{2'}(C_G(x)) = \langle \mathfrak{L} \rangle$. Set $\overline{C_G(x)} := C_G(x)/O(C_G(x))$.

Assume that $L \notin \mathfrak{L}$. Let J be an element of $\mathfrak{L}$. Since $L \neq J$ and since L and J are 2-components of $C_G(x)$, we have $\overline{L} \neq \overline{J}$ by Proposition 2.4. Also, since $\overline{L}$ and $\overline{J}$ are components of $\overline{C_G(x)}$, we have $[\overline{L}, \overline{J}] = 1$ by [37, 6.5.3]. Since $\overline{L} \in E(\overline{C_G(x)}) = \overline{L_{2'}(C_G(x))} = \overline{\langle \mathfrak{L} \rangle} = \langle \overline{J} \mid J \in \mathfrak{L} \rangle$, it follows that $\overline{L}$ lies in the center of $E(\overline{C_G(x)})$. This is impossible since $\overline{L}$ is nontrivial and perfect. So we have $L \in \mathfrak{L}$. $\square$

The concepts introduced by the following two definitions will play a crucial role in the proof of Theorem A (see [31] for a detailed study of these concepts).

Definition 2.6. Let G be a finite group, k be a positive integer and A be an elementary abelian 2-subgroup of G.

(i) For each nontrivial elementary abelian 2-subgroup E of G, we define

$$\Delta_G(E) := \bigcap_{a \in E^\#} O(C_G(a)).$$

(ii) We say that G is *k-balanced with respect to A* if, whenever E is a subgroup of A of rank k and a is a nontrivial element of A, we have

$$\Delta_G(E) \cap C_G(a) \leq O(C_G(a)).$$

(iii) We say that G is *k-balanced* if, whenever E is an elementary abelian 2-subgroup of G of rank k and a is an involution of G centralizing E, we have

$$\Delta_G(E) \cap C_G(a) \leq O(C_G(a)).$$

(iv) By saying that G is *balanced* (respectively, *balanced with respect to A*), we mean that G is 1-balanced (respectively, 1-balanced with respect to A).

Definition 2.7. Let G be a finite quasisimple group, and let k be a positive integer. Then G is said to be *locally k-balanced* if whenever H is a subgroup of $\mathrm{Aut}(G)$ containing $\mathrm{Inn}(G)$, we have

$$\Delta_H(E) = 1$$

for any elementary abelian 2-subgroup E of H of rank k. We say that G is *locally balanced* if G is locally 1-balanced.

We need the following proposition for the proof of Theorem A. It includes [31, Theorem 6.10] and some additional statements, which should be also known. We include a proof for the convenience of the reader.

Proposition 2.8. *Let k be a positive integer, and let G be a finite group. For each elementary abelian 2-subgroup A of G of rank at least $k+1$, let*

$$W_A := \langle \Delta_G(E) \mid E \leq A, m(E) = k \rangle.$$

Then, for any elementary abelian 2-subgroup A of G of rank at least $k+1$, the following hold:

(i) $(W_A)^g = W_{A^g}$ *for all $g \in G$.*
(ii) *Suppose that A has rank at least $k+2$ and that G is k-balanced with respect to A. Then W_A has odd order. Moreover, if A_0 is a subgroup of A of rank at least $k+1$, then we have $W_A = W_{A_0}$ and $N_G(A_0) \leq N_G(W_A)$.*

In order to prove Proposition 2.8, we need the following theorem.

Theorem 2.9 [31, Theorem 6.9]. *Let k be a positive integer, G be a finite group and A be an elementary abelian 2-subgroup of G of rank at least $k+2$. Suppose that G is k-balanced with respect to A. Then we obtain an A-signalizer functor on G (in the sense of [24, Definition 4.37]) by defining*

$$\theta(C_G(a)) := \langle \Delta_G(E) \cap C_G(a) : \; E \leq A, m(E) = k \rangle$$

for each $a \in A^\#$.

We also need the following lemma.

Lemma 2.10. *Let the hypothesis and notation be as in Theorem 2.9. Suppose that A_0 is subgroup of A of rank $k+1$. Then we have*

$$\theta(G, A) := \langle \theta(C_G(a)) \mid a \in A^\# \rangle = \langle \Delta_G(E) \mid E \leq A_0, m(E) = k \rangle =: W_{A_0}.$$

Proof. To prove this, we follow arguments found on pp. 40–41 of [40].

Since θ is an A-signalizer functor on G, $\theta(C_G(a))$ is A-invariant and in particular A_0-invariant for each $a \in A^\#$. Consequently, $\theta(G, A)$ is A_0-invariant. By the solvable signalizer functor theorem [37, 11.3.2], θ is complete (in the sense of [24, Definition 4.37]). In particular, $\theta(G, A)$ has odd order. Applying [27, Proposition 11.23], we conclude that

$$\theta(G, A) = \langle C_{\theta(G,A)}(E) \mid E \leq A_0, m(E) = k \rangle.$$

Since θ is complete, we have $C_{\theta(G,A)}(a) = \theta(C_G(a))$ for each $a \in A^\#$. By definition of θ and since G is k-balanced with respect to A, we have $\theta(C_G(a)) \leq O(C_G(a))$ for each $a \in A^\#$. So, if E is a subgroup of A_0 of rank k, then

$$C_{\theta(G,A)}(E) = \bigcap_{a \in E^\#} C_{\theta(G,A)}(a) = \bigcap_{a \in E^\#} \theta(C_G(a)) \leq \bigcap_{a \in E^\#} O(C_G(a)) = \Delta_G(E).$$

It follows that $\theta(G, A) \leq W_{A_0}$.

Let $E \leq A_0$ with $m(E) = k$. Clearly, $\Delta_G(E)$ is A-invariant. As a consequence of [27, Proposition 11.23], we have

$$\Delta_G(E) = \langle \Delta_G(E) \cap C_G(a) \mid a \in A^\# \rangle.$$

By definition of θ, we have $\Delta_G(E) \cap C_G(a) \leq \theta(C_G(a))$ for each $a \in A^\#$. It follows that $\Delta_G(E) \leq \theta(G, A)$. Consequently, $W_{A_0} \leq \theta(G, A)$. □

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Proof of Proposition 2.8. It is straightforward to verify (i).

To verify (ii), let A be an elementary abelian 2-subgroup of G of rank at least $k+2$ such that G is k-balanced with respect to A. Let θ be the A-signalizer functor on G given by Theorem 2.9, and let $\theta(G,A) := \langle \theta(C_G(a)) \mid a \in A^{\#} \rangle$. As a consequence of Lemma 2.10, we have $\theta(G,A) = W_A$. By the proof of Lemma 2.10, $W_A = \theta(G,A)$ has odd order.

Now let A_0 be a subgroup of A of rank at least $k+1$. By Lemma 2.10, $W_A = \theta(G,A) \leq W_{A_0} \leq W_A$, and so $W_A = W_{A_0}$. Finally, if $g \in N_G(A_0)$, then $(W_A)^g = (W_{A_0})^g = W_{(A_0)^g} = W_{A_0} = W_A$, and hence, $N_G(A_0) \leq N_G(W_A)$. □

2.2. Preliminaries on fusion systems

Lemma 2.11. *Let p be a prime, G be a finite group, N be a normal subgroup of G and $S \in \mathrm{Syl}_p(G)$. Then the canonical group isomorphism $S/(S\cap N) \to SN/N$ induces an isomorphism from $\mathcal{F}_S(G)/(S\cap N)$ to $\mathcal{F}_{SN/N}(G/N)$.*

Proof. Let φ denote the canonical group isomorphism $S/(S\cap N) \to SN/N$. Let P and Q be two subgroups of S such that $S\cap N$ is contained in both P and Q. Set $\widetilde{P} := P/(S\cap N)$, $\widetilde{Q} := Q/(S\cap N)$, $\overline{P} := PN/N$ and $\overline{Q} := QN/N$. For any $g \in G$, let $\overline{g} := gN$. Moreover, define $\widetilde{\mathcal{F}} := \mathcal{F}_S(G)/(S\cap N)$ and $\overline{\mathcal{F}} := \mathcal{F}_{SN/N}(G/N)$. It is enough to show that

$$\mathrm{Hom}_{\overline{\mathcal{F}}}(\overline{P},\overline{Q}) = \{(\varphi^{-1}|_{\overline{P},\widetilde{P}})\alpha(\varphi|_{\widetilde{Q},\overline{Q}}) \mid \alpha \in \mathrm{Hom}_{\widetilde{\mathcal{F}}}(\widetilde{P},\widetilde{Q})\}.$$

Let $\alpha \in \mathrm{Hom}_{\widetilde{\mathcal{F}}}(\widetilde{P},\widetilde{Q})$. Then there exists $g \in G$ with $P^g \leq Q$ and $\alpha = (c_g|_{P,Q})/(S\cap N)$. By a direct calculation, $(\varphi^{-1}|_{\overline{P},\widetilde{P}})\alpha(\varphi|_{\widetilde{Q},\overline{Q}}) = c_{\overline{g}}|_{\overline{P},\overline{Q}} \in \mathrm{Hom}_{\overline{\mathcal{F}}}(\overline{P},\overline{Q})$.

Now let $\overline{\alpha} \in \mathrm{Hom}_{\overline{\mathcal{F}}}(\overline{P},\overline{Q})$. Then there exists $g \in G$ with $\overline{P}^{\overline{g}} \leq \overline{Q}$ and $\overline{\alpha} = c_{\overline{g}}|_{\overline{P},\overline{Q}}$. Clearly, $P^g \leq QN$. Since $S\cap N \leq Q$, we have that Q is a Sylow p-subgroup of QN. Since P^g is a p-subgroup of QN, it follows that there exists an element $n \in N$ with $P^{gn} \leq Q$. Set $\alpha := (c_{gn}|_{P,Q})/(S\cap N)$. Then a direct calculation shows that $\overline{\alpha} = (\varphi^{-1}|_{\overline{P},\widetilde{P}})\alpha(\varphi|_{\widetilde{Q},\overline{Q}})$. □

Corollary 2.12 [10, Part II, Exercise 2.1]. *Let p be a prime, G be a finite group and $S \in \mathrm{Syl}_p(G)$. Then the canonical group isomorphism $S \to \overline{S} := SO_{p'}(G)/O_{p'}(G)$ induces an isomorphism from $\mathcal{F}_S(G)$ to $\mathcal{F}_{\overline{S}}(G/O_{p'}(G))$.*

Lemma 2.13. *Let G be a finite group and $S \in \mathrm{Syl}_2(G)$. Then $Z(\mathcal{F}_S(G)) = S \cap Z^*(G)$. In particular, if $Z^*(G)$ is 2-closed, then $Z(\mathcal{F}_S(G)) = S\cap Z(G)$.*

Proof. By Glauberman's Z^*-Theorem, more precisely by [22, Corollary 1], we have $Z(\mathcal{F}_S(G)) = S\cap Z^*(G)$. Assume now that $Z^*(G)$ is 2-closed, and let $S_0 := S\cap Z^*(G)$. Then $S_0 \trianglelefteq G$ and hence $[S_0,G] \leq S_0 \cap [Z^*(G),G] \leq S_0 \cap O(G) = 1$. Thus, $Z(\mathcal{F}_S(G)) = S_0 = S\cap Z(G)$. □

Lemma 2.14. *Let K_1 and K_2 be two quasisimple finite groups. If the 2-fusion systems of K_1 and K_2 are isomorphic, then the 2-fusion systems of $K_1/Z(K_1)$ and $K_2/Z(K_2)$ are isomorphic.*

Proof. Suppose that the 2-fusion systems of K_1 and K_2 are isomorphic. Let S_i be a Sylow 2-subgroup of K_i and $\mathcal{F}_i := \mathcal{F}_{S_i}(K_i)$ for $i \in \{1,2\}$. Since K_1 and K_2 are quasisimple, we have $Z^*(K_i) = Z(K_i)$ for $i \in \{1,2\}$. So, by Lemma 2.13, we have $Z(\mathcal{F}_i) = S_i \cap Z(K_i)$ for $i \in \{1,2\}$. Since $\mathcal{F}_1 \cong \mathcal{F}_2$, it follows that

$$\mathcal{F}_1/(S_1\cap Z(K_1)) = \mathcal{F}_1/Z(\mathcal{F}_1) \cong \mathcal{F}_2/Z(\mathcal{F}_2) = \mathcal{F}_2/(S_2\cap Z(K_2)).$$

Applying Lemma 2.11, we may conclude that the 2-fusion system of $K_1/Z(K_1)$ is isomorphic to the 2-fusion system of $K_2/Z(K_2)$. □

Lemma 2.15. *Let S be a finite 2-group, and let A and B be normal subgroups of S such that S is the internal direct product of A and B. Suppose that $A \cong Q_8$. Let $\mathcal{F}$ be a (not necessarily saturated) fusion*

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system on S. Assume that A and B are strongly $\mathcal{F}$-closed and that there is an automorphism $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ such that $\alpha|_{A,A}$ has order 3, while $\alpha|_{B,B} = \mathrm{id}_B$. Then each strongly $\mathcal{F}$-closed subgroup of S contains or centralizes A.

Proof. Let C be a strongly $\mathcal{F}$-closed subgroup of S not containing A. Our task is to show that C centralizes A.

Since A and C are strongly $\mathcal{F}$-closed, we have that $A \cap C$ is strongly $\mathcal{F}$-closed. In particular, α normalizes $A \cap C$. An automorphism of $A \cong Q_8$ of order 3 is irreducible on $A/\Phi(A)$. So, as $\alpha|_{A,A}$ has order 3 and normalizes $A \cap C$, we have that $A \cap C$ has order 1 or 2.

By [37, 8.2.7], we have

$$[C, \langle\alpha\rangle] = [[C, \langle\alpha\rangle], \langle\alpha\rangle].$$

By hypothesis $[S, \alpha] = A$, so $[C, \alpha] \leq [S, \alpha] \cap C = A \cap C$. As $|A \cap C| \leq 2$, $[A \cap C, \alpha] = 1$, so $[C, \alpha] = [C, \alpha, \alpha] = [A \cap C, \alpha] = 1$. Hence, $C \leq C_S(\alpha) = Z(A)B = C_S(A)$. □

We need the following definition in order to state the next proposition.

Definition 2.16. A nonabelian finite simple group G is said to be a *Goldschmidt group* provided that one of the following holds:

(1) G has an abelian Sylow 2-subgroup.
(2) G is isomorphic to a finite simple group of Lie type in characteristic 2 of Lie rank 1.

Proposition 2.17. *Let G be a finite group, and let S be a Sylow 2-subgroup of G. Assume that, for each 2-component L of G, the factor group $L/Z^*(L)$ is a known finite simple group. Let $\mathfrak{L}_{2'}$ denote the set of 2-components L of G such that $L/Z^*(L)$ is not a Goldschmidt group. Then the following hold:*

(i) *Let L be a 2-component of G. Then $\mathcal{F}_{S\cap L}(L)$ is a component of $\mathcal{F}_S(G)$ if and only if $L \in \mathfrak{L}_{2'}$.*
(ii) *The map from $\mathfrak{L}_{2'}$ to the set of components of $\mathcal{F}_S(G)$ sending each element L of $\mathfrak{L}_{2'}$ to $\mathcal{F}_{S\cap L}(L)$ is a bijection.*

Proof. Let L be a 2-component of G. Set $\mathcal{G} := \mathcal{F}_{S\cap L}(L)$. Since L is subnormal in G, we have that $\mathcal{G}$ is subnormal in $\mathcal{F}_S(G)$ (see [10, Part I, Proposition 6.2]). Therefore, $\mathcal{G}$ is a component of $\mathcal{F}_S(G)$ if and only if $\mathcal{G}$ is quasisimple. We have $\mathfrak{foc}(\mathcal{G}) = S \cap L' = S \cap L$ by the focal subgroup theorem [23, Chapter 7, Theorem 3.4], and so $\mathcal{G}$ is quasisimple if and only if $\mathcal{G}/Z(\mathcal{G})$ is simple. By Lemma 2.13, we have $Z(\mathcal{G}) = S \cap Z^*(L)$. Lemma 2.11 implies that $\mathcal{G}/Z(\mathcal{G})$ is isomorphic to the 2-fusion system of $L/Z^*(L)$. By [9, Theorem 5.6.18], the 2-fusion system of $L/Z^*(L)$ is simple if and only if $L \in \mathfrak{L}_{2'}$. So $\mathcal{G}$ is a component of $\mathcal{F}_S(G)$ if and only if $L \in \mathfrak{L}_{2'}$, and (i) holds.

(ii) follows from [8, (1.8)]. □

Lemma 2.18. *Let G be a finite group with $O(G) = 1$, and let S be a Sylow 2-subgroup of G. Let $n \geq 1$ be a natural number, and let $L_1, \ldots, L_n$ be pairwise distinct subgroups of G such that L_i is either a component or a solvable 2-component of G for each $1 \leq i \leq n$. Set $Q := (S \cap L_1) \cdots (S \cap L_n)$. Assume that Q is strongly closed in S with respect to $\mathcal{F}_S(G)$ and that $\mathcal{F}_S(G)/Q$ is nilpotent. Then, if L_0 is a component or a solvable 2-component of G, we have $L_0 = L_i$ for some $1 \leq i \leq n$.*

Proof. Let $L^s(G)$ denote the subgroup of G generated by the components and the solvable 2-components of G. By [37, 6.5.2] and [27, Proposition 13.5], $L^s(G)$ is the central product of the subgroups of G which are components or solvable 2-components. Set $L := L_1 \cdots L_n \trianglelefteq L^s(G)$.

Let $\mathcal{G} := \mathcal{F}_{S\cap L^s(G)}(L^s(G))$. Clearly, $S \cap L = (S \cap L_1) \cdots (S \cap L_n) = Q$. Lemma 2.11 implies that the 2-fusion system of $L^s(G)/L$ is isomorphic to $\mathcal{G}/Q$. By hypothesis, $\mathcal{F}_S(G)/Q$ is nilpotent, and so $\mathcal{G}/Q$ is nilpotent. So the 2-fusion system of $L^s(G)/L$ is nilpotent. Applying [39, Theorem 1.4], we conclude that $L^s(G)/L$ is 2-nilpotent.

Suppose $L_0 \neq L_i$ for any $1 \leq i \leq n$. Then from paragraph one, L_0 centralizes L, so $L \cap L_0 \leq Z(L_0)$, and hence, $L_0L/L \cong L_0/(L \cap L_0)$ is quasisimple, A_4, or $SL_2(3)$. In particular, L_0L/L is not 2-nilpotent, a contradiction. □

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Corollary 2.19. *Let G be a finite group, and let S be a Sylow 2-subgroup of G. Let $n \geq 1$ be a natural number, and let $L_1, \dots, L_n$ be pairwise distinct 2-components of G. Assume that $Q := (S \cap L_1) \cdots (S \cap L_n)$ is strongly closed in S with respect to $\mathcal{F}_S(G)$ and that $\mathcal{F}_S(G)/Q$ is nilpotent. Then, if L_0 is a 2-component of G, we have $L_0 = L_i$ for some $1 \leq i \leq n$.*

Proposition 2.20. *Let p be a prime, and let $\mathcal{E}$ be a simple saturated fusion system on a finite p-group T. Suppose that $\mathcal{E}$ is tamely realized (in the sense of [3, Section 2.2]) by a nonabelian known finite simple group K such that $\mathrm{Out}(K)$ is p-nilpotent. Assume moreover that G is a nonabelian finite simple group containing a Sylow p-subgroup S of G with $T \leq S$ such that $\mathcal{E} \trianglelefteq \mathcal{F}_S(G)$ and $C_S(\mathcal{E}) = 1$. Then $\mathcal{F}_S(G)$ is tamely realized by a subgroup L of $\mathrm{Aut}(K)$ containing $\mathrm{Inn}(K)$ such that the index of $\mathrm{Inn}(K)$ in L is coprime to p.*

Proof. Set $\mathcal{F} := \mathcal{F}_S(G)$. By a result of Bob Oliver, namely by [44, Corollary 2.4], $\mathcal{F}$ is tamely realized by a subgroup L of $\mathrm{Aut}(K)$ containing $\mathrm{Inn}(K)$. We are going to show that the index of $\mathrm{Inn}(K)$ in L is coprime to p.

Let S_0 be a Sylow p-subgroup of L. Then $\mathcal{F} \cong \mathcal{F}_{S_0}(L)$. We have $O^p(G) = G$ since G is nonabelian simple, and so $\mathfrak{hnp}(\mathcal{F}) = S$ by the hyperfocal subgroup theorem [18, Theorem 1.33]. It follows that $\mathfrak{hnp}(\mathcal{F}_{S_0}(L)) = S_0$.

By the hyperfocal subgroup theorem [18, Theorem 1.33], $S_0 = \mathfrak{hnp}(\mathcal{F}_{S_0}(L)) = O^p(L) \cap S_0$. Consequently, $O^p(L)$ has p'-index in L, whence $O^p(L) = L$. So we have $O^p(L/\mathrm{Inn}(K)) = L/\mathrm{Inn}(K)$. On the other hand, $L/\mathrm{Inn}(K)$ is p-nilpotent since $\mathrm{Out}(K)$ is p-nilpotent. It follows that $L/\mathrm{Inn}(K)$ is a p'-group, as claimed. □

3. Auxiliary results on linear and unitary groups

In this section, we collect several results on linear and unitary groups needed for the proofs of our main results. Some of the results stated here are known, while others seem to be new. For the convenience of the reader, we also include proofs of known results when we could not find a reference in which they appear in the form stated here.

3.1. Basic definitions

We begin with some basic definitions. Let q be a nontrivial prime power, and let n be a positive integer. The *general linear group* $GL_n(q)$ is the group of all invertible $n \times n$ matrices over $\mathbb{F}_q$ under matrix multiplication. The *special linear group* $SL_n(q)$ is the subgroup of $GL_n(q)$ consisting of all $n \times n$ matrices over $\mathbb{F}_q$ with determinant 1. The center of $GL_n(q)$ consists of all scalar matrices λI_n with $\lambda \in (\mathbb{F}_q)^*$. We have $Z(SL_n(q)) = SL_n(q) \cap Z(GL_n(q))$. Set $PGL_n(q) := GL_n(q)/Z(GL_n(q))$ and $PSL_n(q) := SL_n(q)/Z(SL_n(q))$. By [35, Kapitel II, Satz 6.10] and [35, Kapitel II, Hauptsatz 6.13], $SL_n(q)$ is quasisimple if $n \geq 2$ and $(n, q) \neq (2, 2), (2, 3)$.

As in [35, Kapitel II, Bemerkung 10.5 (b)], we consider the *general unitary group* $GU_n(q)$ as the subgroup of $GL_n(q^2)$ consisting of all $(a_{ij}) \in GL_n(q^2)$ satisfying the condition $((a_{ij})^q)(a_{ij})^t = I_n$. The *special unitary group* $SU_n(q)$ is the subgroup of $GU_n(q)$ consisting of all elements of $GU_n(q)$ with determinant 1. By [35, Kapitel II, Hilfssatz 8.8], we have $SL_2(q) \cong SU_2(q)$. The center of $GU_n(q)$ consists of all scalar matrices λI_n, where $\lambda \in (\mathbb{F}_{q^2})^*$ and $\lambda^{q+1} = 1$. We have $Z(SU_n(q)) = SU_n(q) \cap Z(GU_n(q))$. Set $PGU_n(q) := GU_n(q)/Z(GU_n(q))$ and $PSU_n(q) := SU_n(q)/Z(SU_n(q))$. By [32, Theorems 11.22 and 11.26], $SU_n(q)$ is quasisimple if $n \geq 2$ and $(n, q) \neq (2, 2), (2, 3), (3, 2)$.

We write $(P)GL_n^+(q)$ and $(P)SL_n^+(q)$ for $(P)GL_n(q)$ and $(P)SL_n(q)$, respectively. Also, we write $(P)GL_n^-(q)$ for $(P)GU_n(q)$ and $(P)SL_n^-(q)$ for $PSU_n(q)$.

3.2. Central extensions of $PSL_n(q)$ and $PSU_n(q)$

In the proofs of the following two lemmas, we use the terminology of [5, Section 33].

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Lemma 3.1. *Let $n \geq 3$ be a natural number, and let q be a nontrivial odd prime power. Let H be a perfect central extension of $PSL_n(q)$. Then there is a subgroup $Z \leq Z(SL_n(q))$ such that $H \cong SL_n(q)/Z$.*

Proof. By [28, pp. 312-313], the Schur multiplier of $PSL_n(q)$ is isomorphic to $C_{(n,q-1)} \cong Z(SL_n(q))$. From [5, 33.6], we see that this is just another way to say that $SL_n(q)$ is the universal covering group of $PSL_n(q)$. Applying [5, 33.6] again, we conclude that $H \cong SL_n(q)/Z$ for some $Z \leq Z(SL_n(q))$. □

Lemma 3.2. *Let $n \geq 3$ be a natural number, and let q be a nontrivial odd prime power. Let H be a perfect central extension of $PSU_n(q)$. Assume that $(n, q) \neq (4, 3)$ or that $Z(H)$ is a 2-group. Then there is a subgroup $Z \leq Z(SU_n(q))$ such that $H \cong SU_n(q)/Z$.*

Proof. Suppose that $(n, q) \neq (4, 3)$. By [28, pp. 312-313], the Schur multiplier of $PSU_n(q)$ is isomorphic to $C_{(n,q+1)} \cong Z(SU_n(q))$. As in the proof of Lemma 3.1, we conclude that $H \cong SU_n(q)/Z$ for some $Z \leq Z(SU_n(q))$.

Suppose now that $(n, q) = (4, 3)$ and that $Z(H)$ is a 2-group. Let $G := PSU_4(3)$, and let $\widetilde{G}$ be the universal covering group of G. Then the Schur multiplier of G is isomorphic to $Z(\widetilde{G})$. By [28, pp. 312-313], the Schur multiplier of G is isomorphic to $C_4 \times C_3 \times C_3$. Thus, $Z(\widetilde{G}) \cong C_4 \times C_3 \times C_3$. Since $\widetilde{G}$ is quasisimple, we have $Z(\widetilde{G}/Z) = Z(\widetilde{G})/Z$ whenever $Z \leq Z(\widetilde{G})$. Let Q be the unique Sylow 3-subgroup of $Z(\widetilde{G})$. By [5, 33.6], $\widetilde{G}$ is a central extension of $SU_4(3)$ and of H. Since $SU_4(3)$ has a center of order 4, we have $SU_4(3) \cong \widetilde{G}/Q$. Let $Z \leq Z(\widetilde{G})$ with $H \cong \widetilde{G}/Z$. As $Z(H)$ is a 2-group, we have $Q \leq Z$, whence $H \cong \widetilde{G}/Z \cong (\widetilde{G}/Q)/(Z/Q)$ is isomorphic to a quotient of $SU_4(3)$ by a central subgroup. □

3.3. Involutions

In this subsection, we collect several results on the involutions of the groups $(P)GL_n^{\varepsilon}(q)$ and $(P)SL_n^{\varepsilon}(q)$, where q is a nontrivial odd prime power, $n \geq 2$ and $\varepsilon \in \{+, -\}$.

Lemma 3.3. *Let q be a nontrivial odd prime power, and let $n \geq 2$. Let T be an element of $GL_n(q)$ such that $T^2 = \lambda I_n$ for some $\lambda \in \mathbb{F}_q^*$. Then one of the following holds:*

(i) *There is some $\mu \in \mathbb{F}_q^*$ such that $\lambda = \mu^2$, and T is $GL_n(q)$-conjugate to a diagonal matrix with diagonal entries in $\{\mu, -\mu\}$.*
(ii) *n is even, λ is a nonsquare element of $\mathbb{F}_q^*$, and T is $GL_n(q)$-conjugate to the matrix*

$$\begin{pmatrix} & I_{n/2} \\ \lambda I_{n/2} & \end{pmatrix}.$$

Moreover, we have $C_{GL_n(q)}(T) \cong GL_{\frac{n}{2}}(q^2)$.

Proof. We identify the field $\mathbb{F}_q$ with the subfield of $\mathbb{F}_{q^2}$ consisting of all $x \in \mathbb{F}_{q^2}$ satisfying $x^q = x$. As $q + 1 = (q^2 - 1)/(q - 1)$ is even, any element of $\mathbb{F}_q^*$ is the square of an element of $\mathbb{F}_{q^2}^*$. Let $\mu \in \mathbb{F}_{q^2}^*$ with $\lambda = \mu^2$.

If $\mu \in \mathbb{F}_q$, then the minimal polynomial of T divides $(x - \mu)(x + \mu)$, so T is diagonalizable over $\mathbb{F}_q$, and it follows that (i) holds.

Assume now that $\mu \notin \mathbb{F}_q$. Then λ is a nonsquare element of $\mathbb{F}_q^*$. Let V be an n-dimensional vector space over $\mathbb{F}_q$, and let B be an ordered basis of V. Let φ be the element of $GL(V)$ such that φ is represented by T with respect to B. Since $\mu \notin \mathbb{F}_q$, we have that 1 and μ are linearly independent; so $(1, \mu)$ is an $\mathbb{F}_q$-basis of $\mathbb{F}_{q^2}$. Using that $\varphi^2 = \lambda \mathrm{id}_V$, one can check that V becomes a vector space over $\mathbb{F}_{q^2}$ by defining

$$(x + y\mu)v := xv + yv^{\varphi}$$

for all $x, y \in \mathbb{F}_q$ and $v \in V$. Let m be the dimension of V over $\mathbb{F}_{q^2}$, and let $(v_1, \ldots, v_m)$ be an $\mathbb{F}_{q^2}$-basis of V. Then $B_0 := (v_1, \ldots, v_m, \mu v_1, \ldots, \mu v_m)$ is an $\mathbb{F}_q$-basis of V. In particular, $n = 2m$ is even. For

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$1 \leq i \leq m$, we have $v_i^{\varphi} = \mu v_i$ and $(\mu v_i)^{\varphi} = (v_i)^{\varphi^2} = \lambda v_i$. So, with respect to B_0, φ is represented by the matrix

$$M := \begin{pmatrix} & I_{n/2} \\ \lambda I_{n/2} & \end{pmatrix}.$$

It follows that T and M are $GL_n(q)$-conjugate.

Let ψ be an automorphism of V as an $\mathbb{F}_q$-vector space centralizing φ. For $x, y \in \mathbb{F}_q$ and $v \in V$, we have

$$((x + y\mu)v)^{\psi} = (xv + yv^{\varphi})^{\psi} = xv^{\psi} + yv^{\psi\varphi} = (x + y\mu)v^{\psi},$$

whence ψ is $\mathbb{F}_{q^2}$-linear. Conversely, if ψ is $\mathbb{F}_{q^2}$-linear, then

$$v_i^{\psi\varphi} = \mu v_i^{\psi} = (\mu v_i)^{\psi} = v_i^{\varphi\psi}$$

and hence $\psi\varphi = \varphi\psi$. It follows that the centralizer of φ in the general linear group of V as an $\mathbb{F}_q$-vector space is equal to the general linear group of V as an $\mathbb{F}_{q^2}$-vector space. Thus, $C_{GL_n(q)}(T) \cong GL_{\frac{n}{2}}(q^2)$. So (ii) holds. □

Lemma 3.4. *Let q be a nontrivial odd prime power, and let $n \geq 2$ be a natural number. Let $T \in GU_n(q)$.*

(i) *If $T^2 = \lambda I_n$ for some $\lambda \in \mathbb{F}_{q^2}^*$, then λ is a square in $\mathbb{F}_{q^2}^*$.*
(ii) *If $T^2 = \rho^2 I_n$ for some $\rho \in \mathbb{F}_{q^2}^*$ with $\rho^{q+1} = 1$, then T is $GU_n(q)$-conjugate to a diagonal matrix with diagonal entries in $\{\rho, -\rho\}$.*
(iii) *If $T^2 = \rho^2 I_n$ for some $\rho \in \mathbb{F}_{q^2}^*$ with $\rho^{q+1} \neq 1$, then n is even, and we have $C_{GU_n(q)}(T) \cong GL_{\frac{n}{2}}(q^2)$.*

Proof. Suppose that $T^2 = \lambda I_n$ for some $\lambda \in \mathbb{F}_{q^2}^*$. Since $T^2 \in GU_n(q)$, we have that $\lambda^{q+1} = 1$, so λ is a square in $\mathbb{F}_{q^2}^*$.

A proof of (ii) and (iii) can be extracted from [47, pp. 314-315]. □

Proposition 3.5. *Let q be a nontrivial odd prime power, and let $n \geq 2$ be a natural number. Let ρ be an element of $\mathbb{F}_q^*$ of order $(n, q-1)$. For each even natural number i with $2 \leq i < n$, let*

$$\widetilde{t_i} := \begin{pmatrix} I_{n-i} & \\ & -I_i \end{pmatrix} \in SL_n(q)$$

and let t_i be the image of $\widetilde{t_i}$ in $PSL_n(q)$.

(i) *Assume that n is odd. Then each involution of $PSL_n(q)$ is $PSL_n(q)$-conjugate to t_i for some even $2 \leq i < n$.*
(ii) *Assume that n is even and that there is some $\mu \in \mathbb{F}_q^*$ with $\rho = \mu^2$. For each odd natural number i with $1 \leq i < n$, the matrix*

$$\widetilde{t_i} := \begin{pmatrix} \mu I_{n-i} & \\ & -\mu I_i \end{pmatrix}$$

lies in $SL_n(q)$. Let t_i denote the image of $\widetilde{t_i}$ in $PSL_n(q)$ for each odd $1 \leq i < n$. Then each involution of $PSL_n(q)$ is $PSL_n(q)$-conjugate to t_i for some (even or odd) $1 \leq i \leq \frac{n}{2}$.
(iii) *Assume that n is even and that ρ is a nonsquare element of $\mathbb{F}_q$. Let*

$$\widetilde{w} := \begin{pmatrix} & I_{n/2} \\ \rho I_{n/2} & \end{pmatrix}.$$

If $\widetilde{w} \in SL_n(q)$, then each involution of $PSL_n(q)$ is $PSL_n(q)$-conjugate to t_i for some even $2 \leq i \leq \frac{n}{2}$ or to $w := \widetilde{w}Z(SL_n(q)) \in PSL_n(q)$. If $\widetilde{w} \notin SL_n(q)$, then each involution of $PSL_n(q)$ is $PSL_n(q)$-conjugate to t_i for some even $2 \leq i \leq \frac{n}{2}$.

Proof. We follow arguments found in the proof of [46, Lemma 1.1].

Assume that n is odd. Then $Z(SL_n(q))$ has odd order, and therefore, any involution of $PSL_n(q)$ is the image of an involution of $SL_n(q)$. As a consequence of Lemma 3.3, each involution of $SL_n(q)$ is $SL_n(q)$-conjugate to $\widetilde{t_i}$ for some even $2 \leq i < n$. So (i) follows.

Assume now that n is even and that $\rho = \mu^2$ for some $\mu \in \mathbb{F}_q^*$. Note that $Z(SL_n(q))$ equals $\langle \rho I_n \rangle$. We claim that $\mu^n = -1$. Since $\mu^{2n} = \rho^n = 1$, we have that $\mu^n = 1$ or -1. If $\mu^n = 1$, then $\mu \in \langle \rho \rangle$, and so ρ is a square in $\langle \rho \rangle$, which is impossible. So we have $\mu^n = -1$. It follows that $\widetilde{t_i} \in SL_n(q)$ for each odd $1 \leq i < n$. Now let $T \in SL_n(q)$ such that $TZ(SL_n(q)) \in PSL_n(q)$ is an involution. Then we have $T^2 = \rho^\ell I_n = \mu^{2\ell} I_n$ for some $1 \leq \ell \leq (n, q-1)$. Using Lemma 3.3, we conclude that T is $SL_n(q)$-conjugate to a diagonal matrix $D \in SL_n(q)$ with diagonal entries in $\{\mu^\ell, -\mu^\ell\}$. Let $1 \leq i < n$ such that $-\mu^\ell$ occurs precisely i times as a diagonal entry of D. If i is odd, we may assume that $D = \mu^{\ell-1}\widetilde{t_i}$, and if i is even, we may assume that $D = \mu^\ell \widetilde{t_i}$. In either case, the image of D in $PSL_n(q)$ is t_i. Hence, $TZ(SL_n(q))$ is $PSL_n(q)$-conjugate to t_i. Noticing that t_i is $PSL_n(q)$-conjugate to t_{n-i}, we conclude that (ii) holds.

Now assume that n is even and that ρ is a nonsquare element of $\mathbb{F}_q$. Again, let T be an element of $SL_n(q)$ such that $TZ(SL_n(q)) \in PSL_n(q)$ is an involution. We have $T^2 = \rho^\ell I_n$ for some $1 \leq \ell \leq (n, q-1)$. Assume that ℓ is even. Then Lemma 3.3 implies that T or $-T$ is $SL_n(q)$-conjugate to $\rho^{\frac{\ell}{2}}\widetilde{t_i}$ for some even $2 \leq i \leq \frac{n}{2}$. It follows that $TZ(SL_n(q))$ is $PSL_n(q)$-conjugate to t_i for some even $2 \leq i \leq \frac{n}{2}$. Assume now that ℓ is odd. As ρ is not a square in $\mathbb{F}_q$, but $\rho^{\ell-1}$ is a square in $\mathbb{F}_q$, ρ^ℓ cannot be a square in $\mathbb{F}_q$. Using Lemma 3.3, we may conclude that T is $GL_n(q)$-conjugate to the matrix

$$M := \begin{pmatrix} 0 & \rho^\ell & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \rho^\ell \\ & & & 1 & 0 \end{pmatrix} \in SL_n(q).$$

It is rather easy to see that T and M are even conjugate in $SL_n(q)$. Let $k := \frac{\ell-1}{2}$. It is not hard to show that the matrices

$$\begin{pmatrix} 0 & \rho^\ell \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \rho^{k+1} \\ \rho^k & 0 \end{pmatrix}$$

are $SL_2(q)$-conjugate. So it follows that M, and hence, T is $SL_n(q)$-conjugate to $\rho^k M_2$, where

$$M_2 := \begin{pmatrix} 0 & \rho & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \rho \\ & & & 1 & 0 \end{pmatrix} \in SL_n(q).$$

Consequently, the images of T and M_2 in $PSL_n(q)$ are conjugate. Furthermore, as $\det(M_2) = \det(\widetilde{w})$, we see that $\widetilde{w} \in SL_n(q)$. Also, $\widetilde{w}$ is $SL_n(q)$-conjugate to M_2, and so $TZ(SL_n(q))$ is $PSL_n(q)$-conjugate to w. □

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Lemma 3.6. *Let q be a nontrivial odd prime power, and let $n \geq 4$ be an even natural number. Let ρ be an element of $\mathbb{F}_q^*$ of order $(n, q-1)$. Suppose that ρ is a nonsquare element of $\mathbb{F}_q$ and that*

$$\widetilde{w} := \begin{pmatrix} & I_{n/2} \\ \rho I_{n/2} & \end{pmatrix}$$

lies in $SL_n(q)$. Denote the image of $\widetilde{w}$ in $PSL_n(q)$ by w. Set $C := C_{PSL_n(q)}(w)$. Let P be a Sylow 2-subgroup of C. Then the following hold:

(i) *C has a unique 2-component J, and J is isomorphic to a nontrivial quotient of $SL_{\frac{n}{2}}(q^2)$.*
(ii) *$P \cap J$ is strongly closed in P with respect to $\mathcal{F}_P(C)$, and the factor system $\mathcal{F}_P(C)/(P \cap J)$ is nilpotent.*
(iii) *If $n \geq 6$, then P has rank at least 4.*

Proof. Set $C_0 := C_{SL_n(q)}(\widetilde{w})/Z(SL_n(q)) \leq C$.

Let $y \in C \setminus C_0$, and let $\widetilde{y}$ be a preimage of y in $SL_n(q)$. Then $\widetilde{w}^{\widetilde{y}} = \lambda\widetilde{w}$ for some $1 \neq \lambda \in \langle\rho\rangle$. The characteristic polynomial of $\widetilde{w}$ is $(x^2 - \rho)^{\frac{n}{2}}$, and $\lambda\widetilde{w}$ has the characteristic polynomial $(x^2 - \lambda^2\rho)^{\frac{n}{2}}$. Since $\widetilde{w}^{\widetilde{y}} = \lambda\widetilde{w}$, both polynomials are equal, and so we have $\lambda^2 = 1$. Thus, $\lambda = -1$ and hence $\widetilde{w}^{\widetilde{y}} = -\widetilde{w}$. If z is another element of $C \setminus C_0$ and if $\widetilde{z}$ is a preimage of z in $SL_n(q)$, then we have $\widetilde{w}^{\widetilde{y}} = -\widetilde{w} = \widetilde{w}^{\widetilde{z}}$, and so $\widetilde{yz^{-1}}$ centralizes $\widetilde{w}$. This implies that $yz^{-1} \in C_0$. It follows that $|C : C_0| \leq 2$ (and one can show that in fact $|C : C_0| = 2$).

By the preceding paragraph, C/C_0 is abelian, and so the 2-components of C are precisely the 2-components of C_0. One may deduce from Lemma 3.3 that $C_{SL_n(q)}(\widetilde{w})$ has a normal subgroup $\widetilde{J}$ isomorphic to $SL_{\frac{n}{2}}(q^2)$ such that the corresponding factor group is cyclic. Let J be the image of $\widetilde{J}$ in $PSL_n(q)$. Then J is isomorphic to a nontrivial quotient of $SL_{\frac{n}{2}}(q^2)$. Moreover, $J \trianglelefteq C_0$ and C_0/J is cyclic. Therefore, J is the only 2-component of C_0 and hence the only 2-component of C. Thus, (i) holds.

Since $J \trianglelefteq C$, we have that $P \cap J$ is strongly closed in P with respect to $\mathcal{F}_P(C)$. By Lemma 2.11, the factor system $\mathcal{F}_P(C)/(P \cap J)$ is isomorphic to the 2-fusion system of C/J. Since C_0 has index ≤ 2 in C and C_0/J is abelian, we have that C/J is 2-nilpotent. So C/J has a nilpotent 2-fusion system, and (ii) follows.

We now prove (iii). Assume that $n \geq 6$. Let u denote the image of

$$\begin{pmatrix} 0 & \rho & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \rho \\ & & & 1 & 0 \end{pmatrix} \in SL_n(q)$$

in $PSL_n(q)$.

We claim that there exist $a, b \in \mathbb{F}_q$ with $a^2\rho - b^2\rho^2 = 1$. The field $\mathbb{F}_q$ has precisely $\frac{q+1}{2}$ square elements. Therefore, each of the sets $M_1 := \{a^2\rho \mid a \in \mathbb{F}_q\}$ and $M_2 := \{1 + b^2\rho^2 \mid b \in \mathbb{F}_q\}$ has cardinality $\frac{q+1}{2}$. It follows that $M_1 \cap M_2 \neq \emptyset$. So there exist $a, b \in \mathbb{F}_q$ with $a^2\rho = 1 + b^2\rho^2$, or in other words $a^2\rho - b^2\rho^2 = 1$.

Let s be the image of

$$\begin{pmatrix} -b\rho & a\rho & & & \\ -a & b\rho & & & \\ & & \ddots & & \\ & & & -b\rho & a\rho \\ & & & -a & b\rho \end{pmatrix} \in SL_n(q)$$

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in $PSL_n(q)$. By a direct calculation, $s \in C_{PSL_n(q)}(u)$. Another direct calculation shows that s is an involution. Let z_1 denote the image of

$$\begin{pmatrix} -I_2 & \\ & I_{n-2} \end{pmatrix} \in SL_n(q)$$

in $PSL_n(q)$, and let z_2 denote the image of

$$\begin{pmatrix} I_2 & & \\ & -I_2 & \\ & & I_{n-4} \end{pmatrix} \in SL_n(q)$$

in $PSL_n(q)$. Then one can easily verify that $\langle s, u, z_1, z_2 \rangle \leq C_{PSL_n(q)}(u)$ is isomorphic to E_{16}. So a Sylow 2-subgroup of $C_{PSL_n(q)}(u)$ has rank at least 4. This is also true for P as w and u are conjugate (see Proposition 3.5). □

Lemma 3.7. *Let $n \geq 2$ be a natural number, and let $\varepsilon \in \{+, -\}$. Also, let $T \in GL_n^\varepsilon(3) \setminus Z(GL_n^\varepsilon(3))$ such that $T^2 \in Z(GL_n^\varepsilon(3))$. Then $C_{GL_n^\varepsilon(3)}(T)$ is core-free.*

Proof. By Lemmas 3.3 and 3.4, we either have $C_{GL_n^\varepsilon(3)}(T) \cong GL_i^\varepsilon(3) \times GL_{n-i}^\varepsilon(3)$ for some $1 \leq i < n$, or n is even and $C_{GL_n^\varepsilon(3)}(T) \cong GL_{n/2}(9)$. So we have that $C_{GL_n^\varepsilon(3)}(T)$ is core-free. □

Noticing that $GL_n^\varepsilon(3)/SL_n^\varepsilon(3)$ and $Z(GL_n^\varepsilon(3))$ are 2-groups for any $n \geq 2$ and $\varepsilon \in \{+, -\}$, one can deduce the following two corollaries from Lemma 3.7.

Corollary 3.8. *Let $n \geq 2$ be a natural number, and let $\varepsilon \in \{+, -\}$. Then any involution centralizer in $SL_n^\varepsilon(3)$ is core-free.*

Corollary 3.9. *Let $n \geq 2$ be a natural number, and let $\varepsilon \in \{+, -\}$. Then any involution centralizer in $PGL_n^\varepsilon(3)$ is core-free.*

3.4. Sylow 2-subgroups and 2-fusion systems

In this subsection, we consider several properties of Sylow 2-subgroups and 2-fusion systems of linear and unitary groups.

Lemma 3.10 [17, p. 142]. *Let q be a nontrivial odd prime power. Let $k, s \in \mathbb{N}$ such that 2^k is the 2-part of $q - 1$ and that 2^s is the 2-part of $q + 1$. Then:*

(i) *Assume that $q \equiv 1 \bmod 4$. Then*

$$\left\{ \begin{pmatrix} \lambda & \\ & \mu \end{pmatrix} \,:\, \lambda, \mu \text{ are 2-elements of } \mathbb{F}_q^* \right\} \cdot \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

is a Sylow 2-subgroup of $GL_2(q)$. In particular, the Sylow 2-subgroups of $GL_2(q)$ are isomorphic to the wreath product $C_{2^k} \wr C_2$.

(ii) *If $q \equiv 3 \bmod 4$, then the Sylow 2-subgroups of $GL_2(q)$ are semidihedral of order 2^{s+2}.*

Lemma 3.11 [17, p. 143]. *Let q be a nontrivial odd prime power. Let $k, s \in \mathbb{N}$ such that 2^k is the 2-part of $q - 1$ and that 2^s is the 2-part of $q + 1$. Then:*

(i) *If $q \equiv 1 \bmod 4$, then the Sylow 2-subgroups of $GU_2(q)$ are semidihedral of order 2^{k+2}.*

(ii) *If $q \equiv 3 \bmod 4$, then the Sylow 2-subgroups of $GU_2(q)$ are isomorphic to the wreath product $C_{2^s} \wr C_2$. If $\varepsilon \in \mathbb{F}_{q^2}^*$ has order 2^s, then a Sylow 2-subgroup of $GU_2(q)$ is concretely given by*

$$W := \left\{ \begin{pmatrix} \lambda & \\ & \mu \end{pmatrix} \,:\, \lambda, \mu \in \langle \varepsilon \rangle \right\} \cdot \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle.$$

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Lemma 3.12 [35, Kapitel II, Satz 8.10 a)]. *If q is a nontrivial odd prime power, then a Sylow 2-subgroup of $SL_2(q)$ is generalized quaternion of order $(q^2-1)_2$.*

Lemma 3.13 [35, Kapitel II, Satz 8.10 b)]. *If q is a nontrivial odd prime power, then $PSL_2(q)$ has dihedral Sylow 2-subgroups of order $\frac{1}{2}(q^2-1)_2$.*

Lemma 3.14 [17, Lemma 1]. *Let q be a nontrivial odd prime power, and let $\varepsilon \in \{+,-\}$. Let r be a positive integer. Let W_r be a Sylow 2-subgroup of $GL^{\varepsilon}_{2^r}(q)$. Then $W_r \wr C_2$ is isomorphic to a Sylow 2-subgroup of $GL^{\varepsilon}_{2^{r+1}}(q)$. A Sylow 2-subgroup of $GL^{\varepsilon}_{2^{r+1}}(q)$ is concretely given by*

$$\left\{\begin{pmatrix} A & \\ & B \end{pmatrix} \;:\; A, B \in W_r\right\} \cdot \left\langle \begin{pmatrix} & I_{2^r} \\ I_{2^r} & \end{pmatrix} \right\rangle.$$

Lemma 3.15 [17, Theorem 1]. *Let q be a nontrivial odd prime power, and let n be a positive integer. Let $\varepsilon \in \{+,-\}$. Let $0 \le r_1 < \cdots < r_t$ such that $n = 2^{r_1} + \cdots + 2^{r_t}$. Let $W_i \in \mathrm{Syl}_2(GL^{\varepsilon}_{2^{r_i}}(q))$ for all $1 \le i \le t$. Then $W_1 \times \cdots \times W_t$ is isomorphic to a Sylow 2-subgroup of $GL^{\varepsilon}_n(q)$. A Sylow 2-subgroup of $GL^{\varepsilon}_n(q)$ is concretely given by*

$$\left\{\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_t \end{pmatrix} \;:\; A_i \in W_i\right\}.$$

Lemma 3.16. *Let q be a prime power with $q \equiv 3 \bmod 4$. Let W be a Sylow 2-subgroup of $GL_2(q)$, and let $m \in \mathbb{N}$ such that $|W| = 2^m$. Then:*

(i) *W is semidihedral. In particular, there are elements $a, b \in W$ with $\mathrm{ord}(a) = 2^{m-1}$ and $\mathrm{ord}(b) = 2$ such that $a^b = a^{2^{m-2}-1}$.*
(ii) *We have $W \cap SL_2(q) = \langle a^2\rangle\langle ab\rangle$.*
(iii) *Let $1 \le \ell \le 2^{m-1}$. If ℓ is odd, then a^ℓ has determinant -1, and $a^\ell b$ has determinant 1. If ℓ is even, then a^ℓ has determinant 1, and $a^\ell b$ has determinant -1.*
(iv) *The involutions of W are precisely the elements $a^{2^{m-2}}$ and $a^\ell b$, where $2 \le \ell \le 2^{m-1}$ is even.*

Proof. By Lemma 3.10 (ii), we have (i).

Let $W_0 := W \cap SL_2(q)$. By Lemma 3.12, W_0 is generalized quaternion. Also, W_0 is a maximal subgroup of W since $SL_2(q)$ has index $q-1$ in $GL_2(q)$ and $q \equiv 3 \bmod 4$. By [23, Chapter 5, Theorem 4.3 (ii) (b)], we have $\Phi(W) = \langle a^2\rangle$. So the maximal subgroups of W are precisely the groups $M_1 := \langle a\rangle$, $M_2 := \langle a^2\rangle\langle b\rangle$ and $M_3 := \langle a^2\rangle\langle ab\rangle$. One can check that $M_1 \cong C_{2^{n-1}}$, $M_2 \cong D_{2^{n-1}}$ and $M_3 \cong Q_{2^{n-1}}$. Consequently, $W_0 = \langle a^2\rangle\langle ab\rangle$, and (ii) holds.

(iii) follows from (ii) since any element of $W \setminus W_0$ has determinant -1.

The proof of (iv) is an easy exercise. □

Lemma 3.17. *Let q be a nontrivial odd prime power, n a positive integer and $\varepsilon \in \{+,-\}$. Let $0 \le r_1 < \cdots < r_t$ such that $n = 2^{r_1} + \cdots + 2^{r_t}$. Then there is a Sylow 2-subgroup W of $G := GL^{\varepsilon}_n(q)$ containing all diagonal matrices in G with 2-power order such that $C_W(W \cap SL^{\varepsilon}_n(q))$ consists precisely of the matrices*

$$\begin{pmatrix} \lambda_1 I_{2^{r_1}} & & \\ & \ddots & \\ & & \lambda_t I_{2^{r_t}} \end{pmatrix},$$

where $\lambda_1, \ldots, \lambda_t$ are 2-elements of $\mathbb{F}_q^*$ if $G = GL_n(q)$ and 2-elements of $\mathbb{F}_{q^2}^*$ with $\lambda_i^{q+1} = 1$ (for each $1 \le i \le t$) if $G = GU_n(q)$.

Proof. Using Lemmas 3.10 and 3.11, one can check that the centralizer of a Sylow 2-subgroup of $SL^{\varepsilon}_2(q)$ inside a Sylow 2-subgroup of $GL^{\varepsilon}_2(q)$ is the Sylow 2-subgroup of $Z(GL^{\varepsilon}_2(q))$. Applying

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Lemma 3.14 and arguing by induction, one can see that a similar statement holds for the centralizer of a Sylow 2-subgroup of $SL_{2^r}^{\varepsilon}(q)$ inside a Sylow 2-subgroup of $GL_{2^r}^{\varepsilon}(q)$ for all $r \geq 0$. Now we may apply Lemma 3.15 to obtain a Sylow 2-subgroup of G with the desired properties. □

Lemma 3.18. *Let q be a nontrivial odd prime power, n a positive integer and $\varepsilon \in \{+,-\}$. Let $G := SL_n^{\varepsilon}(q)$, and let S be a Sylow 2-subgroup of G. Then we have $Z(\mathcal{F}_S(G)) = S \cap Z(G)$.*

Proof. This follows from Lemma 2.13. □

Proposition 3.19. *Let n be a positive integer. Let q, q^* be nontrivial odd prime powers, and let $\varepsilon, \varepsilon^* \in \{+,-\}$. If $\varepsilon q \sim \varepsilon^* q^*$, then the 2-fusion systems of $SL_n^{\varepsilon}(q)$ and $SL_n^{\varepsilon^*}(q^*)$ are isomorphic.*

Proof. Assume that $\varepsilon \neq \varepsilon^*$. From $\varepsilon q \sim \varepsilon^* q^*$, it is easy to deduce that $\varepsilon q \equiv \varepsilon^* q^* \mod 8$ and $(q^2-1)_2 = ((q^*)^2-1)_2$. So, in view of the remarks at the bottom of p. 11 of [14], we may apply [14, Proposition 3.3 (a)] to conclude that the 2-fusion system of $SL_n^{\varepsilon}(q)$ is isomorphic to the 2-fusion system of $SL_n^{\varepsilon^*}(q^*)$.

Assume now that $\varepsilon = \varepsilon^*$. Using Dirichlet's theorem [20, Theorem 3.3.1], one can easily see that there is an odd prime q_0 with $\varepsilon q \sim \varepsilon q^* \sim -\varepsilon q_0$. By the preceding paragraph, both the 2-fusion system of $SL_n^{\varepsilon}(q)$ and the 2-fusion system of $SL_n^{\varepsilon}(q^*)$ are isomorphic to the 2-fusion system of $SL_n^{-\varepsilon}(q_0)$. Consequently, the 2-fusion systems of $SL_n^{\varepsilon}(q)$ and $SL_n^{\varepsilon^*}(q^*)$ are isomorphic. □

Proposition 3.20. *Let n be a positive integer. Let q, q^* be nontrivial odd prime powers, and let $\varepsilon, \varepsilon^* \in \{+,-\}$. If $\varepsilon q \sim \varepsilon^* q^*$, then the 2-fusion systems of $PSL_n^{\varepsilon}(q)$ and $PSL_n^{\varepsilon^*}(q^*)$ are isomorphic.*

Proof. Let S and S^* be Sylow 2-subgroups of $G := SL_n^{\varepsilon}(q)$ and $G^* := SL_n^{\varepsilon^*}(q^*)$, respectively. By Proposition 3.19, $\mathcal{F} := \mathcal{F}_S(G)$ and $\mathcal{F}^* := \mathcal{F}_{S^*}(G^*)$ are isomorphic. Therefore, $\mathcal{F}/Z(\mathcal{F})$ and $\mathcal{F}^*/Z(\mathcal{F}^*)$ are isomorphic. Lemma 3.18 implies that $\mathcal{F}/(S \cap Z(G))$ and $\mathcal{F}^*/(S^* \cap Z(G^*))$ are isomorphic. Now the proposition follows from Lemma 2.11. □

The following lemma shows together with [9, Theorem 5.6.18] that the 2-fusion system of $PSL_n(q)$ is simple whenever q is odd and $n \geq 3$.

Lemma 3.21. *Let q be a nontrivial odd prime power and $n \geq 2$ a natural number such that $(n, q) \neq (2, 3)$. Moreover, let ε be an element of $\{+,-\}$. Then $PSL_n^{\varepsilon}(q)$ is a Goldschmidt group if and only if $n = 2$ and $q \equiv 3$ or $5 \mod 8$.*

Proof. Set $G := PSL_n^{\varepsilon}(q)$.

Assume that $n = 2$. Then $G \cong PSL_2(q)$. By Lemma 3.13, G has dihedral Sylow 2-subgroups of order $\frac{1}{2}(q-1)_2(q+1)_2$. So, if $q \equiv 3$ or $5 \mod 8$, then G has abelian Sylow 2-subgroups and is thus a Goldschmidt group. If $q \equiv 1$ or $7 \mod 8$, then the Sylow 2-subgroups of G are dihedral of order at least 8 and hence nonabelian. Moreover, if $q \equiv 1$ or $7 \mod 8$, then [49, Theorem 37] shows that G is not isomorphic to a finite simple group of Lie type in characteristic 2 of Lie rank 1. So G is not a Goldschmidt group if $q \equiv 1$ or $7 \mod 8$.

Assume now that $n \geq 3$. Again, we see from [49, Theorem 37] that there is no finite simple group of Lie type in characteristic 2 of Lie rank 1 which is isomorphic to G. Also, G has a subgroup isomorphic to $SL_2^{\varepsilon}(q) \cong SL_2(q)$, and therefore, the Sylow 2-subgroups of G are nonabelian. Consequently, G is not a Goldschmidt group. □

Lemma 3.22. *Let n be a positive integer, q a nontrivial odd prime power and $\varepsilon \in \{+,-\}$. Let E be the subgroup of $SL_n^{\varepsilon}(q)$ consisting of the diagonal matrices in $SL_n^{\varepsilon}(q)$ with diagonal entries in $\{1,-1\}$. Then $|E| = 2^{n-1}$. Moreover, any elementary abelian 2-subgroup of $SL_n^{\varepsilon}(q)$ is conjugate to a subgroup of E.*

Proof. It is straightforward to check that $|E| = 2^{n-1}$.

Let E_0 be an elementary abelian 2-subgroup of $SL_n^{\varepsilon}(q)$. We show that E_0 is conjugate to a subgroup of E. Using Dirichlet's theorem [20, Theorem 3.3.1], one can see that there is an odd prime number

q^* with $-q \sim q^*$, and Proposition 3.19 shows that the 2-fusion systems of $SU_n(q)$ and $SL_n(q^*)$ are isomorphic. Therefore, it is enough to consider the case $\varepsilon = +$.

Since E_0 is an elementary abelian 2-group, any two elements of E_0 commute, and any element of E_0 is diagonalizable (see Lemma 3.3). It follows that E_0 is simultaneously diagonalizable, and this implies that E_0 is conjugate to a subgroup of E. □

Lemma 3.23. *Let the notation be as in Lemma 3.22, and set $Y := SL_n^{\varepsilon}(q)$. Moreover, for any $A \subseteq \{1, \dots, n\}$, let t_A be the matrix* $\mathrm{diag}(d_1, \dots, d_n)$*, where $d_i = -1$ if $i \in A$ and $d_i = 1$ if $i \in \{1, \dots, n\} \setminus A$. Then the following hold:*

(i) *For each $\pi \in S_n$, there is a unique $\varphi_\pi \in \mathrm{Aut}_Y(E)$ such that*
$$(t_A)^{\varphi_\pi} = t_{A^\pi}$$
for any $A \subseteq \{1, \dots, n\}$ of even order.

(ii) $\mathrm{Aut}_Y(E) = \{\varphi_\pi \mid \pi \in S_n\}$.

Proof. Let V be the defining module for Y. Let $B = (v_1, \dots, v_n)$ be a basis for V with B orthonormal if V is unitary. For any $A \subseteq \{1, \dots, n\}$, let e_A be the linear map $V \longrightarrow V$ represented by t_A with respect to B. Then $e_A \in GL^{\varepsilon}(V)$.

Let $\pi \in S_n$. To prove (i), it suffices to find some $\alpha_\pi \in SL^{\varepsilon}(V)$ such that $(e_A)^{\alpha_\pi} = e_{A^\pi}$ for any $A \subseteq \{1, \dots, n\}$ of even order. Let $\widetilde{\alpha}_\pi$ be the linear map $V \longrightarrow V$ sending v_i to v_{i^π} for each $1 \le i \le n$. Then $\det(\widetilde{\alpha}_\pi) = \mathrm{sgn}(\pi) \in \{-1, 1\}$. Set $\alpha_\pi := \widetilde{\alpha}_\pi$ if $\det(\widetilde{\alpha}_\pi) = 1$ and $\alpha_\pi := e_{\{1\}}\widetilde{\alpha}_\pi$ if $\det(\widetilde{\alpha}_\pi) = -1$. Then $\alpha_\pi \in SL^{\varepsilon}(V)$. Also, if $A \subseteq \{1, \dots, n\}$ and $1 \le i \le n$, then
$$(v_i)^{(\alpha_\pi)^{-1} e_A \alpha_\pi} = v_i^{(\widetilde{\alpha}_\pi)^{-1} e_A \widetilde{\alpha}_\pi} = \begin{cases} -v_i & \text{if } i \in A^\pi \\ v_i & \text{if } i \notin A^\pi \end{cases}$$
and hence $(e_A)^{\alpha_\pi} = e_{A^\pi}$. The proof of (i) is now complete.

We now prove (ii). If $n \in \{1, 2\}$, then $\mathrm{Aut}_Y(E) = \{\mathrm{id}_E\} = \{\varphi_\pi \mid \pi \in S_n\}$. Assume now that $n \ge 3$. Let $\varphi \in \mathrm{Aut}_Y(E)$, and let $y \in Y$ with $\varphi = c_y|_{E,E}$. We are going to show that y is a generalized permutation matrix, which implies the desired conclusion that $\varphi = \varphi_\pi$ for some $\pi \in S_n$. Let $y_1, \dots, y_n$ denote the columns of y, and let $1 \le j \le n$. To prove that y is a generalized permutation matrix, it suffices to show that y_j has precisely one nonzero entry. Let $1 \le k \ne \ell \le n$ with $k \ne j \ne \ell$. Let $A := \{j, k\}$ and $C := \{j, \ell\}$. As y normalizes E, there exist distinct subsets $A_0, C_0 \subseteq \{1, \dots, n\}$ with $|A_0| = 2 = |C_0|$ and $(t_{A_0})^y = t_A$, $(t_{C_0})^y = t_C$. Hence, $t_{A_0} \cdot y = y \cdot t_A$ and $t_{C_0} \cdot y = y \cdot t_C$, and so y_j is an eigenvector of t_{A_0} and of t_{C_0} with eigenvalue -1. Together with the fact that $|A_0| = 2 = |C_0|$ and $A_0 \ne C_0$, it follows that y_j has only one nonzero entry, as required. □

Lemma 3.24. *Let q be a nontrivial odd prime power, $n \ge 3$ a natural number and S a Sylow 2-subgroup of $PSL_n(q)$. Then $\mathrm{Aut}_{PSL_n(q)}(S) = \mathrm{Inn}(S)$.*

Proof. Let $R \in \mathrm{Syl}_2(SL_n(q))$ such that S is the image of R in $PSL_n(q)$. Let T be a Sylow 2-subgroup of $GL_n(q)$ with $R \le T$. By [36, Theorem 1], we have $N_{GL_n(q)}(R) = TC_{GL_n(q)}(T)$. So we have that $\mathrm{Aut}_{SL_n(q)}(R)$ is a 2-group. Since the image of $N_{SL_n(q)}(R)$ in $PSL_n(q)$ equals $N_{PSL_n(q)}(S)$ (see [35, Kapitel I, Hilfssatz 7.7 c)]), it follows that $\mathrm{Aut}_{PSL_n(q)}(S)$ is a 2-group, and this implies $\mathrm{Aut}_{PSL_n(q)}(S) = \mathrm{Inn}(S)$. □

3.5. k-connectivity

In this subsection, we prove some connectivity properties of the Sylow 2-subgroups of $SL_n(q)$ and $PSL_n(q)$, where q is a nontrivial odd prime power and $n \ge 6$. We will work with the following definition (see [31, Section 8]):

Definition 3.25. Let S be a finite 2-group, and let k be a positive integer. If A and B are elementary abelian subgroups of S of rank at least k, then A and B are said to be *k-connected* if there is a sequence

$$A = A_1, A_2, \dots, A_n = B \quad (n \geq 1)$$

of elementary abelian subgroups A_i, $1 \leq i \leq n$, of S with rank at least k such that

$$A_i \subseteq A_{i+1} \text{ or } A_{i+1} \subseteq A_i$$

for all $1 \leq i \leq n-1$. The group S is said to be *k-connected* if any two elementary abelian subgroups of S of rank at least k are k-connected.

Lemma 3.26 [31, Lemma 8.4]. *Let S be a finite 2-group, and let k be a positive integer. If S has a normal elementary abelian subgroup of rank at least $2^{k-1}+1$, then S is k-connected.*

Lemma 3.27. *Let q be a nontrivial odd prime power with $q \equiv 1 \bmod 4$, and let $n \geq 6$ be a natural number. Then the Sylow 2-subgroups of $PSL_n(q)$ and those of $SL_n(q)$ are 3-connected.*

Proof. Let W_0 be the unique Sylow 2-subgroup of $GL_1(q)$, and let W_1 be the Sylow 2-subgroup of $GL_2(q)$ given in Lemma 3.10 (i). For each $r \geq 2$, let W_r be the Sylow 2-subgroup of $GL_{2^r}(q)$ obtained from W_{r-1} by the construction given in the last statement of Lemma 3.14. Let $0 \leq r_1 < \cdots < r_t$ such that $n = 2^{r_1} + \cdots + 2^{r_t}$, and let W be the Sylow 2-subgroup of $GL_n(q)$ obtained from $W_{r_1}, \dots, W_{r_t}$ by using the last statement of Lemma 3.15.

For any $k \geq 1$, let $R_k(q)$ denote the subgroup of $GL_k(q)$ consisting of all diagonal matrices $D \in GL_k(q)$, where $D^2 \in Z(GL_k(q))$ and any diagonal element of D is a 2-element of $\mathbb{F}_q^*$. Also, let $R := R_6(q)$. By Lemma 3.14 and induction on r, $R_{2^r}(q) \trianglelefteq W_r$, using Lemma 3.10 (i) to anchor the induction. Then $R = R_{2^{r_1}}(q) \times \cdots \times R_{2^{r_t}}(q) \trianglelefteq W$. Let $R_0 := R \cap SL_n(q)$ and $E := \Omega_1(R_0)$. By Lemma 3.22, $m(E) = n-1 \geq 5$, so by Lemma 3.26, $W_0 := W \cap SL_n(q)$ is 3-connected. Set $W^* := W/(W \cap Z(SL_n(q)))$; then $m(E^*) \geq m(E) - 1 = n-2$, so by Lemma 3.26, W_0^* is 3-connected, unless possibly $n = 6$. But if $n = 6$, then $F^* = R_0^*$ is of rank 5, where $F = \langle E, iI_6 \cdot r\rangle$ for some reflection $r \in R$ and some $i \in \mathbb{F}_q^*$ of order 4. □

Lemma 3.26 and the proof of Lemma 3.27 show that we also have the following:

Lemma 3.28. *Let q be a nontrivial odd prime power with $q \equiv 1 \bmod 4$, and let $n \geq 6$ be a natural number. Then the Sylow 2-subgroups of $PSL_n(q)$ and those of $SL_n(q)$ are 2-connected.*

We now study the case $q \equiv 3 \bmod 4$.

Lemma 3.29. *Let q be a nontrivial odd prime power with $q \equiv 3 \bmod 4$, and let $n \geq 6$ be a natural number. Then the Sylow 2-subgroups of $PSL_n(q)$ and those of $SL_n(q)$ are 2-connected. If $n \geq 10$, then we even have that the Sylow 2-subgroups of $PSL_n(q)$ and those of $SL_n(q)$ are 3-connected.*

Proof. Let W_0 denote the unique Sylow 2-subgroup of $GL_1(q)$, and let W_1 be a Sylow 2-subgroup of $GL_2(q)$. By Lemma 3.10 (ii), W_1 is semidihedral. Let $m \in \mathbb{N}$ with $|W_1| = 2^m$. Also, let $h, a \in W_1$ such that $\mathrm{ord}(h) = 2^{m-1}$, $\mathrm{ord}(a) = 2$ and $h^a = h^{2^{m-2}-1}$. Set $z := -I_2 = h^{2^{m-2}}$. For each $r \geq 2$, let W_r be the Sylow 2-subgroup of $GL_{2^r}(q)$ obtained from W_{r-1} by the construction given in the last statement of Lemma 3.14. Let $0 \leq r_1 < \cdots < r_t$ such that $n = 2^{r_1} + \cdots + 2^{r_t}$, and let W be the Sylow 2-subgroup of $GL_n(q)$ obtained from $W_{r_1}, \dots, W_{r_t}$ by using the last statement of Lemma 3.15.

Given a natural number $\ell \geq 1$ and elements $x_1, \dots, x_\ell \in GL_2(q)$, we write $\mathrm{diag}(x_1, \dots, x_\ell)$ for the block diagonal matrix

$$\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_\ell \end{pmatrix}.$$

For each natural number $r \geq 1$, let A_r denote the subgroup of $GL_{2^r}(q)$ consisting of the matrices $\mathrm{diag}(x_1, \dots, x_{2^{r-1}})$, where either $x_i \in \langle z \rangle$ for all $1 \leq i \leq 2^{r-1}$ or x_i is an element of $\langle h \rangle$ with order 4 for all $1 \leq i \leq 2^{r-1}$. By induction over r, one can see that $A_r \trianglelefteq W_r$ for all $r \geq 1$. Also, let $\widetilde{A_r} := \Omega_1(A_r)$ for all $r \geq 1$. Clearly, $\widetilde{A_r} \trianglelefteq W_r$ for all $r \geq 1$.

We now consider two cases.

Case 1: n is even.

Let E be the subgroup of $GL_n(q)$ consisting of the matrices $\mathrm{diag}(x_1, \dots, x_{\frac{n}{2}})$, where either $x_i \in \langle z \rangle$ for all $1 \leq i \leq \frac{n}{2}$ or x_i is an element of $\langle h \rangle$ with order 4 for all $1 \leq i \leq \frac{n}{2}$. Let $\widetilde{E} := \Omega_1(E)$. Since $A_{r_i} \trianglelefteq W_{r_i}$ for all $1 \leq i \leq t$, we have that E and $\widetilde{E}$ are normal subgroups of W. Lemma 3.16 (iii) shows that $E \leq W \cap SL_n(q)$.

As $\widetilde{E}$ is elementary abelian of order $2^{\frac{n}{2}}$, Lemma 3.26 implies that $W \cap SL_n(q)$ is 2-connected and even 3-connected if $n \geq 10$. Since $EZ(SL_n(q))/Z(SL_n(q))$ is a normal elementary abelian subgroup of $(W \cap SL_n(q))Z(SL_n(q))/Z(SL_n(q))$ with order $2^{\frac{n}{2}}$, Lemma 3.26 also shows that a Sylow 2-subgroup is 2-connected, and even 3-connected if $n \geq 10$.

Case 2: n is odd.

Now let E denote the subgroup of $GL_n(q)$ consisting of the matrices

$$\left(\begin{array}{c|ccc} 1 & & & \\ \hline & x_1 & & \\ & & \ddots & \\ & & & x_{\frac{n-1}{2}} \end{array}\right),$$

where $x_i \in \langle z \rangle$ for all $1 \leq i \leq \frac{n-1}{2}$. Since $\widetilde{A_{r_i}} \trianglelefteq W_{r_i}$ for all $2 \leq i \leq t$, we have that E is a normal subgroup of $W \cap SL_n(q)$. Moreover, E is elementary abelian of order $2^{\frac{n-1}{2}}$. Lemma 3.26 implies that $W \cap SL_n(q)$ is 2-connected and even 3-connected if $n \geq 11$. There is nothing else to show since the Sylow 2-subgroups of $PSL_n(q)$ are isomorphic to those of $SL_n(q)$ (as n is odd). $\square$

We show next that the groups $SL_n(q)$, where $6 \leq n \leq 9$ and $q \equiv 3 \mod 4$, and the groups $PSL_n(q)$, where $7 \leq n \leq 9$ and $q \equiv 3 \mod 4$, also have 3-connected Sylow 2-subgroups.

Lemma 3.30. *Let q be a nontrivial odd prime power with $q \equiv 3 \mod 4$. Then the Sylow 2-subgroups of $SL_6(q)$ and those of $SL_7(q)$ are 3-connected.*

Proof. Let W_1 be a Sylow 2-subgroup of $GL_2(q)$, let W_2 be the Sylow 2-subgroup of $GL_4(q)$ obtained from W_1 by the construction given in the last statement of Lemma 3.14 and let W be the Sylow 2-subgroup of $GL_6(q)$ obtained from W_1 and W_2 by using the last statement of Lemma 3.15.

From Lemma 3.15, we see that the Sylow 2-subgroups of $SL_7(q)$ are isomorphic to those of $GL_6(q)$. So it is enough to show that W and $W \cap SL_6(q)$ are 3-connected. Given elements $x_1, x_2, x_3 \in GL_2(q)$, we write $\mathrm{diag}(x_1, x_2, x_3)$ for the block diagonal matrix

$$\begin{pmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{pmatrix}.$$

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Let A be the subgroup of $W \cap SL_6(q)$ consisting of the matrices $\mathrm{diag}(x_1, x_2, x_3)$, where $x_i \in \langle -I_2 \rangle$ for $1 \le i \le 3$. Then $A \cong E_8$. We prove the following:

(1) If E is an elementary abelian subgroup of W of rank at least 3, then E is 3-connected to an elementary abelian subgroup of $W \cap SL_6(q)$ of rank at least 3.
(2) If E is an elementary abelian subgroup of $W \cap SL_6(q)$ of rank at least 3, then E is 3-connected to A in $W \cap SL_6(q)$.

By (1) and (2), any elementary abelian subgroup of W of rank at least 3 is 3-connected to A, and so W is 3-connected. Similarly, (2) implies that $W \cap SL_6(q)$ is 3-connected.

Let $Z := \langle \mathrm{diag}(-I_2, I_2, I_2), \mathrm{diag}(I_2, -I_2, -I_2) \rangle$. Since $Z \le Z(W)$, we have that any elementary abelian subgroup of W of rank at least 3 is 3-connected to an E_8-subgroup of W containing Z. Also, any elementary abelian subgroup of $W \cap SL_6(q)$ of rank at least 3 is 3-connected (in $W \cap SL_6(q)$) to an E_8-subgroup of $W \cap SL_6(q)$ containing Z. Therefore, we only need to consider E_8-subgroups containing Z in order to prove (1) and (2).

So let E be an E_8-subgroup of W with $Z \le E$, and let $s \in E \setminus Z$. Suppose that $s = \mathrm{diag}(s_1, s_2, s_3)$, where $s_1, s_2, s_3 \in W_1$. Then $[E, A] = 1$, and it is easy to deduce that E is 3-connected to A so that E satisfies (1). Also, if $E \le W \cap SL_6(q)$, it is easy to deduce that E satisfies (2).

Suppose now that

$$s = \begin{pmatrix} s_1 & & \\ & & s_2 \\ & s_3 & \end{pmatrix}$$

for some $s_1, s_2, s_3 \in W_1$. Since $s^2 = I_6$, we have $s_2 = s_3^{-1}$. Let a be an involution of W_1 with $a \neq -I_2$. Set $s^* := \mathrm{diag}(I_2, a, a^{s_2})$ and $E^* := \langle Z, s^* \rangle \cong E_8$. Clearly, $E^* \le W \cap SL_6(q)$. It is easy to check that $[E, E^*] = 1$, which implies that E is 3-connected to E^*. So E satisfies (1). If $E \le W \cap SL_6(q)$, then E is 3-connected to E^* in $W \cap SL_6(q)$, and E^* is 3-connected to A in $W \cap SL_6(q)$ since $[E^*, A] = 1$. Therefore, E satisfies (2) when $E \le W \cap SL_6(q)$. □

Let q be a nontrivial odd prime power with $q \equiv 3 \bmod 4$. A Sylow 2-subgroup of $PSL_7(q)$ is isomorphic to a Sylow 2-subgroup of $SL_7(q)$. So, by Lemma 3.30, the Sylow 2-subgroups of $PSL_7(q)$ are 3-connected.

We need the following lemma in order to prove that the Sylow 2-subgroups of $SL_n(q)$ and $PSL_n(q)$ are 3-connected when $n \in \{8, 9\}$.

Lemma 3.31. *Let q be a nontrivial odd prime power with $q \equiv 3 \bmod 4$, and let V be a Sylow 2-subgroup of $GL_4(q)$. Let $u \in V$ with $u^2 = I_4$ or $u^2 = -I_4$. Then there is an involution $v \in V \setminus \langle u, -I_4 \rangle$ which commutes with u.*

Proof. Fix a Sylow 2-subgroup W_1 of $GL_2(q)$, and let W_2 be the Sylow 2-subgroup of $GL_4(q)$ obtained from W_1 by the construction given in the last statement of Lemma 3.14. By Sylow's theorem, we may assume that $V = W_2$. Let a be an involution of W_1 with $a \neq -I_2$.

First, we consider the case that

$$u = \begin{pmatrix} x & \\ & y \end{pmatrix}$$

with elements $x, y \in W_1$. If $x \notin \langle -I_2 \rangle$ or $y \notin \langle -I_2 \rangle$, then

$$\begin{pmatrix} -I_2 & \\ & I_2 \end{pmatrix} \in W_2$$

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is an involution commuting with u and not lying in $\langle u, -I_4\rangle$. If $x, y \in \langle -I_2\rangle$, then we may choose

$$v := \begin{pmatrix} a & \\ & a \end{pmatrix}.$$

Assume now that

$$u = \begin{pmatrix} & x \\ y & \end{pmatrix}$$

with elements $x, y \in W_1$. Let

$$v := \begin{pmatrix} a & \\ & a^x \end{pmatrix}.$$

As a is an involution of W_1, we have that v is an involution of W_2. By a direct calculation (using that $xy \in \langle -I_2\rangle$), v has the desired properties. □

Lemma 3.32. *Let q be a nontrivial odd prime power with $q \equiv 3 \bmod 4$. Then the Sylow 2-subgroups of $SL_8(q)$ and those of $SL_9(q)$ are 3-connected.*

Proof. Fix a Sylow 2-subgroup W_1 of $GL_2(q)$, let W_2 be the Sylow 2-subgroup of $GL_4(q)$ obtained from W_1 by the construction given in the last statement of Lemma 3.14 and let W be the Sylow 2-subgroup of $GL_8(q)$ obtained from W_2 by the construction given in the last statement of Lemma 3.14. Set $S := W \cap SL_8(q)$.

From Lemma 3.15, we see that the Sylow 2-subgroups of $SL_9(q)$ are isomorphic to those of $GL_8(q)$. So it is enough to show that W and S are 3-connected.

Given a natural number $\ell \geq 1$ and $x_1, \ldots, x_\ell \in GL_2(q) \cup GL_4(q)$, we write $\mathrm{diag}(x_1, \ldots, x_\ell)$ for the block diagonal matrix

$$\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_\ell \end{pmatrix}.$$

Set

$$A := \{\mathrm{diag}(x_1, x_2, x_3, x_4) \mid x_i \in \langle -I_2\rangle \ \forall\, 1 \leq i \leq 4\} \leq S$$

and

$$Z := \langle -I_8\rangle \leq S.$$

Then $A \cong E_{16}$. Since $Z \leq Z(W)$, we have that any elementary abelian subgroup of W of rank at least 3 is 3-connected to an E_8-subgroup of W containing Z. Similarly, any elementary abelian subgroup of S of rank at least 3 is 3-connected to an E_8-subgroup of S containing Z. So it suffices to prove that any E_8-subgroup E of W with $Z \leq E$ is 3-connected to A, where E is even 3-connected in S to A if $E \leq S$. Thus, let E be an E_8-subgroup of W containing Z, and let $x, y \in E$ with $E = \langle Z, x, y\rangle$.

We consider a number of cases. Below, a will always denote an involution of W_1 with $a \neq -I_2$.

Case 1: $x = \mathrm{diag}(-I_4, I_4)$ and $y = \mathrm{diag}(b_1, b_2)$ for some $b_1, b_2 \in W_2$.

We determine an involution $y_1 \in C_W(E) \setminus \langle Z, x\rangle$ such that $\langle Z, x, y_1\rangle \cong E_8$ is 3-connected to A. In the case that $E \leq S$, we determine y_1 such that $y_1 \in S$ and such that $\langle Z, x, y_1\rangle$ is 3-connected to A in S. The existence of such an involution y_1 easily implies that E is 3-connected to A and even 3-connected to A in S if $E \leq S$. The involution y_1 is given by the following table in dependence of y. In each row, r_1, r_2, r_3, r_4 are assumed to be elements of W_1 such that y is equal to the matrix given in the column “y”

and such that the conditions in the column 'Conditions' (if any) are satisfied. The column 'y_1' gives the involution y_1 with the desired properties. For each row, one can verify the stated properties of y_1 by a direct calculation or by using the previous rows.

Case	**y**	**Conditions**	y_1
1.1	$\begin{pmatrix} r_1 & & & \\ & r_2 & & \\ & & r_3 & \\ & & & r_4 \end{pmatrix}$		y
1.2	$\begin{pmatrix} r_1 & & & \\ & r_2 & & \\ & & & r_3 \\ & & r_4 & \end{pmatrix}$	$\langle r_1, r_2\rangle \not\leq \langle -I_2\rangle$	$\begin{pmatrix} r_1 & & \\ & r_2 & \\ & & I_4 \end{pmatrix}$
1.3	$\begin{pmatrix} r_1 & & & \\ & r_2 & & \\ & & & r_3 \\ & & r_4 & \end{pmatrix}$	$r_1, r_2 \leq \langle -I_2\rangle$	$\begin{pmatrix} a & & \\ & a & \\ & & I_4 \end{pmatrix}$
1.4	$\begin{pmatrix} & r_1 & & \\ r_2 & & & \\ & & r_3 & \\ & & & r_4 \end{pmatrix}$	$\langle r_3, r_4\rangle \not\leq \langle -I_2\rangle$	$\begin{pmatrix} I_4 & & \\ & r_3 & \\ & & r_4 \end{pmatrix}$
1.5	$\begin{pmatrix} & r_1 & & \\ r_2 & & & \\ & & r_3 & \\ & & & r_4 \end{pmatrix}$	$r_3, r_4 \leq \langle -I_2\rangle$	$\begin{pmatrix} I_4 & & \\ & a & \\ & & a \end{pmatrix}$
1.6	$\begin{pmatrix} & r_1 & & \\ r_2 & & & \\ & & & r_3 \\ & & r_4 & \end{pmatrix}$		$\begin{pmatrix} & r_1 & \\ r_2 & & \\ & & I_4 \end{pmatrix}$

Case 2: $x = \mathrm{diag}(a_1, a_2)$ *and* $y = \mathrm{diag}(b_1, b_2)$ *for some* $a_1, a_2, b_1, b_2 \in W_2$.

Set $x_1 := \mathrm{diag}(-I_4, I_4)$. Since $E = \langle Z, x, y\rangle \cong E_8$, the elements x and y cannot be both contained in $\langle Z, x_1\rangle$. Without loss of generality, we may assume that $y \notin \langle Z, x_1\rangle$. Then $E_1 := \langle Z, x_1, y\rangle \cong E_8$. The group E_1 is 3-connected to A by Case 1, and it is 3-connected to E since E and E_1 commute. Hence, E is 3-connected to A. Clearly, if $E \leq S$, then E is even 3-connected in S to A.

Case 3: There are $a_1, a_2, b_1, b_2 \in W_2$ *with*

$$\{x, y\} = \left\{ \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}, \begin{pmatrix} & b_1 \\ b_2 & \end{pmatrix} \right\}.$$

Without loss of generality, we assume that

$$x = \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \text{ and } y = \begin{pmatrix} & b_1 \\ b_2 & \end{pmatrix}.$$

Since x and y are commuting involutions, we have $b_1 = b_2^{-1}$ and $a_2 = {a_1}^{b_1}$. By Lemma 3.31, there is an involution $\widetilde{a_1} \in W_2 \setminus \langle a_1, -I_4\rangle$ which commutes with a_1. Set

$$y_1 := \begin{pmatrix} \widetilde{a_1} & \\ & \widetilde{a_1}^{b_1} \end{pmatrix}.$$

It is easy to see that $y_1 \in S$, and y_1 is an involution since $\widetilde{a_1}$ is an involution of W_2. We have $[x, y_1] = 1$ since $\widetilde{a_1}$ commutes with a_1 and $\widetilde{a_1}^{b_1}$ commutes with ${a_1}^{b_1} = a_2$. A direct calculation using that $b_1 = b_2^{-1}$ shows that we also have $[y, y_1] = 1$. Thus, $E = \langle Z, x, y\rangle$ commutes with $E_1 := \langle Z, x, y_1\rangle$. Since

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$\widetilde{a_1} \notin \langle a_1, -I_4 \rangle$, we have $y_1 \notin \langle Z, x \rangle$ and hence $E_1 \cong E_8$. Applying Case 2, it follows that E is 3-connected to A (and even 3-connected in S to A when $E \leq S$).

Case 4: There are $a_1, a_2, b_1, b_2 \in W_2$ with

$$x = \begin{pmatrix} & a_1 \\ a_2 & \end{pmatrix} \textit{ and } y = \begin{pmatrix} & b_1 \\ b_2 & \end{pmatrix}.$$

This case can be reduced to Case 3 since $E = \langle Z, x, y \rangle = \langle Z, x, xy \rangle$. □

Let q be a nontrivial odd prime power with $q \equiv 3 \mod 4$. A Sylow 2-subgroup of $PSL_9(q)$ is isomorphic to a Sylow 2-subgroup of $SL_9(q)$. So, by Lemma 3.32, the Sylow 2-subgroups of $PSL_9(q)$ are 3-connected.

Lemma 3.33. *Let q be a nontrivial odd prime power with $q \equiv 3 \mod 4$. Then the Sylow 2-subgroups of $PSL_8(q)$ are 3-connected.*

Proof. Let W_1 be a Sylow 2-subgroup of $GL_2(q)$. Let W_2 be the Sylow 2-subgroup of $GL_4(q)$ obtained from W_1 by the construction given in the last statement of Lemma 3.14, and let W_3 be the Sylow 2-subgroup of $GL_8(q)$ obtained from W_2 by the construction given in the last statement of Lemma 3.14. Set $S := W_3 \cap SL_8(q)$. For each subgroup or element X of $SL_8(q)$, let $\overline{X}$ denote the image of X in $PSL_8(q)$. We prove that $\overline{S}$ is 3-connected.

Given a natural number $\ell \geq 1$ and $x_1, \ldots, x_\ell \in GL_2(q) \cup GL_4(q)$, we write $\mathrm{diag}(x_1, \ldots, x_\ell)$ for the block diagonal matrix

$$\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_\ell \end{pmatrix}.$$

Set

$$A := \{\mathrm{diag}(x_1, x_2, x_3, x_4) \mid x_i \in \langle -I_2 \rangle \; \forall \; 1 \leq i \leq 4\} \leq S.$$

We have $\overline{A} \cong E_8$.

Set

$$Z := \langle \mathrm{diag}(-I_4, I_4) \rangle.$$

We have $\overline{Z} \leq Z(\overline{S})$. Using this, it is easy to note that any elementary abelian subgroup of $\overline{S}$ of rank at least 3 is 3-connected to an E_8-subgroup of $\overline{S}$ containing $\overline{Z}$. Hence, it suffices to prove that any E_8-subgroup of $\overline{S}$ containing $\overline{Z}$ is 3-connected to $\overline{A}$.

Let $x, y \in S$ and $B := \langle \overline{Z}, \overline{x}, \overline{y} \rangle$. Suppose that $B \cong E_8$. Considering a number of cases, we will prove that B is 3-connected to $\overline{A}$. Below, a will always denote an involution of W_1 with $a \neq -I_2$.

Case 1: $x = \mathrm{diag}(r_1, r_2, r_3, r_4)$ and $y = \mathrm{diag}(m_1, m_2)$ for some $r_1, r_2, r_3, r_4 \in W_1$ and $m_1, m_2 \in W_2$.

We consider a number of subcases. These subcases are given by the rows of the table below. In each row, we assume that s_1, s_2, s_3, s_4 are elements of W_1 such that y is equal to the matrix given in the column "y". We also assume that the conditions in the column 'Conditions' (if any) are satisfied. The column 'y_1' gives an element of S such that $\overline{y_1}$ is an involution in $C_{\overline{S}}(\overline{E}) \setminus \langle \overline{Z}, \overline{x} \rangle$ and such that $\langle \overline{Z}, \overline{x}, \overline{y_1} \rangle$ is 3-connected to $\overline{A}$. The existence of such an element y_1 easily implies that B is 3-connected to $\overline{A}$.

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Case	**y**	**Conditions**	y_1
1.1	$\begin{pmatrix} s_1 & & & \\ & s_2 & & \\ & & s_3 & \\ & & & s_4 \end{pmatrix}$		y
1.2	$\begin{pmatrix} & s_1 & & \\ s_2 & & & \\ & & s_3 & \\ & & & s_4 \end{pmatrix}$	$x \notin A$	$\begin{pmatrix} I_4 & & \\ & -I_2 & \\ & & I_2 \end{pmatrix}$
1.3	$\begin{pmatrix} & s_1 & & \\ s_2 & & & \\ & & s_3 & \\ & & & s_4 \end{pmatrix}$	$x \in A$	$\begin{pmatrix} a & & \\ & a^{s_2^{-1}} & \\ & & I_4 \end{pmatrix}$
1.4	$\begin{pmatrix} & s_1 & & \\ s_2 & & & \\ & & & s_3 \\ & & s_4 & \end{pmatrix}$	$x \notin A$	$\begin{pmatrix} I_2 & & & \\ & -I_2 & & \\ & & I_2 & \\ & & & -I_2 \end{pmatrix}$
1.5	$\begin{pmatrix} & s_1 & & \\ s_2 & & & \\ & & & s_3 \\ & & s_4 & \end{pmatrix}$	$x \in A$	$\begin{pmatrix} a & & & \\ & a^{s_2^{-1}} & & \\ & & a & \\ & & & a^{s_4^{-1}} \end{pmatrix}$

The subcase that y has the form

$$\begin{pmatrix} s_1 & & & \\ & s_2 & & \\ & & & s_3 \\ & & s_4 & \end{pmatrix}$$

can be easily reduced to Cases 1.2 and 1.3.

Case 2: There are $r_1, r_2, r_3, r_4 \in W_1$ and $m_1, m_2 \in W_2$ with

$$x = \begin{pmatrix} & r_1 & & \\ r_2 & & & \\ & & r_3 & \\ & & & r_4 \end{pmatrix} \text{ and } y = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix}.$$

Case 2.1: There are $s_1, s_2, s_3, s_4 \in W_1$ with

$$y = \begin{pmatrix} s_1 & & & \\ & s_2 & & \\ & & s_3 & \\ & & & s_4 \end{pmatrix} \text{ or } y = \begin{pmatrix} & s_1 & & \\ s_2 & & & \\ & & s_3 & \\ & & & s_4 \end{pmatrix}.$$

Noticing that $\langle \overline{Z}, \bar{x}, \bar{y} \rangle = \langle \overline{Z}, \bar{x}, \bar{x}\bar{y} \rangle$, this case can be reduced to Case 1.

Case 2.2: There are $s_1, s_2, s_3, s_4 \in W_1$ with

$$y = \begin{pmatrix} s_1 & & & \\ & s_2 & & \\ & & & s_3 \\ & & s_4 & \end{pmatrix}.$$

Since $B \cong E_8$, we have $\varepsilon x^y = x$, where $\varepsilon \in \{+, -\}$. By a direct calculation, we have

$$x^y = \begin{pmatrix} & s_1^{-1} r_1 s_2 & & \\ s_2^{-1} r_2 s_1 & & & \\ & & r_4^{s_4} & \\ & & & r_3^{s_3} \end{pmatrix}.$$

As $x = \varepsilon x^y$, we have $r_1 = \varepsilon s_1^{-1} r_1 s_2$, $r_2 = \varepsilon s_2^{-1} r_2 s_1$, $r_3 = \varepsilon r_4^{s_4}$ and $r_4 = \varepsilon r_3^{s_3}$. Note that $\varepsilon s_1^{r_1} = s_2$ and $\varepsilon s_2^{r_2} = s_1$.

We now consider a number of subsubcases. These subsubcases are given by the rows of the table below. The columns 'Condition 1' and 'Condition 2' describe the subsubcase under consideration. The column 'y_1' gives an element $y_1 \in S$ such that $\overline{y_1}$ is an involution in $C_{\overline{S}}(\overline{E}) \setminus \langle \overline{Z}, \overline{x} \rangle$ and such that $\langle \overline{Z}, \overline{x}, \overline{y_1} \rangle$ is 3-connected to $\overline{A}$. In each subsubcase, one can see from the above calculations and from the previous cases that y_1 indeed has the stated properties. The existence of such an element y_1 easily implies that B is 3-connected to $\overline{A}$ in all subsubcases.

Case	**Condition 1**	**Condition 2**	y_1
2.2.1	$x^2 = I_8 = y^2$	$\langle r_3, r_4 \rangle \nleq \langle -I_2 \rangle$	$\begin{pmatrix} \varepsilon s_1 & & & \\ & s_2 & & \\ & & \varepsilon r_3 & \\ & & & r_4 \end{pmatrix}$
.2.2	$x^2 = I_8 = y^2$	$\langle r_3, r_4 \rangle \leq \langle -I_2 \rangle$	$\begin{pmatrix} & r_1 & & \\ r_2 & & & \\ & & \varepsilon a & \\ & & & a^{s_3} \end{pmatrix}$
2.2.3	$x^2 = -I_8 = y^2$		$\begin{pmatrix} \varepsilon s_1 & & & \\ & s_2 & & \\ & & \varepsilon r_3 & \\ & & & r_4 \end{pmatrix}$
2.2.4	$x^2 = I_8, y^2 = -I_8$	$\langle r_3, r_4 \rangle \nleq \langle -I_2 \rangle$	$\begin{pmatrix} I_4 & & \\ & \varepsilon r_3 & \\ & & r_4 \end{pmatrix}$
2.2.5	$x^2 = I_8, y^2 = -I_8$	$\langle r_3, r_4 \rangle \leq \langle -I_2 \rangle$	$\begin{pmatrix} I_4 & & \\ & \varepsilon a & \\ & & \varepsilon a^{s_3} \end{pmatrix}$

The case that $x^2 = -I_8$ and $y^2 = I_8$ can be easily reduced to Cases 2.2.4 and 2.2.5.

Case 2.3: There are $s_1, s_2, s_3, s_4 \in W_1$ with

$$y = \begin{pmatrix} & s_1 & & \\ s_2 & & & \\ & & & s_3 \\ & & s_4 & \end{pmatrix}.$$

Since $\langle \overline{Z}, \overline{x}, \overline{y} \rangle = \langle \overline{Z}, \overline{x}, \overline{x}\overline{y} \rangle$, this case can be reduced to Case 2.2.

Case 3: There are $r_1, r_2, r_3, r_4 \in W_1$ and $m_1, m_2 \in W_2$ with

$$x = \begin{pmatrix} r_1 & & & \\ & r_2 & & \\ & & & r_3 \\ & & r_4 & \end{pmatrix} \text{ and } y = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix}.$$

This case can be reduced to Case 2.

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Case 4: There are $r_1, r_2, r_3, r_4 \in W_1$ and $m_1, m_2 \in W_2$ with

$$x = \begin{pmatrix} & r_1 & & \\ r_2 & & & \\ & & & r_3 \\ & & r_4 & \end{pmatrix} \textit{ and } y = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix}.$$

In view of Cases 1–3, we may assume that

$$y = \begin{pmatrix} & s_1 & & \\ s_2 & & & \\ & & & s_3 \\ & & s_4 & \end{pmatrix}$$

for some $s_1, s_2, s_3, s_4 \in W_1$. Since $\langle \overline{Z}, \bar{x}, \bar{y} \rangle = \langle \overline{Z}, \bar{x}, \bar{x}\bar{y} \rangle$, we can now reduce the given case to Case 1.

Case 5: There are $a_1, a_2, b_1, b_2 \in W_2$ with

$$\{x, y\} = \left\{ \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}, \begin{pmatrix} & b_1 \\ b_2 & \end{pmatrix} \right\}.$$

Without loss of generality, we assume that

$$x = \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \text{ and } y = \begin{pmatrix} & b_1 \\ b_2 & \end{pmatrix}.$$

We have $x^2 \in \langle -I_8 \rangle$ since $B = \langle \overline{Z}, \bar{x}, \bar{y} \rangle \cong E_8$, and hence, ${a_1}^2 \in \langle -I_4 \rangle$. So, by Lemma 3.31, there is an involution $\widetilde{a_1} \in W_2 \setminus \langle a_1, -I_4 \rangle$ which commutes with a_1. Set

$$y_1 := \begin{pmatrix} \widetilde{a_1} & \\ & \widetilde{a_1}^{b_1} \end{pmatrix}.$$

Clearly, $\overline{y_1}$ is an involution of $\bar{S}$. As $[x, y] \in \langle -I_8 \rangle$, we have ${a_1}^{b_1} \in \{a_2, -a_2\}$. Since a_1 and $\widetilde{a_1}$ commute, it follows that $\widetilde{a_1}^{b_1}$ and a_2 commute. So we have $[x, y_1] = 1$ and hence $[\bar{x}, \overline{y_1}] = 1$. Using that $y^2 \in \langle -I_8 \rangle$, one can easily verify that $[y, y_1] = 1$ and hence $[\bar{y}, \overline{y_1}] = 1$. As $\widetilde{a_1} \notin \langle a_1, -I_4 \rangle$, we have $\overline{y_1} \notin \langle \overline{Z}, \bar{x} \rangle$.

Now $\langle \overline{Z}, \bar{x}, \overline{y_1} \rangle$ is an E_8-subgroup of $\bar{S}$ which commutes with B and which is 3-connected to $\bar{A}$ by Cases 1-4. Thus, B is 3-connected to $\bar{A}$.

Case 6: There are $a_1, a_2, b_1, b_2 \in W_2$ with

$$x = \begin{pmatrix} & a_1 \\ a_2 & \end{pmatrix} \textit{ and } y = \begin{pmatrix} & b_1 \\ b_2 & \end{pmatrix}.$$

Noticing that $\langle \overline{Z}, \bar{x}, \bar{y} \rangle = \langle \overline{Z}, \bar{x}, \bar{x}\bar{y} \rangle$, we can reduce this case to Case 5. □

We summarize the above lemmas in the following corollary.

Corollary 3.34. *Let q be a nontrivial odd prime power and $n \geq 6$. Then the following hold:*

(i) *The Sylow 2-subgroups of $SL_n(q)$ and those of $PSL_n(q)$ are 2-connected.*
(ii) *The Sylow 2-subgroups of $SL_n(q)$ are 3-connected.*
(iii) *If $q \equiv 1 \bmod 4$ or $n \geq 7$, then the Sylow 2-subgroups of $PSL_n(q)$ are 3-connected.*

Unfortunately, the Sylow 2-subgroups of $PSL_6(q)$ are not 3-connected when $q \equiv 3 \bmod 4$ (this is not terribly difficult to observe).

Corollary 3.35. *Let q be a nontrivial odd prime power and $n \geq 6$. Let $G = SL_n(q)$, or $G = PSL_n(q)$ and $n \geq 7$ if $q \equiv 3 \bmod 4$. For any Sylow 2-subgroup S of G and any elementary abelian subgroup A of S with $m(A) \leq 3$, there is some elementary abelian subgroup B of S with $A < B$ and $m(B) = 4$.*

Proof. By Corollary 3.34, S is 2-connected and 3-connected. Applying [31, Lemma 8.7], the claim follows. □

3.6. Generation

Next, we discuss some generational properties of $(P)SL_n(q)$ and $(P)SU_n(q)$, where $n \geq 3$ and q is a nontrivial odd prime power. We need the following definition (see [31, Section 8]).

Definition 3.36. Let G be a finite group, let S be a Sylow 2-subgroup of G and let k be a positive integer. We say that G is *k-generated* if

$$G = \Gamma_{S,k}(G) := \langle N_G(T) \mid T \leq S, m(T) \geq k \rangle.$$

The following two lemmas will later prove to be useful.

Lemma 3.37 (see [4]). *Let q be a nontrivial odd prime power. Then the groups $SL_3(q)$, $PSL_3(q)$, $SU_3(q)$ and $PSU_3(q)$ are 2-generated.*

Lemma 3.38. *Let q be a nontrivial odd prime power, and let $n \geq 4$ be a natural number. Moreover, let $\varepsilon \in \{+,-\}$ and $Z \leq Z(SL_n^{\varepsilon}(q))$. Assume that one of the following holds:*

(i) *$n \geq 5$,*
(ii) *$q \equiv \varepsilon \bmod 8$,*
(iii) *$Z = 1$.*

Then $SL_n^{\varepsilon}(q)/Z$ is 3-generated.

We need the following lemma in order to prove Lemma 3.38.

Lemma 3.39 (see [45], [13]). *Let $q > 2$ be a prime power, and let $n \geq 3$ be a natural number. Let $\varepsilon \in \{+,-\}$. Define*

$$U_1 := \left\{ \begin{pmatrix} A & \\ & I_{n-2} \end{pmatrix} \ : \ A \in SL_2^{\varepsilon}(q) \right\}$$

and

$$U_{n-1} := \left\{ \begin{pmatrix} I_{n-2} & \\ & A \end{pmatrix} \ : \ A \in SL_2^{\varepsilon}(q) \right\}.$$

Moreover, for each $2 \leq i \leq n-2$, let

$$U_i := \left\{ \begin{pmatrix} I_{i-1} & & \\ & A & \\ & & I_{n-i-1} \end{pmatrix} \ : \ A \in SL_2^{\varepsilon}(q) \right\}.$$

Then the following hold:

(i) *We have $SL_n^{\varepsilon}(q) = \langle U_i \ : \ 1 \leq i \leq n-1 \rangle$.*
(ii) *For each $1 \leq i \leq n-2$, there is a monomial matrix m_i in $SL_n^{\varepsilon}(q)$ with $U_i^{m_i} = U_{i+1}$.*

Proof of Lemma 3.38. Let q be a nontrivial odd prime power, $n \geq 4$ be a natural number, $\varepsilon \in \{+,-\}$, $L := SL_n^{\varepsilon}(q)$ and $Z \leq Z(L)$. Suppose that one of the conditions $n \geq 5$, $q \equiv \varepsilon \bmod 8$ or $Z = 1$ is satisfied. We have to show that L/Z is 3-generated.

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Let $U_1, \ldots, U_{n-1}$ denote the $SL_2^\varepsilon(q)$-subgroups of L corresponding to the 2×2 blocks along the main diagonal (as in Lemma 3.39). Let E be the subgroup of L consisting of the diagonal matrices in L with diagonal entries in $\{-1, 1\}$.

Assume that $n \geq 5$. We claim that there is an E_8-subgroup E_i of E with $E_i \cap Z(L) = 1$ and $[E_i, U_i] = 1$ for each $i \in \{1, \ldots, n-1\}$. Let V be the module defining L. Let $X = \{x_1, \ldots, x_n\}$ be a basis for V, with X orthonormal if V is unitary, and let $V_i := \langle x_i, x_{i+1}\rangle$ for $1 \leq i < n$. Then $U_i = SL^\varepsilon(V_i)$. We have $m(E) = n - 1 \geq 4$. For $1 \leq i < n$ let $e_i \in E$ invert V_i and centralize x_j for $x_j \notin V_i$, and set $F_i := C_E(V_i)\langle e_i\rangle$. Then F_i is a hyperplane of E centralizing U_i. If n is odd, then $|Z(L)|$ is odd, so we may choose $E_i \leq F_i$. Thus, we may take n even, so $n \geq 6$. Choose a hyperplane D_i of F_i with $D_i \cap Z(L) = 1$, and take $E_i \leq D_i$.

For $1 \leq i < n$, U_iZ/Z centralizes $E_iZ/Z \cong E_8$ since $[E_i, U_i] = 1$. Now, if S is a Sylow 2-subgroup of L/Z containing EZ/Z, we have $U_iZ/Z \leq \Gamma_{S,3}(L/Z)$ for each $i \in \{1, \ldots, n-1\}$, and Lemma 3.39 (i) implies that L/Z is 3-generated.

We now consider the case $n = 4$. By hypothesis, $Z = 1$ or $q \equiv \varepsilon \bmod 8$. Let

$$U := \left\{ \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{A}} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array} \right) \; : \; A \in SL_3^\varepsilon(q) \right\}.$$

If $Z = 1$, set $y := -I_4$. If $q \equiv \varepsilon \bmod 8$, let λ be an element of $\mathbb{F}_{q^2}^*$ of order 8 such that $\lambda^{q-\varepsilon} = 1$. Note that $\lambda \in \mathbb{F}_q^*$ if $\varepsilon = +$. Also, if $q \equiv \varepsilon \bmod 8$ and $|Z| = 2$, let $y := \lambda^2 I_4 \in L$, and if $q \equiv \varepsilon \bmod 8$ and $|Z| = 4$, let $y := \mathrm{diag}(\lambda, \lambda, \lambda, -\lambda) \in L$.

Let S_0 be a Sylow 2-subgroup of U containing $E \cap U$. Let $\widetilde{S}$ be a Sylow 2-subgroup of L containing S_0 and y. Denote the image of $\widetilde{S}$ in L/Z by S. We have $S \cap UZ/Z = S_0Z/Z \in \mathrm{Syl}_2(UZ/Z)$. By Lemma 3.37, $UZ/Z \cong U \cong SL_3^\varepsilon(q)$ is 2-generated. So we have

$$UZ/Z = \Gamma_{S_0Z/Z,2}(UZ/Z) = \langle N_{UZ/Z}(T) \mid T \leq S_0Z/Z, m(T) \geq 2\rangle.$$

Let $T \leq S_0Z/Z$ with $m(T) \geq 2$ and $\widehat{T} := \langle T, yZ\rangle$. Clearly, yZ is an involution of S not contained in UZ/Z and centralizing UZ/Z. Therefore, we have that $m(\widehat{T}) \geq 3$ and $N_{UZ/Z}(T) \leq N_{L/Z}(\widehat{T})$. It follows that $UZ/Z \leq \Gamma_{S,3}(L/Z)$. In particular, $U_iZ/Z \leq \Gamma_{S,3}(L/Z)$ for $i \in \{1, 2\}$.

From Lemma 3.39 (ii), we see that there is some $m \in L$ such that $U_2{}^m = U_3$ and such that m normalizes $\langle E, y\rangle$. So mZ normalizes $\langle EZ/Z, yZ\rangle$. It is easy to note that $\langle EZ/Z, yZ\rangle \cong E_8$, and so we have $mZ \in \Gamma_{S,3}(L/Z)$. It follows that $U_3Z/Z = (U_2Z/Z)^{mZ} \leq \Gamma_{S,3}(L/Z)$.

So we have $U_iZ/Z \leq \Gamma_{S,3}(L/Z)$ for $i \in \{1, 2, 3\}$, and Lemma 3.39 (i) implies that L/Z is 3-generated. □

3.7. *Automorphisms of* $(P)SL_n(q)$

Fix a prime number p, a positive integer f and a natural number $n \geq 2$. Set $q := p^f$ and $T := SL_n(q)$. We now briefly describe the structure of $\mathrm{Aut}(T/Z)$, where $Z \leq Z(T)$, referring to [19] and [16, Section 2.1] for further details.

Let $\mathrm{Inndiag}(T) := \mathrm{Aut}_{GL_n(q)}(T)$. Note that

$$\mathrm{Inndiag}(T)/\mathrm{Inn}(T) \cong C_{(n,q-1)}.$$

The map

$$\phi : T \to T, (a_{ij}) \mapsto (a_{ij}{}^p)$$

is an automorphism of T with order f. One can check that ϕ normalizes $\mathrm{Inndiag}(T)$. Set

$$P\Gamma L_n(q) := \mathrm{Inndiag}(T)\langle\phi\rangle.$$

It is easy to note that $\langle\phi\rangle \cap \mathrm{Inndiag}(T) = 1$ so that $P\Gamma L_n(q)$ is the inner semidirect product of $\mathrm{Inndiag}(T)$ and $\langle\phi\rangle$.

The map

$$\iota : T \to T, a \mapsto (a^t)^{-1}$$

is an automorphism of T with order 2. If $n = 2$, then ι turns out to be an inner automorphism of T, while we have $\iota \notin P\Gamma L_n(q)$ when $n \geq 3$. By a direct calculation,ι normalizes $\mathrm{Inndiag}(T)$ and commutes with ϕ. In particular, $A := P\Gamma L_n(q)\langle\iota\rangle$ is a subgroup of $\mathrm{Aut}(T)$, and we have

$$A/\mathrm{Inndiag}(T) \cong C_f \times C_a,$$

where $a = 1$ if $n = 2$ and $a = 2$ if $n \geq 3$.

Now let Z be a central subgroup of T. As $Z(T)$ is cyclic, Z is characteristic in T. Then as T is perfect, $SL_2(2)$ or $SL_2(3)$, the natural homomorphism $\mathrm{Aut}(T) \to \mathrm{Aut}(T/Z)$ is injective. The image of $\mathrm{Inndiag}(T)$ under this homomorphism will be denoted by $\mathrm{Inndiag}(T/Z)$. By abuse of notation, we denote the image of $P\Gamma L_n(q)$ in $\mathrm{Aut}(T/Z)$ again by $P\Gamma L_n(q)$ and the images of ι and ϕ again by ι and ϕ, respectively.

With this notation, we have

$$\mathrm{Aut}(T/Z) = P\Gamma L_n(q)\langle\iota\rangle.$$

Note that the natural homomorphism $\mathrm{Aut}(T) \to \mathrm{Aut}(T/Z)$ is an isomorphism and that it induces an isomorphism $\mathrm{Out}(T) \to \mathrm{Out}(T/Z)$.

The elements of $\mathrm{Inndiag}(T/Z) \setminus \mathrm{Inn}(T/Z)$ are said to be the (nontrivial) *diagonal automorphisms* of T/Z. An automorphism of T/Z is called a *field automorphism* if it is conjugate to ϕ^i for some $1 \leq i < f$. The automorphisms of the form $\alpha\iota$, where $\alpha \in \mathrm{Inndiag}(T/Z)$, are said to be the *graph automorphisms* of T/Z. An automorphism of T/Z is said to be a *graph-field automorphism* if it is conjugate to an automorphism of the form $\phi^i\iota$ for some $1 \leq i < f$. We remark that these definitions are particular cases of more general definitions; see [49, Chapter 10].

Proposition 3.40. *Let q be a nontrivial prime power, and let $n \geq 2$. Then* $\mathrm{Out}(PSL_n(q))$ *is 2-nilpotent.*

Proof. By the above remarks, $\mathrm{Out}(PSL_n(q))$ has a chief series with cyclic factors. Consequently, $\mathrm{Out}(PSL_n(q))$ is supersolvable. By [38, Lemma 2.4 (4)], any supersolvable finite group is 2-nilpotent, and so the proposition follows. $\square$

The following proposition also follows from the above remarks.

Proposition 3.41. *Let $n \geq 2$ be a natural number. Then* $\mathrm{Out}(SL_n(3))$ *is a 2-group.*

3.8. Automorphisms of $(P)SU_n(q)$

Let p be a prime number, f be a positive integer and $n \geq 3$ be a natural number. Set $q := p^f$ and $T := SU_n(q)$. We now briefly describe the structure of $\mathrm{Aut}(T/Z)$, where $Z \leq Z(T)$, referring to [19] and [16, Section 2.3] for further details.

Let $\mathrm{Inndiag}(T) := \mathrm{Aut}_{GU_n(q)}(SU_n(q))$. It is rather easy to note that

$$\mathrm{Inndiag}(T)/\mathrm{Inn}(T) \cong C_{(n,q+1)}.$$

The map

$$\phi : T \to T, (a_{ij}) \mapsto ({a_{ij}}^p)$$

is an automorphism of T with order $2f$. One can check that ϕ normalizes $\mathrm{Inndiag}(T)$. Set

$$P\Gamma U_n(q) := \mathrm{Inndiag}(T)\langle\phi\rangle.$$

It is rather easy to note that $\langle\phi\rangle \cap \mathrm{Inndiag}(T) = 1$ so that $P\Gamma U_n(q)$ is the inner semidirect product of $\mathrm{Inndiag}(T)$ and $\langle\phi\rangle$. Note that

$$P\Gamma U_n(q)/\mathrm{Inndiag}(T) \cong C_{2f}.$$

Now let Z be a central subgroup of T. As in the case $T = SL_n(q)$, the natural homomorphism $\mathrm{Aut}(T) \to \mathrm{Aut}(T/Z)$ is injective. The image of $\mathrm{Inndiag}(T)$ under this homomorphism will be denoted by $\mathrm{Inndiag}(T/Z)$. By abuse of notation, we denote the image of $P\Gamma U_n(q)$ in $\mathrm{Aut}(T/Z)$ again by $P\Gamma U_n(q)$ and the image of ϕ again by ϕ.

With this notation, we have

$$\mathrm{Aut}(T/Z) = P\Gamma U_n(q).$$

Note that the natural homomorphism $\mathrm{Aut}(T) \to \mathrm{Aut}(T/Z)$ is an isomorphism and that it induces an isomorphism $\mathrm{Out}(T) \to \mathrm{Out}(T/Z)$.

The elements of $\mathrm{Inndiag}(T/Z) \setminus \mathrm{Inn}(T/Z)$ are said to be the (nontrivial) *diagonal automorphisms* of T/Z. An automorphism of T/Z is called a *field automorphism* if it is conjugate to ϕ^i for some $1 \le i < 2f$ such that ϕ^i has odd order. The automorphisms of the form $\alpha\phi^i$, where ϕ^i has even order and $\alpha \in \mathrm{Inndiag}(T/Z)$, are said to be the *graph automorphisms* of T/Z. There are no graph-field automorphisms of T/Z.

Proposition 3.42. *Let q be a nontrivial prime power, and let $n \ge 3$. Then $\mathrm{Out}(PSU_n(q))$ is 2-nilpotent.*

Proof. We see from the above remarks that $\mathrm{Out}(PSU_n(q))$ is supersolvable. So $\mathrm{Out}(PSU_n(q))$ is 2-nilpotent by [38, Lemma 2.4 (4)]. □

The following proposition also follows from the above remarks.

Proposition 3.43. *Let $n \ge 3$ be a natural number. Then $\mathrm{Out}(SU_n(3))$ is a 2-group.*

3.9. Some lemmas

We now prove several results on the automorphism groups of $(P)SL_n(q)$ and $(P)SU_n(q)$, where $n \ge 2$ and q is a nontrivial odd prime power.

Lemma 3.44. *Let q be a nontrivial odd prime power. Also, let $T := SL_2(q)$ and $S \in \mathrm{Syl}_2(T)$. Suppose that α and β are 2-elements of $\mathrm{Aut}(T)$ such that $S^\alpha = S = S^\beta$ and $\alpha|_{S,S} = \beta|_{S,S}$. Then $\alpha = \beta$.*

Proof. Let $\gamma := \alpha\beta^{-1} \in C_{\mathrm{Aut}(T)}(S)$. We have $C_{\mathrm{Inndiag}(T)}(S) = 1$ by [28, Lemma 4.10.10]. Therefore, it suffices to show that $\gamma \in \mathrm{Inndiag}(T)$. Clearly, the images of α and β^{-1} in $\mathrm{Aut}(T)/\mathrm{Inndiag}(T)$ are 2-elements of $\mathrm{Aut}(T)/\mathrm{Inndiag}(T)$. Since $\mathrm{Aut}(T)/\mathrm{Inndiag}(T)$ is abelian,

$$\gamma \cdot \mathrm{Inndiag}(T) = (\alpha \cdot \mathrm{Inndiag}(T)) \cdot (\beta^{-1} \cdot \mathrm{Inndiag}(T))$$

is still a 2-element of $\mathrm{Aut}(T)/\mathrm{Inndiag}(T)$. By [28, Lemma 4.10.10], $C_{\mathrm{Aut}(T)}(S)$ is a $2'$-group, and so γ has odd order. Therefore, $\gamma \cdot \mathrm{Inndiag}(T)$ has odd order. It follows that $\gamma \in \mathrm{Inndiag}(T)$, as required. □

Lemma 3.45. *Let $q = p^f$, where p is an odd prime and f is a positive integer. Let $T := PSL_2(q)$, and let α be an involution of $\mathrm{Aut}(T)$. Suppose that $C_T(\alpha)$ has a 2-component K. Then we have $2 \mid f$, $(f, p) \neq (2, 3)$ and $K \cong PSL_2(p^{\frac{f}{2}})$. In particular, K is a component of $C_T(\alpha)$.*

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Proof. Note that $C_T(\alpha) \cong C_{\mathrm{Inn}(T)}(\alpha)$.

Assume that $\alpha \in \mathrm{Inndiag}(T)$. Noticing that $\mathrm{Inndiag}(T) \cong PGL_2(q)$, we see from Lemma 3.3 that $C_{\mathrm{Inndiag}(T)}(\alpha)$ is solvable. Thus, $C_T(\alpha) \cong C_{\mathrm{Inn}(T)}(\alpha)$ is solvable, and $C_T(\alpha)$ has no 2-components, a contradiction to the choice of α.

So we have $\alpha \notin \mathrm{Inndiag}(T)$. By the structure of $\mathrm{Aut}(PSL_2(q))$ and since α has order 2, we can write α as a product of an inner-diagonal automorphism and a field automorphism of order 2. In particular, f must be even. Consulting [28, Proposition 4.9.1 (d)], we see that α itself is a field automorphism. So we can apply [28, Proposition 4.9.1 (b)] to conclude that $C_{\mathrm{Inndiag}(T)}(\alpha) \cong \mathrm{Inndiag}(PSL_2(p^{\frac{f}{2}})) \cong PGL_2(p^{\frac{f}{2}})$. Consequently, K is isomorphic to a 2-component of $PGL_2(p^{\frac{f}{2}})$. It follows that $(f, p) \neq (2, 3)$ and $K \cong PSL_2(p^{\frac{f}{2}})$. □

Before we state the next lemma, we introduce some notational conventions for adjoint Chevalley groups. Given a nontrivial prime power q, we denote $A_1(q)$ also by $B_1(q)$ and by $C_1(q)$. Moreover, $B_2(q)$ will be also denoted by $C_2(q)$, and $A_3(q)$ will be also denoted by $D_3(q)$. We also set $D_2(q) := A_1(q) \times A_1(q)$ and ${}^2D_2(q) := A_1(q^2)$.

Lemma 3.46. *Let $q = p^f$, where p is an odd prime and f is a positive integer. Let $n \geq 3$ be a natural number and $\varepsilon \in \{+, -\}$. Let $T := PSL_n^{\varepsilon}(q)$, and let α be an involution of* $\mathrm{Aut}(T)$. *Suppose that $C_T(\alpha)$ has a 2-component K. Then K is in fact a component, and one of the following holds:*

(i) *$K \cong SL_i^{\varepsilon}(q)$ for some $2 \leq i < n$, where $i > 2$ if $q = 3$;*
(ii) *n is even, and K is isomorphic to a nontrivial quotient of $SL_{\frac{n}{2}}(q^2)$;*
(iii) *$\varepsilon = +$, f is even, $K \cong PSL_n(p^{\frac{f}{2}})$ or $K \cong PSU_n(p^{\frac{f}{2}})$;*
(iv) *$q \neq 3$, $n = 3$ or 4, and $K \cong PSL_2(q)$;*
(v) *n is odd, $n \geq 5$ and $K \cong B_{\frac{n-1}{2}}(q)$;*
(vi) *n is even and $K \cong C_{\frac{n}{2}}(q)$;*
(vii) *n is even, $n \geq 6$ and $K \cong D_{\frac{n}{2}}(q)$;*
(viii) *n is even, $n \geq 6$ and $K \cong {}^2D_{\frac{n}{2}}(q)$.*

Here, the (twisted) Chevalley groups appearing in (v)–(viii) are adjoint.

Proof. It can be shown that any involution of $\mathrm{Aut}(T)$ is an inner-diagonal automorphism, a field automorphism, a graph automorphism or a graph-field automorphism (see [16, Section 3.1.3] or [28, Section 4.9]).

Case 1: $\alpha \in$ $\mathrm{Inndiag}(T)$, *or α is a graph automorphism.*

Set $C^* := C_{\mathrm{Inndiag}(T)}(\alpha)$ and $L^* := O^{p'}(C^*)$. One can see from [28, Theorem 4.2.2 and Table 4.5.1] that C^*/L^* is solvable and that one of the following holds:

(1) L^* is the central product of two subgroups isomorphic to $SL_i^{\varepsilon}(q)$ and $SL_{n-i}^{\varepsilon}(q)$ for some natural number i with $1 \leq i \leq \frac{n}{2}$,
(2) n is even and L^* is isomorphic to a nontrivial quotient of $SL_{\frac{n}{2}}(q^2)$,
(3) n is odd and $L^* \cong B_{\frac{n-1}{2}}(q)$,
(4) n is even and $L^* \cong C_{\frac{n}{2}}(q)$,
(5) n is even and $L^* \cong D_{\frac{n}{2}}(q)$,
(6) n is even and $L^* \cong {}^2D_{\frac{n}{2}}(q)$,

where the (twisted) Chevalley groups appearing in the last four cases are adjoint. Since $C_T(\alpha)$ is isomorphic to $C_{\mathrm{Inn}(T)}(\alpha) \trianglelefteq C^*$, we have that K is isomorphic to a 2-component of C^* and thus isomorphic to a 2-component of L^*. Therefore, one of the conditions (i)-(viii) is satisfied.

Case 2: α is a field automorphism or a graph-field automorphism.

Again, let $C^* := C_{\mathrm{Inndiag}(T)}(\alpha)$. Since the field automorphisms of $PSU_n(q)$ have odd order and $PSU_n(q)$ has no graph-field automorphisms, we have $\varepsilon = +$. Also, f is even since α is a field automorphism or a graph-field automorphism of order 2. From [28, Proposition 4.9.1 (a), (b)], we see

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that $C^* \cong PGL_n(p^{\frac{f}{2}})$ if α is a field automorphism and that $C^* \cong PGU_n(p^{\frac{f}{2}})$ if α is a graph-field automorphism. Since K is isomorphic to a 2-component of C^*, it follows that (iii) is satisfied. □

Corollary 3.47. *Let $q = p^f$, where p is an odd prime and f is a positive integer. Let $n \geq 2$ be a natural number and $\varepsilon \in \{+, -\}$. Let Z be a central subgroup of $SL_n^\varepsilon(q)$, and let $T := SL_n^\varepsilon(q)/Z$. Let α be an involution of $\mathrm{Aut}(T)$, and let K be a 2-component of $C_T(\alpha)$. Then the following hold:*

(i) *K is a component of $C_T(\alpha)$, and $K/Z(K)$ is a known finite simple group.*
(ii) *$K/Z(K) \not\cong M_{11}$.*
(iii) *Assume that $K/Z(K) \cong PSL_k^{\varepsilon^*}(q^*)$ for some positive integer $2 \leq k \leq n$, some nontrivial odd prime power q^* and some $\varepsilon^* \in \{+, -\}$. Then one of the following holds:*
  (a) *$q^* = q$;*
  (b) *$q^* = q^2$, $n \geq 4$ is even, $k = \frac{n}{2}$ and $\varepsilon^* = +$ if $n \geq 6$;*
  (c) *f is even, $k = n$, $q^* = p^{\frac{f}{2}}$.*

Proof. Set $\overline{T} := T/Z(T) \cong PSL_n^\varepsilon(q)$. Let $\overline{\alpha}$ be the automorphism of $\overline{T}$ induced by α.

As $\overline{K}$ is a 2-component of $\overline{C_T(\alpha)}$ and $\overline{C_T(\alpha)} \trianglelefteq C_{\overline{T}}(\overline{\alpha})$, it follows that $\overline{K}$ is a 2-component of $C_{\overline{T}}(\overline{\alpha})$. Lemmas 3.45 and 3.46 imply that $\overline{K}$ is a component of $C_{\overline{T}}(\overline{\alpha})$ and that $\overline{K}/Z(\overline{K})$ is a known finite simple group. Applying [37, 6.5.1], we conclude that K' is a component of $C_T(\alpha)$. We have $K = K'$ since K is a 2-component of $C_T(\alpha)$, and so it follows that K is a component of $C_T(\alpha)$. Also, $K/Z(K) \cong \overline{K}/Z(\overline{K})$ so that $K/Z(K)$ is a known finite simple group. Hence, (i) holds.

If $K/Z(K) \cong M_{11}$, then $\overline{K}/Z(\overline{K}) \cong M_{11}$, which is not possible by Lemmas 3.45 and 3.46. So (ii) holds.

Suppose that $K/Z(K) \cong PSL_k^{\varepsilon^*}(q^*)$ for some positive integer $2 \leq k \leq n$, some nontrivial odd prime power q^* and some $\varepsilon^* \in \{+, -\}$. By Lemmas 3.45 and 3.46, one of the following holds:

(1) $\overline{K}/Z(\overline{K}) \cong PSL_i^\varepsilon(q)$ for some $2 \leq i < n$;
(2) n is even, and $\overline{K}/Z(\overline{K})$ is isomorphic to $PSL_{\frac{n}{2}}(q^2)$;
(3) f is even, $\overline{K} \cong PSL_n(p^{\frac{f}{2}})$ or $PSU_n(p^{\frac{f}{2}})$;
(4) $q \neq 3$, $n = 3$ or 4, $\overline{K} \cong PSL_2(q)$;
(5) n is odd, $n \geq 5$, $\overline{K} \cong B_{\frac{n-1}{2}}(q)$;
(6) n is even, $n \geq 4$, $\overline{K} \cong C_{\frac{n}{2}}(q)$;
(7) n is even, $n \geq 6$, $\overline{K} \cong D_{\frac{n}{2}}(q)$;
(8) n is even, $n \geq 6$, $\overline{K} \cong {}^2D_{\frac{n}{2}}(q)$.

Here, the (twisted) Chevalley groups appearing in (5)–(8) are adjoint. On the other hand, we have $\overline{K}/Z(\overline{K}) \cong PSL_k^{\varepsilon^*}(q^*)$. Now, if (1) holds, then $PSL_k^{\varepsilon^*}(q^*) \cong PSL_i^\varepsilon(q)$ for some $2 \leq i < n$, and [49, Theorem 37] shows that this is only possible when $q^* = q$ so that (a) holds. Similarly, if (2) holds, then we have (b). Moreover, (3) implies (c) and (4) implies (a). As Theorem [49, Theorem 37] shows, the cases (5) and (6) cannot occur, while (7) and (8) can only occur when $n = 6$. As above, one can see that if $n = 6$ and (7) or (8) holds, then we have (a). □

Lemma 3.48. *Let $n \geq 3$ and $\varepsilon \in \{+, -\}$. Then $SL_n^\varepsilon(3)$ is locally balanced (in the sense of Definition 2.7).*

Proof. Set $T := SL_n^\varepsilon(3)$. Let H be a subgroup of $\mathrm{Aut}(T)$ containing $\mathrm{Inn}(T)$, and let x be an involution of H. It is enough to show that $O(C_H(x)) = 1$.

Assume that $O(C_H(x)) \neq 1$. Then $x \in \mathrm{Inndiag}(T)$ by [28, Theorem 7.7.1]. By Propositions 3.41 and 3.43, $\mathrm{Out}(T)$ is a 2-group. This implies $O(C_H(x)) = O(C_{\mathrm{Inn}(T)}(x)) = O(C_{\mathrm{Inndiag}(T)}(x))$. Since x is an involution of $\mathrm{Inndiag}(T) \cong PGL_n^\varepsilon(3)$, we have $O(C_{\mathrm{Inndiag}(T)}(x)) = 1$ by Corollary 3.9. Thus, $O(C_H(x)) = 1$. This contradiction completes the proof. □

Lemma 3.49. *Let $n \geq 3$ be a natural number, let q be a nontrivial odd power, and let $\varepsilon \in \{+, -\}$. Then any nontrivial quotient of $SL_n^\varepsilon(q)$ is locally 2-balanced (in the sense of Definition 2.7).*

Proof. By [24, Theorem 4.61] or [28, Theorem 7.7.4], $PSL_n^\varepsilon(q)$ is locally 2-balanced. Let K be a nontrivial quotient of $SL_n^\varepsilon(q)$. As we have seen, there is an isomorphism $\mathrm{Aut}(K) \to \mathrm{Aut}(PSL_n^\varepsilon(q))$ mapping $\mathrm{Inn}(K)$ to $\mathrm{Inn}(PSL_n^\varepsilon(q))$. So the local 2-balance of K follows from the local 2-balance of $PSL_n^\varepsilon(q)$. □

Lemma 3.50. *Let q be a nontrivial odd prime power and $n \geq 4$ be a natural number. Let $Z \leq Z(SL_n(q))$ and $T := SL_n(q)/Z$. Let K_1 be the image of*

$$\left\{\begin{pmatrix} A & \\ & I_{n-2}\end{pmatrix} : A \in SL_2(q)\right\}$$

in T, and let K_2 be the image of

$$\left\{\begin{pmatrix} I_2 & \\ & B\end{pmatrix} : B \in SL_{n-2}(q)\right\}$$

in T. Let α be an automorphism of T with odd order such that α normalizes K_1 and centralizes K_2. Then $\alpha|_{K_1,K_1}$ is an inner automorphism.

Proof. By hypothesis, $q = p^f$ for some odd prime number p and some positive integer f. We have $\alpha \in P\Gamma L_n(q)$ since α has odd order and $|\mathrm{Aut}(T)/P\Gamma L_n(q)| = 2$. So there are some $m \in GL_n(q)$ and some $1 \leq r \leq f$ such that, for each element (a_{ij}) of $SL_n(q)$, α maps $(a_{ij})Z$ to $((a_{ij})^{p^r})^m Z$.

Let x be the image of $\mathrm{diag}(-1,-1,1,\ldots,1) \in SL_n(q)$ in T. Then x is the unique involution of K_1, and so we have $x^\alpha = x$. This easily implies that

$$m = \begin{pmatrix} m_1 & \\ & m_2\end{pmatrix}$$

for some $m_1 \in GL_2(q)$ and some $m_2 \in GL_{n-2}(q)$.

Since α centralizes K_2, we have $((a_{ij})^{p^r})^{m_2} = (a_{ij})$ for all $(a_{ij}) \in SL_{n-2}(q)$. Therefore, the automorphism $SL_{n-2}(q) \to SL_{n-2}(q), (a_{ij}) \mapsto (a_{ij})^{p^r}$ is an element of $\mathrm{Inndiag}(SL_{n-2}(q))$. This implies $r = f$.

Thus, under the isomorphism $\mathrm{Aut}(SL_2(q)) \to \mathrm{Aut}(K_1)$ induced by the canonical isomorphism $SL_2(q) \to K_1$, the automorphism $\alpha|_{K_1,K_1}$ of K_1 corresponds to the inner-diagonal automorphism $\widetilde{\alpha} : SL_2(q) \to SL_2(q), a \mapsto a^{m_1}$, and this automorphism has odd order since α has odd order. The index of $\mathrm{Inn}(SL_2(q))$ in $\mathrm{Inndiag}(SL_2(q))$ is 2, and so it follows that $\widetilde{\alpha} \in \mathrm{Inn}(SL_2(q))$. Consequently, $\alpha|_{K_1,K_1} \in \mathrm{Inn}(K_1)$. □

By using similar arguments as in the proof of Lemma 3.50, one can prove the following lemma.

Lemma 3.51. *Let q be a nontrivial odd prime power and $n \geq 4$ be a natural number. Let $Z \leq Z(SU_n(q))$ and $T := SU_n(q)/Z$. Let K_1 be the image of*

$$\left\{\begin{pmatrix} A & \\ & I_{n-2}\end{pmatrix} : A \in SU_2(q)\right\}$$

in T, and let K_2 be the image of

$$\left\{\begin{pmatrix} I_2 & \\ & B\end{pmatrix} : B \in SU_{n-2}(q)\right\}$$

in T. Let α be an automorphism of T with odd order such that α normalizes K_1 and centralizes K_2. Then $\alpha|_{K_1,K_1}$ is an inner automorphism.

Our next goal is to prove the following lemma.

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Lemma 3.52. *Let q and q^* be nontrivial odd prime powers. Let L be a group isomorphic to $SL_2(q^*)$. Let Q be a Sylow 2-subgroup of L. Moreover, let V be a Sylow 2-subgroup of $GL_2(q)$ and $V_0 := V \cap SL_2(q)$. Suppose that there is a group isomorphism $\psi : V_0 \to Q$. Let v_1, v_2, v_3 be elements of V such that $v_3 = v_1 v_2$ and such that the square of any element of $\{v_1, v_2, v_3\}$ lies in $Z(GL_2(q))$. For each $i \in \{1,2,3\}$, let α_i be a 2-element of $\mathrm{Aut}(L)$ normalizing Q such that*

$$\alpha_i|_{Q,Q} = \psi^{-1}(c_{v_i}|_{V_0,V_0})\psi.$$

Then we have

$$\bigcap_{i=1}^{3} O(C_L(\alpha_i)) = 1.$$

To prove Lemma 3.52, we need to prove some other lemmas.

Lemma 3.53. *Let q be a nontrivial odd prime power with $q \equiv 1 \bmod 4$, and let $k \in \mathbb{N}$ with $(q-1)_2 = 2^k$. Let G be a group isomorphic to $SL_2(q)$ and $Q \in \mathrm{Syl}_2(G)$. Then the following hold:*

(i) *There are elements a, b generating Q such that* $\mathrm{ord}(a) = 2^k$, $\mathrm{ord}(b) = 4$, $a^b = a^{-1}$ *and* $b^2 = a^{2^{k-1}}$.
(ii) *Let a and b be as in (i). Then there is a group isomorphism $\varphi : G \to SL_2(q)$ such that*

$$a^\varphi = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

for some $\lambda \in \mathbb{F}_q^*$ with order 2^k and

$$b^\varphi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proof. (i) follows from Lemma 3.12.

We now prove (ii). Assume that $k \geq 3$. By Lemma 3.10 (i),

$$R = \left\{ \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \ : \ \mu \text{ is a 2-element of } \mathbb{F}_q^* \right\} \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

is a Sylow 2-subgroup of $SL_2(q)$. Choose a group isomorphism $\psi : G \to SL_2(q)$ such that $Q^\psi = R$. Since $k \geq 3$, Q has only one cyclic subgroup of order 2^k. This implies that

$$a^\psi = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

for some $\lambda \in \mathbb{F}_q^*$ with order 2^k. Since $b \notin \langle a \rangle$, we have

$$b^\psi = \begin{pmatrix} 0 & \mu \\ -\mu^{-1} & 0 \end{pmatrix}$$

for some 2-element μ of $\mathbb{F}_q^*$. Composing ψ with the automorphism

$$SL_2(q) \to SL_2(q), \ A \mapsto \begin{pmatrix} \mu^{-1} & 0 \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix},$$

we get a group isomorphism $\varphi : G \to SL_2(q)$ with the desired properties. This completes the proof of (ii) for the case $k \geq 3$.

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Assume now that $k = 2$. Let $\psi : G \longrightarrow SL_2(q)$ be a group isomorphism. We have $(a^\psi)^2 = -I_2$ since $-I_2$ is the only involution of $SL_2(q)$ and $\mathrm{ord}(a^2) = 2$. So, by Lemma 3.3, we may assume that

$$a^\psi = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

for some $\lambda \in \mathbb{F}_q^*$ with order 4. Since $a^b = a^{-1}$, we have

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}^{b^\psi} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}.$$

This implies that

$$b^\psi = \begin{pmatrix} 0 & \mu \\ -\mu^{-1} & 0 \end{pmatrix}$$

for some $\mu \in \mathbb{F}_q^*$. Again, we may compose ψ with a suitable diagonal automorphism of $SL_2(q)$ to obtain a group isomorphism $\varphi : G \longrightarrow SL_2(q)$ with the desired properties. □

By using similar arguments as in the proof of Lemma 3.53, one can prove the following lemma.

Lemma 3.54. *Let q be a nontrivial odd prime power with $q \equiv 3 \bmod 4$, and let $s \in \mathbb{N}$ with $(q+1)_2 = 2^s$. Let G be a group isomorphic to $SU_2(q)$ and $Q \in \mathrm{Syl}_2(G)$. Then the following hold:*

(i) *There are elements $a, b \in Q$ such that $\mathrm{ord}(a) = 2^s$, $\mathrm{ord}(b) = 4$, $a^b = a^{-1}$ and $b^2 = a^{2^{s-1}}$.*
(ii) *Let a and b be as in (i). Then there is a group isomorphism $\varphi : G \longrightarrow SU_2(q)$ such that*

$$a^\varphi = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

for some $\lambda \in \mathbb{F}_{q^2}^*$ with order 2^s and

$$b^\varphi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Lemma 3.55. *Let q be a nontrivial odd prime power with $q \equiv 1 \bmod 4$. Let ρ be a generating element of the Sylow 2-subgroup of $\mathbb{F}_q^*$, and let*

$$a := \begin{pmatrix} \rho & \\ & \rho^{-1} \end{pmatrix}, \quad b := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let V be the Sylow 2-subgroup of $GL_2(q)$ given by Lemma 3.10 (i), and let $v, w \in V$ such that $v^2, w^2, (vw)^2 \in Z(GL_2(q))$. Then one of the following holds:

(i) *$\{v, w, vw\} \cap Z(GL_2(q)) \neq \emptyset$.*
(ii) *There exist $r, s \in \{v, w, vw\}$ with $a^r = a$, $b^r = b^3$ and $a^s = a^{-1}$.*

Proof. It is easy to note that (i) holds if v and w are diagonal matrices.

Suppose now that v or w is not a diagonal matrix. If neither v nor w is a diagonal matrix, then vw is a diagonal matrix. So there exist $r, s \in \{v, w, vw\}$ such that

$$r = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}, \quad s = \begin{pmatrix} & \mu_1 \\ \mu_2 & \end{pmatrix},$$

where $\lambda_1, \lambda_2, \mu_1$ and μ_2 are 2-elements of $\mathbb{F}_q^*$.

If $\lambda_1 = \lambda_2$, then (i) holds. Assume now that $\lambda_1 \neq \lambda_2$. Then $\lambda_2 = -\lambda_1$ since $r^2 \in Z(GL_2(q))$, and a direct calculation shows that $a^r = a$, $b^r = b^3$ and $a^s = a^{-1}$. □

Lemma 3.56. *Let q be a nontrivial odd prime power with $q \equiv 3 \bmod 4$, and let $k \in \mathbb{N}$ with $(q+1)_2 = 2^k$. Let V be a Sylow 2-subgroup of $GL_2(q)$.*

(i) *There exist $x, y \in V$ with $\operatorname{ord}(x) = 2^{k+1}$, $\operatorname{ord}(y) = 2$ and $x^y = x^{-1+2^k}$. We have $V \cap SL_2(q) = \langle x^2\rangle\langle xy\rangle$.*
(ii) *Let x and y be as above, and let $a := x^2$ and $b := xy$. Let $v, w \in V$ with $v^2, w^2, (vw)^2 \in Z(GL_2(q))$. Then one of the following holds:*
   (a) *$\{v, w, vw\} \cap Z(GL_2(q)) \neq \emptyset$.*
   (b) *There exist $r, s \in \{v, w, vw\}$ such that $a^r = a$, $b^r = b^3$ and $a^s = a^{-1}$.*

Proof. (i) follows from Lemma 3.16 (i), (ii).

We now prove (ii). We have $Z(V) = \langle x^{2^k}\rangle$ by Lemma [23, Chapter 5, Theorem 4.3]. Thus, $Z(GL_2(q)) \cap V = \langle x^{2^k}\rangle$. Clearly, $\{v, w, vw\} \cap \langle x\rangle \subseteq \langle x^{2^{k-1}}\rangle$.

If $v, w \in \langle x\rangle$, then $v, w \in \langle x^{2^{k-1}}\rangle$, and it easily follows that (a) holds.

Assume now that $v \notin \langle x\rangle$ or $w \notin \langle x\rangle$. If neither v nor w lies in $\langle x\rangle$, then $vw \in \langle x\rangle$. Consequently, $\{v, w, vw\}$ has an element r of the form $x^{\ell 2^{k-1}}$ for some $1 \leq \ell \leq 4$ and an element s of the form $x^i y$ for some $1 \leq i \leq 2^{k+1}$. If $\ell = 2$ or 4, then (a) holds. Assume now that $\ell = 1$ or 3. It is clear that $a^r = a$. Furthermore, we have

$$
\begin{aligned}
b^r &= (xy)^{x^{\ell 2^{k-1}}} \\
&= xy^{x^{\ell 2^{k-1}}} \\
&= xx^{-\ell 2^{k-1}} y x^{\ell 2^{k-1}} y^2 \\
&= x^{1-\ell 2^{k-1}} (x^y)^{\ell 2^{k-1}} y \\
&= x^{1-\ell 2^{k-1}} (x^{-1+2^k})^{\ell 2^{k-1}} y \\
&= x^{1-\ell 2^k + \ell 2^{2k-1}} y \\
&= x^{1-\ell 2^k} y \\
&\overset{\ell \text{ odd}}{=} x^{1+2^k} y.
\end{aligned}
$$

On the other hand, we have

$$
b^3 = (xy)^3 = x^{2^k} xy = x^{1+2^k} y.
$$

Consequently, $b^r = b^3$. Finally, we also have

$$
a^s = (x^2)^{x^i y} = (x^2)^y = (x^y)^2 = (x^{-1+2^k})^2 = x^{-2} = a^{-1}.
$$

Thus, (b) holds. □

Proof of Lemma 3.52. If $\alpha_j|_{Q,Q} = \mathrm{id}_Q$ for some $j \in \{1, 2, 3\}$, then $\alpha_j = \mathrm{id}_L$ by Lemma 3.44, which implies that

$$
\bigcap_{i=1}^{3} O(C_L(\alpha_i)) \leq O(C_L(\alpha_j)) = O(L) = 1.
$$

Suppose now that α_i acts nontrivially on Q for all $i \in \{1,2,3\}$. Let $m \in \mathbb{N}$ with $|Q| = 2^m$. Using Lemma 3.55 (together with Sylow's theorem) and Lemma 3.56, we see that there exist $a, b \in Q$ and $i, j \in \{1,2,3\}$ such that the following hold:

(i) $\mathrm{ord}(a) = 2^{m-1}$, $\mathrm{ord}(b) = 4$, $a^b = a^{-1}$, $b^2 = a^{2^{m-2}}$;
(ii) $a^{\alpha_i} = a$, $b^{\alpha_i} = b^3$, $a^{\alpha_j} = a^{-1}$.

Clearly, $b^{\alpha_j} = a^{\ell} b$ for some $1 \leq \ell \leq 2^{m-1}$.

Assume that $q^* \equiv 1 \bmod 4$. By Lemma 3.53, there is group isomorphism $\varphi : L \to SL_2(q^*)$ with

$$a^{\varphi} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

for some generator λ of the Sylow 2-subgroup of $(\mathbb{F}_{q^*})^*$ and

$$b^{\varphi} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Set $\beta_k := \varphi^{-1}\alpha_k\varphi$ for $k \in \{1,2,3\}$. Let i and j be as in (ii). Also, let

$$m_i := \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

Then β_i and c_{m_i} normalize Q^{φ}, and we have $\beta_i|_{Q^{\varphi},Q^{\varphi}} = c_{m_i}|_{Q^{\varphi},Q^{\varphi}}$. Applying Lemma 3.44, we conclude that $\beta_i = c_{m_i}$.

Clearly,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\beta_j} = \begin{pmatrix} 0 & \mu \\ -\mu^{-1} & 0 \end{pmatrix}$$

for some 2-element μ of $(\mathbb{F}_{q^*})^*$. Set

$$m_j := \begin{pmatrix} 0 & \mu \\ -1 & 0 \end{pmatrix}.$$

Then β_j and c_{m_j} normalize Q^{φ}, and we have $\beta_j|_{Q^{\varphi},Q^{\varphi}} = c_{m_j}|_{Q^{\varphi},Q^{\varphi}}$. Applying Lemma 3.44, we conclude that $\beta_j = c_{m_j}$.

It follows that $C_{SL_2(q^*)}(\beta_i) \cap C_{SL_2(q^*)}(\beta_j) = Z(SL_2(q^*))$. So we have $C_L(\alpha_i) \cap C_L(\alpha_j) = Z(L)$, and this implies that

$$\bigcap_{k=1}^{3} O(C_L(\alpha_k)) = 1$$

since $|Z(L)| = 2$.

If $q^* \equiv 3 \bmod 4$, then a very similar argumentation shows that the same conclusion holds. Here, one has to use Lemma 3.54 instead of Lemma 3.53, together with the fact that $SL_2(q^*) \cong SU_2(q^*)$. □

We bring this section to a close with a proof of the following lemma, which will play an important role in the proof of Theorem B.

Lemma 3.57. *Let q be a nontrivial odd prime power, $\varepsilon \in \{+,-\}$ and $n \geq 2$ a natural number. Set $T := \mathrm{Inn}(PSL_n^{\varepsilon}(q))$. Let A be a subgroup of $\mathrm{Aut}(PSL_n^{\varepsilon}(q))$ such that $T \leq A$ and such that the index of T in A is odd. Let S be a Sylow 2-subgroup of T. Then we have $\mathcal{F}_S(T) = \mathcal{F}_S(A)$.*

To prove Lemma 3.57, we need to prove some other lemmas first.

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Lemma 3.58. *Let q be a nontrivial odd prime power, $\varepsilon \in \{+,-\}$, and let r be positive integer. Also, let W be a Sylow 2-subgroup of $GL^{\varepsilon}_{2^r}(q)$. Then* $\mathrm{Aut}(W)$ *is a 2-group.*

Proof. We proceed by induction over r.

Suppose that $r = 1$. If $q \equiv -\varepsilon \bmod 4$, then W is semidihedral by Lemmas 3.10 and 3.11, and so $\mathrm{Aut}(W)$ is a 2-group by [18, Proposition 4.53]. If $q \equiv \varepsilon \bmod 4$, then $W \cong C_{2^k} \wr C_2$ for some positive integer k by Lemmas 3.10 and 3.11, and so $\mathrm{Aut}(W)$ is a 2-group as a consequence of [21, Theorem 2].

Assume now that $r > 1$ and that the lemma is true with $r-1$ instead of r. Let W_0 be a Sylow 2-subgroup of $GL^{\varepsilon}_{2^{r-1}}(q)$. Hence, $\mathrm{Aut}(W_0)$ is a 2-group. By Lemma 3.14, we have $W \cong W_0 \wr C_2$. Applying [21, Theorem 2], we conclude that $\mathrm{Aut}(W)$ is a 2-group. □

Lemma 3.59. *Let q be a nontrivial odd prime power, $\varepsilon \in \{+,-\}$, and let $n \geq 3$ be a natural number. Let $T := SL^{\varepsilon}_n(q)$, and let S be a Sylow 2-subgroup of* $\mathrm{Inndiag}(T)$. *Then* $\mathrm{Aut}_{P\Gamma L^{\varepsilon}_n(q)}(S)$ *is a 2-group.*

Proof. Let $\alpha \in N_{P\Gamma L^{\varepsilon}_n(q)}(S)$. It suffices to show that $c_\alpha|_{S,S}$ is a 2-automorphism of S.

Let $0 \leq r_1 < \cdots < r_t$ such that $n = 2^{r_1} + \cdots + 2^{r_t}$. Let $W_i \in \mathrm{Syl}_2(GL^{\varepsilon}_{2^{r_i}}(q))$ for all $1 \leq i \leq t$. By Lemma 3.15,

$$W = \left\{ \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_t \end{pmatrix} : A_i \in W_i \right\}$$

is a Sylow 2-subgroup of $GL^{\varepsilon}_n(q)$.

We have that $\{c_w|_{T,T} \mid w \in W\}$ is a Sylow 2-subgroup of $\mathrm{Inndiag}(T)$ since it is the image of W under the canonical group epimorphism $GL^{\varepsilon}_n(q) \longrightarrow \mathrm{Inndiag}(T)$. Without loss of generality, we assume that $S = \{c_w|_{T,T} \mid w \in W\}$.

Let p be the odd prime number and f be the positive integer with $q = p^f$. Since $\alpha \in P\Gamma L^{\varepsilon}_n(q)$, there exist some $m \in GL^{\varepsilon}_n(q)$ and some natural number ℓ, where $1 \leq \ell \leq f$ if $\varepsilon = +$ and $1 \leq \ell \leq 2f$ if $\varepsilon = -$, such that

$$(a_{ij})^{\alpha} = (a_{ij}^{p^{\ell}})^m$$

for all $(a_{ij}) \in T$.

Let

$$\overline{\alpha} : GL^{\varepsilon}_n(q) \longrightarrow GL^{\varepsilon}_n(q), (a_{ij}) \mapsto (a_{ij}^{p^{\ell}})^m.$$

Observe that $\overline{\alpha}$ is the product of a field automorphism with an inner automorphism of $GL^{\varepsilon}_n(q)$. Using this fact, one can see that $\alpha^{-1}(c_w|_{T,T})\alpha = c_{w^{\overline{\alpha}}}|_{T,T}$ for all $w \in W$.

Let $w \in W$. Since α normalizes S, there is some $\widetilde{w} \in W$ with $c_{w^{\overline{\alpha}}}|_{T,T} = \alpha^{-1}(c_w|_{T,T})\alpha = c_{\widetilde{w}}|_{T,T}$. It follows that $w^{\overline{\alpha}} \in \widetilde{w}Z(GL^{\varepsilon}_n(q)) \subseteq WZ(GL^{\varepsilon}_n(q))$. This implies $w^{\overline{\alpha}} \in W$ since W is the unique Sylow 2-subgroup of $WZ(GL^{\varepsilon}_n(q))$. In particular, $\overline{\alpha}$ induces an automorphism of W.

Let

$$d_i := \begin{pmatrix} I_{2^{r_1}} & & & & \\ & \ddots & & & \\ & & -I_{2^{r_i}} & & \\ & & & \ddots & \\ & & & & I_{2^{r_t}} \end{pmatrix}$$

for each $1 \leq i \leq t$. Then d_i is a central involution of W for each $1 \leq i \leq t$ and centralized by the field automorphism $(a_{ij}) \mapsto (a_{ij}^p)$. So we have that $(d_i)^{\overline{\alpha}} = (d_i)^m$ is a central involution of W for each $1 \leq i \leq t$. As we see from Lemma 3.17, this already implies that $(d_i)^m = d_i$ for each $1 \leq i \leq t$. For

d_i is the unique member d of $\langle d_1, \ldots, d_n\rangle$ with $m([V,d]) = 2^{r_i}$, where V is the defining module for $GL_n^\varepsilon(q)$. So there is some $m_i \in GL_{2^{r_i}}^\varepsilon(q)$ for each $1 \le i \le t$ such that

$$m = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_t \end{pmatrix}.$$

Now

$$W_r \to W_r, (a_{ij}) \mapsto (a_{ij}^{p^\ell})^{m_i}$$

is an automorphism of W_r for each $1 \le r \le t$. Applying Lemma 3.58, we conclude that $\overline{\alpha}|_{W,W}$ is a 2-automorphism of W. Since $\alpha^{-1}(c_w|_{T,T})\alpha = c_{w^{\overline{\alpha}}}|_{T,T}$ for all $w \in W$, it follows that $c_\alpha|_{S,S}$ is a 2-automorphism of S, as required. □

Corollary 3.60. *Let q be a nontrivial odd prime power, $\varepsilon \in \{+,-\}$, and let $n \ge 3$ be a natural number. Let $T := PSL_n^\varepsilon(q)$, and let S be a Sylow 2-subgroup of* $\mathrm{Inndiag}(T)$. *Then* $\mathrm{Aut}_{P\Gamma L_n^\varepsilon(q)}(S)$ *is a 2-group.*

Lemma 3.61. *Let q be a nontrivial odd prime power, $\varepsilon \in \{+,-\}$, and $n \ge 3$ be a natural number. Let S be a Sylow 2-subgroup of $PSL_n^\varepsilon(q)$, and let S_1 be a Sylow 2-subgroup of $PGL_n^\varepsilon(q)$ containing S. Then $N_{PGL_n^\varepsilon(q)}(S) = N_{PGL_n^\varepsilon(q)}(S_1)$.*

Proof. Let T_1 be a Sylow 2-subgroup of $GL_n^\varepsilon(q))$ such that $S_1 = T_1Z(GL_n^\varepsilon(q))/Z(GL_n^\varepsilon(q))$. Let $T := T_1 \cap SL_n^\varepsilon(q)$. Then $S = TZ(GL_n^\varepsilon(q))/Z(GL_n^\varepsilon(q))$. It is rather easy to show $N_{PGL_n^\varepsilon(q)}(S) = N_{GL_n^\varepsilon(q)}(T)Z(GL_n^\varepsilon(q))/Z(GL_n^\varepsilon(q))$. By [36, Theorem 1], $N_{GL_n^\varepsilon(q)}(T) = T_1C_{GL_n^\varepsilon(q)}(T_1) \le N_{GL_n^\varepsilon(q)}(T_1)$. It follows that $N_{PGL_n^\varepsilon(q)}(S) \le N_{PGL_n^\varepsilon(q)}(S_1)$. It is clear that we also have $N_{PGL_n^\varepsilon(q)}(S_1) \le N_{PGL_n^\varepsilon(q)}(S)$. □

Corollary 3.62. *Let q be a nontrivial odd prime power, $\varepsilon \in \{+,-\}$, and let $n \ge 3$ be a natural number. Let $T := PSL_n^\varepsilon(q)$, let S be a Sylow 2-subgroup of* $\mathrm{Inn}(T)$ *and let S_1 be a Sylow 2-subgroup of* $\mathrm{Inndiag}(T)$ *containing S. Then $N_{\mathrm{Inndiag}(T)}(S) = N_{\mathrm{Inndiag}(T)}(S_1)$.*

We are now ready to prove Lemma 3.57.

Proof of Lemma 3.57. Assume that $n = 2$ and $q \equiv 3$ or $5 \bmod 8$. Then $S \cong C_2 \times C_2$ by Lemma 3.13. There is only one nonnilpotent fusion system on S. Since T and A are not 2-nilpotent, we have that $\mathcal{F}_S(T)$ and $\mathcal{F}_S(A)$ are not nilpotent (see [39, Theorem 1.4]). It follows that $\mathcal{F}_S(T) = \mathcal{F}_S(A)$.

From now on, we assume that either $n \ge 3$, or $n = 2$ and $q \equiv 1$ or $7 \bmod 8$. Let $P, Q \le S$ and $a \in A$ such that $P^a \le Q$. We are going to show that $c_a|_{P,Q}$ is a morphism in $\mathcal{F}_S(T)$. By the Frattini argument, we have $a = wu$ for some $w \in N_A(S)$ and some $u \in T$. We prove that $c_w|_{S,S} \in \mathrm{Inn}(S)$ so that $c_a|_{P,Q}$ is a morphism in $\mathcal{F}_S(T)$.

Suppose that $n = 2$. Then S is dihedral of order at least 8 by Lemma 3.13, and so $\mathrm{Aut}(S)$ is a 2-group by [18, Proposition 4.53]. This implies that $\mathrm{Aut}_A(S) = \mathrm{Inn}(S)$, whence $c_w|_{S,S} \in \mathrm{Inn}(S)$.

Suppose now that $n \ge 3$. Let S_1 be a Sylow 2-subgroup of $\mathrm{Inndiag}(PSL_n^\varepsilon(q))$ containing S. Since T has odd index in A, we have that $A \le P\Gamma L_n^\varepsilon(q)$. By the Frattini argument, $w = w_1w_2$ for some $w_1 \in N_{P\Gamma L_n^\varepsilon(q)}(S_1)$ and some $w_2 \in \mathrm{Inndiag}(PSL_n^\varepsilon(q))$. Since w_1 normalizes both S_1 and T, we have that w_1 normalizes S. And since $w = w_1w_2$ normalizes S, we also have that w_2 normalizes S. So w_2 normalizes S_1 by Corollary 3.62. Consequently, $w = w_1w_2 \in N_{P\Gamma L_n^\varepsilon(q)}(S_1)$. By Corollary 3.60, $c_w|_{S_1,S_1}$ is a 2-automorphism of S_1. So $c_w|_{S,S}$ is a 2-automorphism of S. Since $S \in \mathrm{Syl}_2(A)$ and $w \in A$, this implies that $c_w|_{S,S} \in \mathrm{Inn}(S)$, as required. □

4. The case $n \le 5$

In this section, we verify Theorem A for the case $n \le 5$.

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Proposition 4.1. *Let q be a nontrivial odd prime power, and let G be a finite simple group. Then the following are equivalent:*

(i) *the 2-fusion system of G is isomorphic to the 2-fusion system of $PSL_2(q)$;*
(ii) *the Sylow 2-subgroups of G are isomorphic to those of $PSL_2(q)$;*
(iii) *$G \cong PSL_2^{\varepsilon}(q^*)$ for some $\varepsilon \in \{+,-\}$ and some odd prime power $q^* \geq 5$ with $\varepsilon q^* \sim q$, or $|PSL_2(q)|_2 = 8$ and $G \cong A_7$.*

In particular, Theorem A holds for $n = 2$.

Proof. The implication (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii): Assume that the Sylow 2-subgroups of G are isomorphic to those of $PSL_2(q)$. Hence, G has dihedral Sylow 2-subgroups of order $\frac{1}{2}(q-1)_2(q+1)_2$. Applying a result of Gorenstein and Walter [30, Theorem 1], we may conclude that $G \cong PSL_2(q^*)$ for some odd prime power $q^* \geq 5$, or $G \cong A_7$. Suppose that the former holds. Then $(q^*-1)_2(q^*+1)_2 = 2|G|_2 = (q-1)_2(q+1)_2$, whence either $q^* \sim q$ or $-q^* \sim q$. Since $PSL_2(q^*) \cong PSU_2(q^*)$, this implies that the first statement in (iii) is satisfied. If $G \cong A_7$, then $|PSL_2(q)|_2 = |G|_2 = 8$ so that the second statement in (iii) is satisfied.

(iii) $\Rightarrow$ (i): Assume that (iii) holds. Set $G_1 := G$ and $G_2 := PSL_2(q)$. For $i \in \{1,2\}$, let $S_i \in \mathrm{Syl}_2(G_i)$ and $\mathcal{F}_i := \mathcal{F}_{S_i}(G_i)$. Clearly, S_1 and S_2 are dihedral groups of the same order. Let $i \in \{1,2\}$. By [23, Chapter 5, Theorem 4.3], any subgroup of S_i is cyclic or dihedral. By [18, Proposition 4.53], a dihedral subgroup of S_i with order greater than 4 cannot be $\mathcal{F}_i$-essential. Since the automorphism group of a finite cyclic 2-group is itself a 2-group, a cyclic subgroup of S_i cannot be $\mathcal{F}_i$-essential either. So we have that any $\mathcal{F}_i$-essential subgroup of S_i is a Klein four group. Alperin's fusion theorem [10, Part I, Theorem 3.5] implies that

$$\mathcal{F}_i = \langle \mathrm{Aut}_{\mathcal{F}_i}(P) \mid P \leq S_i, P \cong C_2 \times C_2 \text{ or } P = S_i \rangle_{S_i}.$$

If $|S_i| = 4$, then $\mathrm{Aut}_{\mathcal{F}_i}(S_i)$ is the unique subgroup of $\mathrm{Aut}(S_i)$ with order 3, because otherwise $\mathrm{Aut}_{\mathcal{F}_i}(S_i) = \mathrm{Inn}(S_i)$, so that [39, Theorem 1.4] would imply that G_i is 2-nilpotent. If $|S_i| \geq 8$, then $\mathrm{Aut}_{\mathcal{F}_i}(S_i) = \mathrm{Inn}(S_i)$ since $\mathrm{Aut}(S_i)$ is a 2-group by [18, Proposition 4.53], and for any Klein four subgroup P of S_i, we have $\mathrm{Aut}_{\mathcal{F}_i}(P) = \mathrm{Aut}(P)$ by [23, Chapter 7, Theorem 7.3]. As $S_1 \cong S_2$ and as the preceding observations do not depend on whether i is 1 or 2, we may conclude that $\mathcal{F}_1 \cong \mathcal{F}_2$, as required. □

Proposition 4.2. *Let q be a nontrivial odd prime power, and let G be a finite simple group. Then the following are equivalent:*

(i) *the 2-fusion system of G is isomorphic to the 2-fusion system of $PSL_3(q)$;*
(ii) *the Sylow 2-subgroups of G are isomorphic to those of $PSL_3(q)$;*
(iii) *$G \cong PSL_3^{\varepsilon}(q^*)$ for some $\varepsilon \in \{+,-\}$ and some nontrivial odd prime power q^* with $\varepsilon q^* \sim q$, or $(q+1)_2 = 4$ and $G \cong M_{11}$.*

In particular, Theorem A holds for $n = 3$.

Proof. The implication (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii): Assume that the Sylow 2-subgroups of G are isomorphic to those of $PSL_3(q)$. Hence, a Sylow 2-subgroup of G is wreathed (i.e., isomorphic to $C_{2^k} \wr C_2$ for some positive integer k) if $q \equiv 1 \bmod 4$ and semidihedral if $q \equiv 3 \bmod 4$. Applying work of Alperin, Brauer and Gorenstein, namely [1, Third Main Theorem] and [2, First Main Theorem], we may conclude that either $G \cong PSL_3^{\varepsilon}(q^*)$ for some $\varepsilon \in \{+,-\}$ and some nontrivial odd prime power q^* with $\varepsilon q^* \equiv q \bmod 4$ or $q \equiv 3 \bmod 4$ and $G \cong M_{11}$. If the former holds, then $((q^*-\varepsilon)_2)^2(q^*+\varepsilon)_2 = |G|_2 = ((q-1)_2)^2(q+1)_2$, and it easily follows that $\varepsilon q^* \sim q$. If $G \cong M_{11}$, then $16 = |G|_2 = ((q-1)_2)^2(q+1)_2$, and hence, $(q+1)_2 = 4$.

(iii) $\Rightarrow$ (i): Assume that (iii) holds. If $q \equiv 1 \bmod 4$, then Proposition 3.20 implies that the 2-fusion system of G is isomorphic to the 2-fusion system of $PSL_3(q)$. Alternatively, this can be seen from [18, Proposition 5.87]. Now suppose that $q \equiv 3 \bmod 4$. If $(q+1)_2 \neq 4$, then we could apply Proposition 3.20 again, but we are going to argue in a more elementary way. Let $G_1 := G$ and $G_2 := PSL_3(q)$. For

$i \in \{1,2\}$, let $S_i \in \mathrm{Syl}_2(G_i)$ and $\mathcal{F}_i := \mathcal{F}_{S_i}(G_i)$. Clearly, S_1 and S_2 are semidihedral groups of the same order. Let $i \in \{1,2\}$. By [23, Chapter 5, Theorem 4.3], any proper subgroup of S_i is cyclic, dihedral or generalized quaternion. By [18, Proposition 4.53], dihedral subgroups of S_i with order greater than 4 and generalized quaternion subgroups of S_i with order greater than 8 cannot be $\mathcal{F}_i$-essential. Since the automorphism group of a finite cyclic 2-group is itself a 2-group, a cyclic subgroup of S_i cannot be $\mathcal{F}_i$-essential either. Alperin's fusion theorem [10, Part I, Theorem 3.5] implies that

$$\mathcal{F}_i = \langle \mathrm{Aut}_{\mathcal{F}_i}(P) \mid P \cong C_2 \times C_2, P \cong Q_8, \text{ or } P = S_i \rangle_{S_i}.$$

Since $\mathrm{Aut}(S_i)$ is a 2-group by [18, Proposition 4.53], we have $\mathrm{Aut}_{\mathcal{F}_i}(S_i) = \mathrm{Inn}(S_i)$. From [1, pp. 10-11, Proposition 1], one can see that $\mathrm{Aut}_{\mathcal{F}_i}(P) = \mathrm{Aut}(P)$ for any subgroup P of S_i isomorphic to $C_2 \times C_2$ or Q_8. As $S_1 \cong S_2$ and as the preceding observations do not depend on whether i is 1 or 2, we may conclude that $\mathcal{F}_1 \cong \mathcal{F}_2$, as required. □

The following lemma is required to verify Theorem A for the case $n = 4$.

Lemma 4.3. *Let q be a nontrivial odd prime power. Assume that G is $A_{10}, A_{11}, M_{22}, M_{23}$ or McL. Then the 2-fusion system of G is not isomorphic to the 2-fusion system of $PSL_4(q)$.*

Proof. Assume otherwise. Let $L := PSL_4(q)$ and $S \in \mathrm{Syl}_2(L)$. Then $|S| = |G|_2 = 2^7$, so $q \equiv \pm 3 \bmod 8$. Let $\mathcal{E} := \mathcal{F}_S(L)$.

Take a Sylow 2-subgroup V of $GL_2(q)$, and let W be the Sylow 2-subgroup of $GL_4(q)$ obtained from V by the construction given in the last statement of Lemma 3.14. Then $S_0 := W \cap SL_4(q)$ is a Sylow 2-subgroup of $SL_4(q)$, and we assume without loss of generality that S is the image of S_0 in L.

Now $Z(S) = \langle z \rangle$, where z is the image of $\mathrm{diag}(1,1,-1,-1)$ in L. Let F be the image of

$$\left\{ \begin{pmatrix} A & \\ & B \end{pmatrix} : A, B \in SL_2(q) \right\}$$

in L, and let Q be the image of

$$\left\{ \begin{pmatrix} A & \\ & B \end{pmatrix} : A, B \in V_0 \right\},$$

in L, where $V_0 := V \cap SL_2(q)$. Then $F \trianglelefteq C_L(z)$, and so $Q = S \cap F$ is strongly closed in S with respect to $C_{\mathcal{E}}(z)$. Also, $Q' = \langle z \rangle$ is strongly closed in S with respect to $C_{\mathcal{E}}(z)$, and we have $[Q, \langle z \rangle] = 1$. Applying [10, Part I, Proposition 4.6], we conclude that $Q \trianglelefteq C_{\mathcal{E}}(z)$. Since Q is a self-centralizing subgroup of S, it follows that $C_{\mathcal{E}}(z)$ is constrained. Set $M := N_{C_L(z)}(Q)$. Then M is the image of

$$\left\{ \begin{pmatrix} A & \\ & B \end{pmatrix} : A, B \in N_{GL_2(q)}(V_0), \det(AB) = 1 \right\} \left\langle \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \right\rangle$$

in L. Since $q \equiv \pm 3 \bmod 8$, we have $N_{SL_2(q)}(V_0) \cong SL_2(3)$ by [48, Proposition 3.1]. Thus, $|N_{GL_2(q)}(V_0)| = 24(q-1)$, and it follows that $|M| = 2^7 \cdot 3^2$. By [34, Proposition 8.8], M is a model of $C_{\mathcal{E}}(z)$.

Now let $R \in \mathrm{Syl}_2(G)$ and $\mathcal{F} := \mathcal{F}_R(G)$. Also, let u be the central involution of R. Then $C_{\mathcal{F}}(u) \cong C_{\mathcal{E}}(z)$ is constrained with M a model $C_{\mathcal{F}}(u)$. By [34, Proposition 8.8], there is a core-free section of $C_G(u)$ which is isomorphic to M.

If G is McL, then $C_G(u)/\langle u \rangle \cong A_8$ by [28, Table 5.3] so that $C_{\mathcal{F}}(u)/\langle u \rangle$ is nonsolvable. On the other hand, $C_{\mathcal{E}}(z)/\langle z \rangle \cong C_{\mathcal{F}}(u)/\langle u \rangle$ is solvable. Thus, $G \neq McL$.

If G is A_{10}, M_{22} or M_{23}, then $|C_G(u)| = 2^7 \cdot 3$ or $2^7 \cdot 21$ so that $|M|$ does not divide $|C_G(u)|$, a contradiction.

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So G must be A_{11}. Then $|C_G(u)| = 2^7 \cdot 3^2$, and $C_G(u)$ has a normal subgroup of order 3. Therefore, M cannot be isomorphic to a core-free section of $C_G(u)$, which is again a contradiction.

The proof is now complete. □

Proposition 4.4. *Let q be a nontrivial odd prime power, and let G be a finite simple group. Then the following are equivalent:*

(i) *the 2-fusion system of G is isomorphic to the 2-fusion system of $PSL_4(q)$;*
(ii) *$G \cong PSL_4^{\varepsilon}(q^*)$ for some $\varepsilon \in \{+, -\}$ and some nontrivial odd prime power q^* with $\varepsilon q^* \sim q$.*

In particular, Theorem A holds for $n = 4$.

Proof. The implication (ii) ⇒ (i) is given by Proposition 3.20.

(i) ⇒ (ii): Assume that the 2-fusion system of G is isomorphic to the 2-fusion system of $PSL_4(q)$. Then the Sylow 2-subgroups of G are isomorphic to those of $PSL_4(q)$. Applying Mason's results [41, Theorem 1.1 and Corollary 1.3] and [40, Theorems 1.1 and 3.15], the latter together with [28, Theorem 4.10.5 (f)], we see that one of the following holds:

(1) $G \cong PSL_4^{\varepsilon}(q^*)$ for some nontrivial odd prime power q^* and some $\varepsilon \in \{+, -\}$ with $\varepsilon q^* \equiv q \bmod 4$;
(2) $G \cong A_{10}$ or A_{11};
(3) $G \cong M_{22}$, M_{23} or McL.

However, we know from Lemma 4.3 that the 2-fusion system of G is not isomorphic to the 2-fusion system of $PSL_4(q)$ when (2) or (3) holds. Thus, (1) holds.

Let q_0 be a nontrivial odd prime power, $\varepsilon_0 \in \{+, -\}$, and $k_0, s_0 \in \mathbb{N}$ such that $2^{k_0} = (q_0 - \varepsilon_0)_2$ and $2^{s_0} = (q_0 + \varepsilon_0)_2$. Then we have

$$|PSL_4^{\varepsilon_0}(q_0)|_2 = \frac{|GL_4^{\varepsilon_0}(q_0)|_2}{2^{k_0}(4, 2^{k_0})} = \frac{2(|GL_2^{\varepsilon_0}(q_0)|_2)^2}{2^{k_0}(4, 2^{k_0})} = \frac{2^{3k_0+2s_0+1}}{(4, 2^{k_0})}.$$

Let $k, k^*, s, s^* \in \mathbb{N}$ such that $2^k = (q-1)_2$, $2^{k^*} = (q^* - \varepsilon)_2$, $2^s = (q+1)_2$ and $2^{s^*} = (q^* + \varepsilon)_2$. Then we have

$$\frac{2^{3k^*+2s^*+1}}{(4, 2^{k^*})} = |G|_2 = \frac{2^{3k+2s+1}}{(4, 2^k)}.$$

Since $\varepsilon q^* \equiv q \bmod 4$, it follows that $\varepsilon q^* \sim q$. □

Proposition 4.5. *Let q be a nontrivial odd prime power, and let G be a finite simple group. Then the following are equivalent:*

(i) *the 2-fusion system of G is isomorphic to the 2-fusion system of $PSL_5(q)$;*
(ii) *the Sylow 2-subgroups of G are isomorphic to those of $PSL_5(q)$;*
(iii) *$G \cong PSL_5^{\varepsilon}(q^*)$ for some nontrivial odd prime power q^* and some $\varepsilon \in \{+, -\}$ with $\varepsilon q^* \sim q$.*

In particular, Theorem A holds for $n = 5$.

Proof. The implication (i) ⇒ (ii) is clear, and the implication (iii) ⇒ (i) is given by Proposition 3.20.

(ii) ⇒ (iii): Assume that the Sylow 2-subgroups of G are isomorphic to those of $PSL_5(q)$. Applying work of Mason [42, Theorem 1.1], it follows that $G \cong PSL_5^{\varepsilon}(q^*)$ for some $\varepsilon \in \{+, -\}$ and some nontrivial odd prime power q^*. In view of Lemma 3.15, it is easy to see that a Sylow 2-subgroup of G is isomorphic to a Sylow 2-subgroup of $GL_4^{\varepsilon}(q^*)$, while a Sylow 2-subgroup of $PSL_5(q)$ is isomorphic to a Sylow 2-subgroup of $GL_4(q)$. Now it is easy to deduce from Lemmas 3.10, 3.11 and 3.14 that a Sylow 2-subgroup of G has a center of order $(q^* - \varepsilon)_2$, while a Sylow 2-subgroup of

$PSL_5(q)$ has a center of order $(q-1)_2$. It follows that $(q^* - \varepsilon)_2 = (q-1)_2$. Let $k, s, k^*, s^* \in \mathbb{N}$ with $2^k = (q-1)_2, 2^s = (q+1)_2, 2^{k^*} = (q^* - \varepsilon)_2$ and $2^{s^*} = (q^* + \varepsilon)_2$. Then

$$2^{4k^*+2s^*+1} = |GL_4^{\varepsilon}(q^*)|_2 = |G|_2 = |GL_4(q)|_2 = 2^{4k+2s+1}.$$

Since $2^{k^*} = 2^k$, we thus have $k = k^*$ and $s = s^*$. This implies $\varepsilon q^* \sim q$. □

5. The case $n \geq 6$: preliminary discussion and notation

Given a natural number $k \geq 6$, we say that $P(k)$ is satisfied if whenever q_0 is a nontrivial odd prime power and H is a finite simple group satisfying ($\mathcal{CK}$) and realizing the 2-fusion system of $PSL_k(q_0)$, we have $H \cong PSL_k^{\varepsilon}(q^*)$ for some nontrivial odd prime power q^* and some $\varepsilon \in \{+, -\}$ with $\varepsilon q^* \sim q_0$.

In order to establish Theorem A for $n \geq 6$, we are going to prove by induction that $P(k)$ is satisfied for all $k \geq 6$. From now on until the end of Section 8, we will assume the following hypothesis.

Hypothesis 5.1. *Let $n \geq 6$ be a natural number such that $P(k)$ is satisfied for all natural numbers k with $6 \leq k < n$, and let q be a nontrivial odd prime power. Moreover, let G be a finite group satisfying the following properties:*

(i) *G realizes the 2-fusion system of $PSL_n(q)$;*
(ii) *$O(G) = 1$;*
(iii) *G satisfies ($\mathcal{CK}$).*

We will prove the following theorem.

Theorem 5.2. *There is a normal subgroup G_0 of G isomorphic to a nontrivial quotient of $SL_n^{\varepsilon}(q^*)$ for some nontrivial odd prime power q^* and some $\varepsilon \in \{+, -\}$ with $\varepsilon q^* \sim q$. In particular, $P(n)$ is satisfied.*

The proof of Theorem 5.2 will occupy Sections 5-8. In this section, we introduce some notation and prove some preliminary results needed for the proof.

For each $A \subseteq \{1, \ldots, n\}$ of even order, let t_A be the image of the diagonal matrix $\mathrm{diag}(d_1, \ldots, d_n)$ in $PSL_n(q)$, where

$$d_i = \begin{cases} -1 & \text{if } i \in A \\ 1 & \text{if } i \notin A \end{cases}$$

for each $1 \leq i \leq n$. If i is an even natural number with $2 \leq i < n$ and $A = \{n-i+1, \ldots, n\}$, then we write t_i for t_A. We denote t_2 by t, and we write u for $t_{\{1,2\}}$.

We assume ρ to be an element of $\mathbb{F}_q^*$ of order $(n, q-1)$. If ρ is a square in $\mathbb{F}_q$, then we assume μ to be a fixed element of $\mathbb{F}_q$ with $\rho = \mu^2$.

If n is even, ρ is a square in $\mathbb{F}_q$, and i is an odd natural number with $1 \leq i < n$, then

$$\begin{pmatrix} \mu I_{n-i} & \\ & -\mu I_i \end{pmatrix}$$

is an element of $SL_n(q)$ by Proposition 3.5, and we will denote its image in $PSL_n(q)$ by t_i.

If n is even and ρ is a nonsquare element of $\mathbb{F}_q$, then we denote the matrix

$$\begin{pmatrix} & I_{n/2} \\ \rho I_{n/2} & \end{pmatrix}$$

by $\widetilde{w}$, and if $\widetilde{w} \in SL_n(q)$, then we use w to denote its image in $PSL_n(q)$.

Note that, by Proposition 3.5, any involution of $PSL_n(q)$ is conjugate to t_i for some $1 \leq i < n$ such that t_i is defined, or to w (if defined).

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Next, we construct a Sylow 2-subgroup of $C_{PSL_n(q)}(t)$ containing some 'nice' elements of $PSL_n(q)$. Take a Sylow 2-subgroup V of $GL_2(q)$ containing each diagonal matrix in $GL_2(q)$ with 2-elements of $\mathbb{F}_q^*$ along the main diagonal. Similarly, we assume V_2 to be a Sylow 2-subgroup of $GL_{n-4}(q)$ containing each diagonal matrix in $GL_{n-4}(q)$ with 2-elements of $\mathbb{F}_q^*$ along the main diagonal. Now let W be a Sylow 2-subgroup of $GL_{n-2}(q)$ containing

$$\left\{\begin{pmatrix} A & \\ & B \end{pmatrix} \ :\ A \in V, B \in V_2\right\}.$$

If $n = 6$, then we assume that $V = V_2$ and that W is the Sylow 2-subgroup

$$\left\{\begin{pmatrix} A & \\ & B \end{pmatrix} \ :\ A, B \in V\right\} \cdot \left\langle \begin{pmatrix} & I_2 \\ I_2 & \end{pmatrix} \right\rangle$$

of $GL_4(q)$.

Let $\widetilde{t} := \mathrm{diag}(1, \dots, 1, -1, -1) \in SL_n(q)$. Then we have

$$C_{SL_n(q)}(\widetilde{t}) = \left\{\begin{pmatrix} A & \\ & B \end{pmatrix} \ :\ A \in GL_{n-2}(q), B \in GL_2(q), \det(A)\det(B) = 1\right\}.$$

It is easy to note that

$$\widetilde{T} := \left\{\begin{pmatrix} A & \\ & B \end{pmatrix} \ :\ A \in W, B \in V, \det(A)\det(B) = 1\right\}$$

is a Sylow 2-subgroup of $C_{SL_n(q)}(\widetilde{t})$. Let T denote the image of $\widetilde{T}$ in $PSL_n(q)$. As the centralizer of t in $PSL_n(q)$ is the image of $C_{SL_n(q)}(\widetilde{t})$ in $PSL_n(q)$, we have that T is a Sylow 2-subgroup of $C_{PSL_n(q)}(t)$. We assume S to be a Sylow 2-subgroup of $PSL_n(q)$ containing T. Since $C_S(t) = T \in \mathrm{Syl}_2(C_{PSL_n(q)}(t))$, we have that $\langle t \rangle$ is fully $\mathcal{F}_S(PSL_n(q))$-centralized.

Let K_1 be the image of

$$\left\{\begin{pmatrix} A & \\ & I_2 \end{pmatrix} \ :\ A \in SL_{n-2}(q)\right\}$$

in $PSL_n(q)$, and let K_2 be the image of

$$\left\{\begin{pmatrix} I_{n-2} & \\ & B \end{pmatrix} \ :\ B \in SL_2(q)\right\}$$

in $PSL_n(q)$. Clearly, K_1 and K_2 are normal subgroups of $C_{PSL_n(q)}(t)$ isomorphic to $SL_{n-2}(q)$ and $SL_2(q)$, respectively. Define X_1 to be the image of

$$\left\{\begin{pmatrix} A & \\ & I_2 \end{pmatrix} \ :\ A \in W \cap SL_{n-2}(q)\right\}$$

in $PSL_n(q)$, and define X_2 to be the image of

$$\left\{\begin{pmatrix} I_{n-2} & \\ & B \end{pmatrix} \ :\ B \in V \cap SL_2(q)\right\}$$

in $PSL_n(q)$.

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Note that $X_1 = T \cap K_1 \in \mathrm{Syl}_2(K_1)$ and $X_2 = T \cap K_2 \in \mathrm{Syl}_2(K_2)$. Define

$$\mathcal{C}_i := \mathcal{F}_{X_i}(K_i)$$

for $i \in \{1, 2\}$. By [10, Part I, Proposition 6.2], $\mathcal{C}_1$ and $\mathcal{C}_2$ are normal subsystems of $\mathcal{F}_T(C_{PSL_n(q)}(t))$.

Lemma 5.3. *Let $\mathcal{F} := \mathcal{F}_S(PSL_n(q))$. If $q \equiv 1$ or $7 \bmod 8$, then the components of $C_{\mathcal{F}}(\langle t\rangle)$ are precisely the subsystems $\mathcal{C}_1$ and $\mathcal{C}_2$. If $q \equiv 3$ or $5 \bmod 8$, then $\mathcal{C}_1$ is the only component of $C_{\mathcal{F}}(\langle t\rangle)$.*

Proof. Set $C := C_{PSL_n(q)}(t)$. Observe that the 2-components of C are precisely the quasisimple members of $\{K_1, K_2\}$. As $n \geq 6$ and as $K_1 \cong SL_{n-2}(q)$ and $K_2 \cong SL_2(q)$, it follows that the 2-components of C are K_1 and K_2 if $q \neq 3$ and that K_1 is the only 2-component of C if $q = 3$.

By Lemma 3.21, $K_1/Z(K_1)$ is not a Goldschmidt group. If $q \neq 3$, then the lemma just cited also shows that $K_2/Z(K_2)$ is a Goldschmidt group if and only if $q \equiv 3$ or $5 \bmod 8$.

Now we apply Proposition 2.17 to conclude that $\mathcal{F}_{T\cap K_1}(K_1)$ and $\mathcal{F}_{T\cap K_2}(K_2)$ are precisely the components of $\mathcal{F}_T(C)$ if $q \equiv 1$ or $7 \bmod 8$ and that $\mathcal{F}_{T\cap K_1}(K_1)$ is the only component of $\mathcal{F}_T(C)$ if $q \equiv 3$ or $5 \bmod 8$. This completes the proof because $C_{\mathcal{F}}(\langle t\rangle) = \mathcal{F}_T(C)$, $\mathcal{C}_1 = \mathcal{F}_{T\cap K_1}(K_1)$ and $\mathcal{C}_2 = \mathcal{F}_{T\cap K_2}(K_2)$. □

Lemma 5.4. *Let $\mathcal{F} := \mathcal{F}_S(PSL_n(q))$. Then the factor system $C_{\mathcal{F}}(\langle t\rangle)/X_1X_2$ is nilpotent.*

Proof. Set $C := C_{PSL_n(q)}(t)$. As $X_i = K_i \cap T$ is Sylow in K_i, $X_1X_2 = K_1K_2 \cap T$. By Lemma 2.11, $C_{\mathcal{F}}(\langle t\rangle)/X_1X_2$ is isomorphic to the 2-fusion system of C/K_1K_2. The factor group C/K_1K_2 is 2-nilpotent by Propositions 3.40 and 3.42, and so the 2-fusion system of C/K_1K_2 is nilpotent. Hence, $C_{\mathcal{F}}(\langle t\rangle)/X_1X_2$ is nilpotent. □

Lemma 5.5. *Let $A \in W$ and $B \in V$ such that* $\det(A)\det(B) = 1$. *Let*

$$m := \begin{pmatrix} A & \\ & B \end{pmatrix} Z(SL_n(q)) \in T.$$

Then we have $m \in Z(\mathcal{C}_1\langle m\rangle)$ if and only if $A \in Z(GL_{n-2}(q))$.

Proof. By [33, Proposition 1], we have $\mathcal{C}_1\langle m\rangle = \mathcal{F}_{X_1\langle m\rangle}(K_1\langle m\rangle)$. So, by Lemma 2.13, $m \in Z(\mathcal{C}_1\langle m\rangle)$ if and only if $m \in Z^*(K_1\langle m\rangle)$. This is the case if and only if $[K_1, \langle m\rangle] \leq O(K_1)$. If the latter holds, then $[\langle m\rangle, K_1, K_1] = [K_1, \langle m\rangle, K_1] = 1$ as $O(K_1) \leq Z(K_1)$, and so $[K_1, \langle m\rangle] = 1$ by the three subgroups lemma. Thus, the condition $[K_1, \langle m\rangle] \leq O(K_1)$ is satisfied if and only if $[K_1, \langle m\rangle] = 1$. So we have $m \in Z(\mathcal{C}_1\langle m\rangle)$ if and only if m centralizes K_1, and this is the case if and only if $A \in Z(GL_{n-2}(q))$. □

Lemma 5.6. *Set $\mathcal{F} := \mathcal{F}_S(PSL_n(q))$ and $\mathcal{G} := C_{\mathcal{F}}(\langle t\rangle)$. Then $\mathfrak{hnp}(C_{\mathcal{G}}(X_1)) = X_2$.*

Proof. Set $C := C_{PSL_n(q)}(t)$. Note that $C' = K_1K_2$.

By [23, Chapter 7, Theorem 3.4], we have $\mathfrak{foc}(C_{\mathcal{G}}(X_1)) = C_T(X_1) \cap C_C(X_1)' \leq C_T(X_1) \cap C' = C_T(X_1) \cap X_1X_2 = Z(X_1)X_2$. As $\mathfrak{hnp}(C_{\mathcal{G}}(X_1)) \leq \mathfrak{foc}(C_{\mathcal{G}}(X_1))$, it follows that $\mathfrak{hnp}(C_{\mathcal{G}}(X_1)) \leq Z(X_1)X_2$.

Let P be a subgroup of $C_T(X_1)$, and let φ be a $2'$-element of $\mathrm{Aut}_{C_C(X_1)}(P)$. By [37, 8.2.7], we have

$$[P, \langle\varphi\rangle] = [P, \langle\varphi\rangle, \langle\varphi\rangle] \leq [\mathfrak{hnp}(C_{\mathcal{G}}(X_1)) \cap P, \langle\varphi\rangle] \leq [Z(X_1)X_2 \cap P, \langle\varphi\rangle].$$

Since $\varphi \in \mathrm{Aut}_{C_C(X_1)}(P)$, $K_2 \trianglelefteq C$ and $X_2 = T \cap K_2$, it follows $[P, \langle\varphi\rangle] \leq X_2$. Consequently, $\mathfrak{hnp}(C_{\mathcal{G}}(X_1)) \leq X_2$.

On the other hand, since $K_2 \leq O^2(C_C(X_1))$, we have $X_2 \leq \mathfrak{hnp}(C_{\mathcal{G}}(X_1))$ by [18, Theorem 1.33]. □

Lemma 5.7. *Set $C := C_{PSL_n(q)}(t)$. Then $\mathrm{Aut}_C(X_1)$ is a 2-group.*

Proof. Let $m \in N_C(X_1)$. We have

$$m = \begin{pmatrix} M_1 & \\ & M_2 \end{pmatrix} Z(SL_n(q))$$

for some $M_1 \in GL_{n-2}(q)$ and some $M_2 \in GL_2(q)$ with $\det(M_1)\det(M_2) = 1$. Let $A \in W \cap SL_{n-2}(q)$ and

$$x := \begin{pmatrix} A & \\ & I_2 \end{pmatrix} Z(SL_n(q)) \in X_1.$$

As m normalizes X_1, we have

$$\begin{pmatrix} A^{M_1} & \\ & I_2 \end{pmatrix} Z(SL_n(q)) = x^m \in X_1.$$

This easily implies that $A^{M_1} \in W \cap SL_{n-2}(q)$. It follows that M_1 normalizes $W \cap SL_{n-2}(q)$. By [36, Theorem 1], we have $N_{GL_{n-2}(q)}(W \cap SL_{n-2}(q)) = WC_{GL_{n-2}(q)}(W)$. It follows that $c_m|_{X_1,X_1}$ is a 2-automorphism. □

Define T_1 to be the image of

$$\left\{ \begin{pmatrix} A & \\ & I_{n-2} \end{pmatrix} \ : \ A \in V \cap SL_2(q) \right\}$$

in $PSL_n(q)$ and T_2 to be the image of

$$\left\{ \begin{pmatrix} I_2 & & \\ & B & \\ & & I_2 \end{pmatrix} \ : \ B \in V_2 \cap SL_{n-4}(q) \right\}$$

in $PSL_n(q)$. By the definitions of X_1 and of W, T_1 and T_2 are subgroups of X_1. Recall that we use u to denote $t_{\{1,2\}} \in X_1$. The following lemma sheds light on some properties of the centralizer fusion system $C_{\mathcal{C}_1}(\langle u \rangle)$.

Lemma 5.8. *The following hold.*

(i) *We have $C_{X_1}(u) \in \mathrm{Syl}_2(C_{K_1}(u))$. In particular, $\langle u \rangle$ is fully $\mathcal{C}_1$-centralized.*
(ii) $\mathfrak{foc}(C_{\mathcal{C}_1}(\langle u \rangle)) = T_1T_2$.
(iii) *If $n = 6$ and $q \equiv 3$ or $5 \bmod 8$, then T_1 and T_2 are the only subgroups of $\mathfrak{foc}(C_{\mathcal{C}_1}(\langle u \rangle))$ which are isomorphic to Q_8 and strongly closed in $C_{\mathcal{C}_1}(\langle u \rangle)$.*
(iv) *If $n \geq 7$ and $q \equiv 3$ or $5 \bmod 8$, then T_1 is the only subgroup of the intersection $\mathfrak{foc}(C_{\mathcal{C}_1}(\langle u \rangle)) \cap C_{X_1}(T_2)$ which is isomorphic to Q_8 and strongly closed in $C_{\mathcal{C}_1}(\langle u \rangle)$.*
(v) *Let C_1 be the image of*

$$\left\{ \begin{pmatrix} A & \\ & I_{n-2} \end{pmatrix} \ : \ A \in SL_2(q) \right\}$$

in $PSL_n(q)$ and C_2 be the image of

$$\left\{ \begin{pmatrix} I_2 & & \\ & B & \\ & & I_2 \end{pmatrix} \ : \ A \in SL_{n-4}(q) \right\}$$

in $PSL_n(q)$. Then any component of $C_{\mathcal{C}_1}(\langle u \rangle)$ lies in $\{\mathcal{F}_{T_1}(C_1), \mathcal{F}_{T_2}(C_2)\}$. Moreover, $\mathcal{F}_{T_1}(C_1)$ is a component if and only if $q \equiv 1$ or $7 \bmod 8$, and $\mathcal{F}_{T_2}(C_2)$ is a component if and only if $n \geq 7$ or $q \equiv 1$ or $7 \bmod 8$.

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Proof. Clearly, $C_{K_1}(u)$ is the image of

$$\left\{\begin{pmatrix} A & & \\ & B & \\ & & I_2 \end{pmatrix} : A \in GL_2(q), B \in GL_{n-4}(q), \det(A)\det(B) = 1\right\}$$

in $PSL_n(q)$. Let $\widetilde{W}$ be the image of

$$\left\{\begin{pmatrix} A & & \\ & B & \\ & & I_2 \end{pmatrix} : A \in V, B \in V_2, \det(A)\det(B) = 1\right\}$$

in $PSL_n(q)$. By definition of X_1, we have $\widetilde{W} \leq C_{X_1}(u)$. We have $|C_{K_1}(u)| = |GL_2(q)||SL_{n-4}(q)|$ and $|\widetilde{W}| = |V||V_2 \cap SL_{n-4}(q)|$; so $\widetilde{W}$ is a Sylow 2-subgroup of $C_{K_1}(u)$. Thus, $C_{X_1}(u) = \widetilde{W} \in \mathrm{Syl}_2(C_{K_1}(u))$. Hence, (i) holds.

We have $C_{\mathcal{C}_1}(\langle u \rangle) = \mathcal{F}_{C_{X_1}(u)}(C_{K_1}(u)) = \mathcal{F}_{\widetilde{W}}(C_{K_1}(u))$. The focal subgroup theorem [23, Chapter 7, Theorem 3.4] implies that $\mathfrak{foc}(C_{\mathcal{C}_1}(\langle u \rangle)) = \widetilde{W} \cap (C_{K_1}(u))'$. It is easy to see that $(C_{K_1}(u))' = C_1C_2$, where C_1 and C_2 are as in (v). We thus have $\mathfrak{foc}(C_{\mathcal{C}_1}(\langle u \rangle)) = T_1T_2$. Hence, (ii) holds.

Now we turn to the proofs of (iii) and (iv). Assume that $q \equiv 3$ or $5 \bmod 8$. Clearly, C_1 and C_2 are normal subgroups of $C_{K_1}(u)$ and we have $T_1 = C_1 \cap \widetilde{W}$, $T_2 = C_2 \cap \widetilde{W}$. This implies that T_1 and T_2 are strongly closed in $C_{\mathcal{C}_1}(\langle u \rangle)$. By Lemma 3.12, we have $T_1 \cong Q_8$ and, if $n = 6$, we also have $T_2 \cong Q_8$. Clearly, any strongly $C_{\mathcal{C}_1}(\langle u \rangle)$-closed subgroup of $\mathfrak{foc}(C_{\mathcal{C}_1}(\langle u \rangle)) = T_1T_2$ is strongly closed in $\mathcal{F}_{T_1T_2}(C_1C_2)$. Hence, in order to prove (iii), it suffices to show that if $n = 6$, then T_1 and T_2 are the only strongly $\mathcal{F}_{T_1T_2}(C_1C_2)$-closed subgroups of T_1T_2 which are isomorphic to Q_8. Similarly, in order to prove (iv), it suffices to show that if $n \geq 7$, then T_1 is the only subgroup of T_1T_2 which centralizes T_2, which is isomorphic to Q_8, and which is strongly closed in $\mathcal{F}_{T_1T_2}(C_1C_2)$.

Continue to assume that $q \equiv 3$ or $5 \bmod 8$. In order to prove the two statements just mentioned, we need some observations. As $C_1 \cong SL_2(q)$, we have that C_1 is not 2-nilpotent. So $\mathcal{F}_{T_1}(C_1)$ is not nilpotent by [39, Theorem 1.4]. Again, by [39, Theorem 1.4], it follows that $\mathrm{Aut}_{C_1}(T_1)$ is not a 2-group. So $\mathrm{Aut}_{C_1}(T_1)$ has an element of order 3. Similarly, if $n = 6$, then $\mathrm{Aut}_{C_2}(T_2)$ has an element of order 3. It follows that there is an element $\alpha \in \mathrm{Aut}_{C_1C_2}(T_1T_2)$ such that $\alpha|_{T_1,T_1}$ has order 3, while $\alpha|_{T_2,T_2} = \mathrm{id}_{T_2}$. Moreover, if $n = 6$, then there is an element $\beta \in \mathrm{Aut}_{C_1C_2}(T_1T_2)$ such that $\beta|_{T_1,T_1} = \mathrm{id}_{T_1}$, while $\beta|_{T_2,T_2}$ has order 3.

Continue to assume that $q \equiv 3$ or $5 \bmod 8$. If $n = 6$, then the observations in the preceding two paragraphs show together with Lemma 2.15 that T_1 and T_2 are the only strongly $\mathcal{F}_{T_1T_2}(C_1C_2)$-closed subgroups of T_1T_2 which are isomorphic to Q_8. As observed above, this is enough to conclude that (iii) holds. If $n \geq 7$, then we may apply the observations in the preceding two paragraphs together with Lemma 2.15 to conclude that if T_0 is a strongly $\mathcal{F}_{T_1T_2}(C_1C_2)$-closed subgroup of T_1T_2 such that $T_0 \cong Q_8$ and such that T_0 centralizes T_2, then $T_0 = T_1$. As observed above, this is enough to conclude that (iv) holds.

Noticing that the 2-components of $C_{K_1}(u)$ are precisely the quasisimple members of $\{C_1, C_2\}$, we obtain (v) from Proposition 2.17 and Lemma 3.21. $\square$

Let G be as in Hypothesis 5.1. The group G realizes the 2-fusion system of $PSL_n(q)$. So, if R is a Sylow 2-subgroup of G, then $\mathcal{F}_S(PSL_n(q)) \cong \mathcal{F}_R(G)$. For the sake of simplicity, we will identify S with a Sylow 2-subgroup R of G and $\mathcal{F}_S(PSL_n(q))$ with $\mathcal{F}_R(G)$. Hence, we will work under the following hypothesis.

Hypothesis 5.9. *We will treat G as a group with $S \in \mathrm{Syl}_2(G)$ and $\mathcal{F}_S(G) = \mathcal{F}_S(PSL_n(q))$.*

The following lemma will play a key role in the proof of Theorem 5.2.

Lemma 5.10. *Let x be an involution of S such that $C_S(x) \in \mathrm{Syl}_2(C_G(x))$. Let $\mathcal{C}$ be a component of $\mathcal{F}_{C_S(x)}(C_G(x))$, and let k be a natural number with $3 \leq k < n$. Then the following hold.*

(i) *There is a unique 2-component Y of $C_G(x)$ such that $\mathcal{C} = \mathcal{F}_{C_S(x)\cap Y}(Y)$.*

(ii) *If $\mathcal{C}$ is isomorphic to the 2-fusion system of $SL_k(q)$, then we either have that $Y/O(Y) \cong SL_k^{\varepsilon}(q^*)/O(SL_k^{\varepsilon}(q^*))$ for some nontrivial odd prime power q^* and some $\varepsilon \in \{+,-\}$ with $q \sim \varepsilon q^*$; or $k = 3$, $(q+1)_2 = 4$, and $Y/Z^*(Y) \cong M_{11}$.*

(iii) *If $\mathcal{C}$ is isomorphic to the 2-fusion system of a nontrivial quotient of $SL_k(q^2)$, then $Y/O(Y)$ is isomorphic to a nontrivial quotient of $SL_k^{\varepsilon}(q^*)$ for some nontrivial odd prime power q^* and some $\varepsilon \in \{+,-\}$ with $q^2 \sim \varepsilon q^*$.*

In order to prove Lemma 5.10, we need the following observation.

Lemma 5.11. *Let $k \geq 6$ be a natural number satisfying $P(k)$. If q_0 is a nontrivial odd prime power and H is a known finite simple group realizing the 2-fusion system of $PSL_k(q_0)$, then $H \cong PSL_k^{\varepsilon}(q^*)$ for some $\varepsilon \in \{+,-\}$ and some nontrivial odd prime power q^* with $\varepsilon q^* \sim q_0$.*

Proof. It suffices to show that any known finite simple group H satisfies ($\mathcal{CK}$). Without using the CFSG, this is a priori not clear. It can be deduced from [28, Proposition 5.2.9] if H is alternating, from [28, Table 4.5.1] if H is a finite simple group of Lie type in odd characteristic, and from [28, Table 5.3] if H is sporadic. If H is a finite simple group of Lie type in characteristic 2, then H satisfies ($\mathcal{CK}$) since, in this case, no involution centralizer in H has a 2-component (see [5, 47.8 (3)]). □

Proof of Lemma 5.10. Since G satisfies ($\mathcal{CK}$), we have that $Y/Z^*(Y)$ is a known finite simple group for each 2-component Y of $C_G(x)$. Proposition 2.17 implies that there is a unique 2-component Y of $C_G(x)$ with $\mathcal{C} = \mathcal{F}_{C_S(x)\cap Y}(Y)$. Thus, (i) holds.

Suppose that $\mathcal{C}$ is isomorphic to the 2-fusion system of $SL_k(q_0)/Z$, where either $q_0 = q$ and $Z = 1$, or $q_0 = q^2$ and $Z \leq Z(SL_k(q^2))$. In order to prove (ii) and (iii), we need the following three claims.

(1) *The 2-fusion systems of $Y/Z^*(Y)$ and $PSL_k(q_0)$ are isomorphic.*

As $\mathcal{C} = \mathcal{F}_{C_S(x)\cap Y}(Y)$, we have that the 2-fusion system of Y is isomorphic to the 2-fusion system of $SL_k(q_0)/Z$. So, by Corollary 2.12, the 2-fusion system of $Y/O(Y)$ is isomorphic to the 2-fusion system of $SL_k(q_0)/Z$. Lemma 2.14 implies that the 2-fusion systems of $Y/Z^*(Y)$ and $PSL_k(q_0)$ are isomorphic.

(2) *We have $Y/Z^*(Y) \cong PSL_k^{\varepsilon}(q^*)$ for some nontrivial odd prime power q^* and some $\varepsilon \in \{+,-\}$* with *$q_0 \sim \varepsilon q^*$, or $k = 3$, $(q_0+1)_2 = 4$ and $Y/Z^*(Y) \cong M_{11}$.*

If $k = 3$, then it follows from (1) and Proposition 4.2. If $k \in \{4,5\}$, then it follows from (1) together with Propositions 4.4 and 4.5. Assume now that $k \geq 6$. By Hypothesis 5.1 and since $k < n$, we have that k satisfies $P(k)$. Since $Y/Z^*(Y)$ is a known finite simple group, the claim follows from (1) and Lemma 5.11.

(3) *Suppose that $Y/Z^*(Y) \cong PSL_k^{\varepsilon}(q^*)$, where q^* and ε are as in (2). Then we have $Y/O(Y) \cong SL_k^{\varepsilon}(q^*)/U$, where $U \leq Z(SL_k^{\varepsilon}(q^*))$ and the index of U in $Z(SL_k^{\varepsilon}(q^*))$ is equal to the 2-part of $|Z(SL_k(q_0))/Z|$.*

The group $Y/O(Y)$ is a perfect central extension of $PSL_k^{\varepsilon}(q^*)$. Since $Y/O(Y)$ is core-free, the center of $Y/O(Y)$ is a 2-group. So, by Lemmas 3.1 and 3.2, there is a central subgroup U of $SL_k^{\varepsilon}(q^*)$ with $Y/O(Y) \cong SL_k^{\varepsilon}(q^*)/U$. The claim now follows from

$$
\begin{aligned}
|PSL_k(q_0)|_2 |Z(SL_k(q_0))/Z|_2 &= |SL_k(q_0)/Z|_2 \\
&= |Y|_2 \\
&= |Y/Z^*(Y)|_2 |Z(Y/O(Y))| \\
&= |PSL_k(q_0)|_2 |Z(SL_k^{\varepsilon}(q^*))/U|.
\end{aligned}
$$

Here, the second equality follows from the fact that Y realizes $\mathcal{C}$, the third one holds since $|Z^*(Y)|_2 = |Z^*(Y)/O(Y)|_2 = |Z(Y/O(Y))|_2 = |Z(Y/O(Y))|$, and the fourth one follows from (1).

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Assume that $q_0 = q$ and $Z = 1$. By (2) and (3), one of the following holds: either $k = 3$, $(q+1)_2 = 4$ and $Y/Z^*(Y) \cong M_{11}$ or $Y/O(Y) \cong SL_k^\varepsilon(q^*)/U$, where q^* is a nontrivial odd prime power, $\varepsilon \in \{+,-\}$, $q \sim \varepsilon q^*$, $U \leq Z(SL_k^\varepsilon(q^*))$ and the index of U in $Z(SL_k^\varepsilon(q^*))$ is equal to the 2-part of $|Z(SL_k(q))|$. Assume that the latter holds. As $q \sim \varepsilon q^*$, we have $(q-1)_2 = (\varepsilon q^* - 1)_2 = (q^* - \varepsilon)_2$. Since $|Z(SL_k(q))| = (k, q-1)$ and $|Z(SL_k^\varepsilon(q^*))| = (k, q^* - \varepsilon)$, it follows that the 2-part of $|Z(SL_k(q))|$ is equal to the 2-part of $|Z(SL_k^\varepsilon(q^*))|$. It follows that $U = O(Z(SL_k^\varepsilon(q^*))) = O(SL_k^\varepsilon(q^*))$. This completes the proof of (ii).

Assume now that $q_0 = q^2$. Then, since $q^2 \equiv 1 \bmod 4$, (2) and (3) imply that $Y/O(Y)$ is isomorphic to a nontrivial quotient of $SL_k^\varepsilon(q^*)$ for some nontrivial odd prime power q^* and some $\varepsilon \in \{+,-\}$ with $q^2 \sim \varepsilon q^*$. Thus, (iii) holds. □

6. 2-components of involution centralizers

In this section, we continue to assume Hypotheses 5.1 and 5.9. We will use the notation introduced in the last section without further explanation.

The main goal of this section is to describe the 2-components and the solvable 2-components of the centralizers of involutions of G.

6.1. The subgroups K and L of $C_G(t)$

We start by considering $C_G(t)$. Let $\mathcal{F} := \mathcal{F}_S(G) = \mathcal{F}_S(PSL_n(q))$. Since $\langle t \rangle$ is fully $\mathcal{F}$-centralized, we have that $T = C_S(t) \in \mathrm{Syl}_2(C_G(t))$. Also, note that $\mathcal{F}_T(C_G(t)) = C_{\mathcal{F}}(\langle t \rangle) = \mathcal{F}_T(C_{PSL_n(q)}(t))$.

Proposition 6.1. *There is a unique 2-component K of $C_G(t)$ such that $\mathcal{C}_1 = \mathcal{F}_{T \cap K}(K)$. We have $K/O(K) \cong SL_{n-2}^\varepsilon(q^*)/O(SL_{n-2}^\varepsilon(q^*))$ for some nontrivial odd prime power q^* and some $\varepsilon \in \{+,-\}$ with $q \sim \varepsilon q^*$. Moreover, K is a normal subgroup of $C_G(t)$.*

Proof. Set $\mathcal{F} := \mathcal{F}_S(G)$. By Lemma 5.3, $\mathcal{C}_1$ is a component of $C_{\mathcal{F}}(\langle t \rangle)$. Lemma 5.10 (i) implies that there is a unique 2-component K of $C_G(t)$ such that $\mathcal{C}_1 = \mathcal{F}_{T \cap K}(K)$. By definition, the component $\mathcal{C}_1$ is isomorphic to the 2-fusion system of $SL_{n-2}(q)$. Lemma 5.10 (ii) implies that $K/O(K) \cong SL_{n-2}^\varepsilon(q^*)/O(SL_{n-2}^\varepsilon(q^*))$ for some nontrivial odd prime power q^* and some $\varepsilon \in \{+,-\}$ with $q \sim \varepsilon q^*$.

It remains to show that K is a normal subgroup of $C_G(t)$. Suppose that $\widetilde{K}$ is a 2-component of $C_G(t)$ such that $K \cong \widetilde{K}$. Set $\widetilde{\mathcal{C}} := \mathcal{F}_{T \cap \widetilde{K}}(\widetilde{K})$. Since $\widetilde{K}$ is subnormal in $C_G(t)$, it easily follows from [10, Part I, Proposition 6.2] that $\widetilde{\mathcal{C}}$ is subnormal in $C_{\mathcal{F}}(\langle t \rangle)$. Moreover, $\widetilde{\mathcal{C}} \cong \mathcal{C}_1$ as $\widetilde{K} \cong K$. Hence, $\widetilde{\mathcal{C}}$ is a component of $C_{\mathcal{F}}(\langle t \rangle)$. But as a consequence of Lemma 5.3, there is no component of $C_{\mathcal{F}}(\langle t \rangle)$ which is isomorphic to $\mathcal{C}_1$ and different from $\mathcal{C}_1$. So we have $\mathcal{C}_1 = \widetilde{\mathcal{C}}$. The uniqueness in the first statement of the proposition implies that $K = \widetilde{K}$. Consequently, $C_G(t)$ has no 2-component which is different from K and isomorphic to K. So K is characteristic and hence normal in $C_G(t)$. □

From now on, K, q^* and ε will always have the meanings given to them by Proposition 6.1.

Our next goal is to prove the existence and uniqueness of a normal subgroup $\overline{L}$ of $\overline{C_G(t)} := C_G(t)/O(C_G(t))$ such that $\overline{L} \cong SL_2(q^*)$ and to show that the image $\overline{K}$ of K in $\overline{C_G(t)}$ and $\overline{L}$ are the only subgroups of $\overline{C_G(t)}$ which are components or solvable 2-components of $\overline{C_G(t)}$. First, we need to prove some lemmas.

Lemma 6.2. *Let $A \in W$ and $B \in V$ such that* $\det(A)\det(B) = 1$. *Let*

$$m := \begin{pmatrix} A & \\ & B \end{pmatrix} Z(SL_n(q)) \in T.$$

Set $\overline{C_G(t)} := C_G(t)/O(C_G(t))$. Then $\overline{m}$ centralizes $\overline{K}$ if and only if $A \in Z(GL_{n-2}(q))$.

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Proof. Let $\overline{\mathcal{C}_1}$ be the subsystem of $\mathcal{F}_{\overline{T}}(\overline{C_G(t)})$ corresponding to $\mathcal{C}_1$ under the isomorphism from $\mathcal{F}_T(C_G(t))$ to $\mathcal{F}_{\overline{T}}(\overline{C_G(t)})$ given by Corollary 2.12. By [33, Proposition 1], we have

$$\overline{\mathcal{C}_1}\langle\overline{m}\rangle = \mathcal{F}_{\overline{X_1}\langle\overline{m}\rangle}(\overline{K}\langle\overline{m}\rangle).$$

Since $\overline{m}$ is a 2-element of $\overline{C_G(t)}$, we have $O(\overline{K}\langle\overline{m}\rangle) = O(\overline{K}) = 1$. Applying Lemma 2.13, it follows that the center of the fusion system $\overline{\mathcal{C}_1}\langle\overline{m}\rangle$ is equal to the center of $\overline{K}\langle\overline{m}\rangle$. In particular, $\overline{m}$ centralizes $\overline{K}$ if and only if $\overline{m} \in Z(\overline{\mathcal{C}_1}\langle\overline{m}\rangle)$. By Lemma 5.5, this is the case if and only if $A \in Z(GL_{n-2}(q))$. □

Lemma 6.3. *Suppose that $q^* = 3$. Let $C := C_G(t)$ and $\overline{C} := C/O(C)$. Then:*

(i) *The factor group $\overline{C}/\overline{K}C_{\overline{C}}(\overline{K})$ is a 2-group.*
(ii) *The centralizer $C_{\overline{C}}(\overline{u})$ is core-free.*
(iii) *The factor group $C_{\overline{C}}(\overline{u})/C_{\overline{C}}(\overline{K})$ is core-free.*

Proof. Clearly, $\overline{C}/\overline{K}C_{\overline{C}}(\overline{K})$ is isomorphic to a subgroup of $\mathrm{Out}(\overline{K})$. Since $q^* = 3$, we have that $\overline{K} \cong SL^{\varepsilon}_{n-2}(3)$. By Propositions 3.41 and 3.43, $\mathrm{Out}(\overline{K})$ is a 2-group. So (i) holds.

Set $\overline{C}_0 := \overline{K}C_{\overline{C}}(\overline{K})$. As a consequence of (i), $C_{\overline{C}}(\overline{u})/C_{\overline{C}_0}(\overline{u})$ is a 2-group. Hence, in order to prove (ii), it suffices to show that $C_{\overline{C}_0}(\overline{u})$ is core-free. As $\overline{u} \in \overline{K}$, we have $C_{\overline{C}_0}(\overline{u}) = C_{\overline{K}}(\overline{u})C_{\overline{C}}(\overline{K})$. It follows that $C_{\overline{C}_0}(\overline{u})/C_{\overline{C}}(\overline{K}) \cong C_{\overline{K}}(\overline{u})/(C_{\overline{K}}(\overline{u}) \cap C_{\overline{C}}(\overline{K})) = C_{\overline{K}}(\overline{u})/Z(\overline{K})$. By Corollary 3.8, $C_{\overline{K}}(\overline{u})$ is core-free. This easily implies that $C_{\overline{K}}(\overline{u})/Z(\overline{K})$ is core-free. It follows that $C_{\overline{C}_0}(\overline{u})/C_{\overline{C}}(\overline{K})$ is core-free. Consequently, $O(C_{\overline{C}_0}(\overline{u})) = O(C_{\overline{C}}(\overline{K})) = 1$. So (ii) follows.

Finally, (iii) is true since $C_{\overline{C}}(\overline{u})/C_{\overline{C}_0}(\overline{u})$ is a 2-group and $C_{\overline{C}_0}(\overline{u})/C_{\overline{C}}(\overline{K})$ is core-free. □

Lemma 6.4. *Let $\overline{C_G(t)} := C_G(t)/O(C_G(t))$. Then there is a unique pair $({A_1}^+, {A_2}^+)$ of normal subgroups ${A_1}^+$, ${A_2}^+$ of $C_{\overline{K}}(\overline{u})'$ such that $C_{\overline{K}}(\overline{u})' = {A_1}^+ \times {A_2}^+$, ${A_1}^+ \cong SL^{\varepsilon}_2(q^*)$, ${A_2}^+ \cong SL^{\varepsilon}_{n-4}(q^*)$ and $\overline{u} \in {A_1}^+$. Moreover, the following hold.*

(i) ${A_1}^+ \cap \overline{X_1} = \overline{T_1}$.
(ii) ${A_2}^+ \cap \overline{X_1} = \overline{T_2}$.
(iii) *There is a group isomorphism $\varphi : \overline{K} \longrightarrow SL^{\varepsilon}_{n-2}(q^*)/O(SL^{\varepsilon}_{n-2}(q^*))$ under which ${A_1}^+$ corresponds to the image of*

$$\left\{\begin{pmatrix} A & \\ & I_{n-4}\end{pmatrix} : A \in SL^{\varepsilon}_2(q^*)\right\}$$

in $SL^{\varepsilon}_{n-2}(q^*)/O(SL^{\varepsilon}_{n-2}(q^*))$ and under which ${A_2}^+$ corresponds to the image of

$$\left\{\begin{pmatrix} I_2 & \\ & B\end{pmatrix} : B \in SL^{\varepsilon}_{n-4}(q^*)\right\}$$

in $SL^{\varepsilon}_{n-2}(q^*)/O(SL^{\varepsilon}_{n-2}(q^*))$.

Proof. For each subsystem $\mathcal{G}$ of $\mathcal{F}_T(C_G(t))$, we use $\overline{\mathcal{G}}$ to denote the subsystem of $\mathcal{F}_{\overline{T}}(\overline{C_G(t)})$ corresponding to $\mathcal{G}$ under the isomorphism from $\mathcal{F}_T(C_G(t))$ to $\mathcal{F}_{\overline{T}}(\overline{C_G(t)})$ given by Corollary 2.12. Note that $\overline{\mathcal{C}_1} = \mathcal{F}_{\overline{X_1}}(\overline{K})$.

Set $H := SL^{\varepsilon}_{n-2}(q^*)/O(SL^{\varepsilon}_{n-2}(q^*))$. For each even natural number i with $2 \le i \le n-2$, let h_i be the image of $\widetilde{h_i} := \mathrm{diag}(-1, \dots, -1, 1, \dots, 1) \in SL^{\varepsilon}_{n-2}(q^*)$ in H, where -1 occurs precisely i times as a diagonal entry.

We claim that there is a group isomorphism $\varphi : \overline{K} \longrightarrow H$ such that $\overline{u}^{\varphi} = h_i$ for some even $2 \le i < n-2$. By Proposition 6.1, we have $\overline{K} \cong K/O(K) \cong H$. As a consequence of Lemmas 3.3 and 3.4, any involution of $SL^{\varepsilon}_{n-2}(q^*)$ is conjugate to $\widetilde{h_i}$ for some even $2 \le i \le n-2$. Since any involution of H is induced by an involution of $SL^{\varepsilon}_{n-2}(q^*)$, it follows that any involution of H is conjugate to h_i for some even $2 \leq i \leq n-2$. As $\bar{u}$ is an involution of $\overline{K}$, it follows that there is an isomorphism $\varphi : \overline{K} \longrightarrow H$ mapping $\bar{u}$ to h_i for some even $2 \leq i \leq n-2$. Assume that $i = n-2$. Then $\bar{u}$ is central in $\overline{K}$. Thus, $\bar{u} \in Z(\overline{\mathcal{C}_1})$ and hence $u \in Z(\mathcal{C}_1)$. This is a contradiction to Lemma 3.18 and the definition of $\mathcal{C}_1$. So we have $i < n-2$.

Set $h := \bar{u}^{\varphi} = h_i$. Also, let H_1 be the image of

$$\left\{ \begin{pmatrix} A & \\ & I_{n-2-i} \end{pmatrix} \; : \; A \in SL^{\varepsilon}_i(q^*) \right\}$$

in H, and let H_2 be the image of

$$\left\{ \begin{pmatrix} I_i & \\ & B \end{pmatrix} \; : \; B \in SL^{\varepsilon}_{n-2-i}(q^*) \right\}$$

in H. For $j \in \{1,2\}$, let ${A_j}^+$ be the subgroup of $\overline{K}$ corresponding to H_j under φ.

We now proceed in a number of steps in order to complete the proof.

(1) *We have* $C_{\overline{K}}(\bar{u})' = {A_1}^+{A_2}^+$, $[{A_1}^+, {A_2}^+] = 1$, ${A_1}^+, {A_2}^+ \trianglelefteq C_{\overline{K}}(\bar{u})$, $\bar{u} \in {A_1}^+$ *and* $\bar{u} \notin {A_2}^+$.

It is easy to note that $C_H(h)'$ is the central product of H_1 and H_2 and that H_1 and H_2 are normal in $C_H(h)$. Therefore, $C_{\overline{K}}(\bar{u})'$ is the central product of ${A_1}^+$ and ${A_2}^+$, and ${A_1}^+, {A_2}^+$ are normal in $C_{\overline{K}}(\bar{u})$. By definition of H_1 and H_2, we have $h \in H_1$ and $h \notin H_2$. Thus, $\bar{u} \in {A_1}^+$ and $\bar{u} \notin {A_2}^+$.

(2) *We have* $C_{\overline{X_1}}(\bar{u}) \in \mathrm{Syl}_2(C_{\overline{K}}(\bar{u}))$, *and* $\{\mathcal{F}_{\overline{X_1} \cap {A_1}^+}({A_1}^+), \mathcal{F}_{\overline{X_1} \cap {A_2}^+}({A_2}^+)\}$ *contains every component of* $C_{\overline{\mathcal{C}_1}}(\langle \bar{u} \rangle)$.

By Lemma 5.8 (i), we have that $\langle \bar{u} \rangle$ is fully $\overline{\mathcal{C}_1}$-centralized. So we have $C_{\overline{X_1}}(\bar{u}) \in \mathrm{Syl}_2(C_{\overline{K}}(\bar{u}))$.

Set $P := C_{\overline{X_1}}(\bar{u})^{\varphi} \in \mathrm{Syl}_2(C_H(h))$. Noticing that the 2-components of $C_H(h)$ are precisely the quasisimple members of $\{H_1, H_2\}$, we see from Proposition 2.17 that the components of $\mathcal{F}_P(C_H(h))$ are precisely the quasisimple members of $\{\mathcal{F}_{P \cap H_1}(H_1), \mathcal{F}_{P \cap H_2}(H_2)\}$.

Thus, the components of $C_{\overline{\mathcal{C}_1}}(\langle \bar{u} \rangle) = \mathcal{F}_{C_{\overline{X_1}}(\bar{u})}(C_{\overline{K}}(\bar{u}))$ are precisely the quasisimple members of $\{\mathcal{F}_{\overline{X_1} \cap {A_1}^+}({A_1}^+), \mathcal{F}_{\overline{X_1} \cap {A_2}^+}({A_2}^+)\}$.

(3) $\overline{X_1} \cap {A_1}^+$ *and* $\overline{X_1} \cap {A_2}^+$ *are subgroups of* $\mathfrak{foc}(C_{\overline{\mathcal{C}_1}}(\langle \bar{u} \rangle))$ *and are strongly closed in* $C_{\overline{\mathcal{C}_1}}(\langle \bar{u} \rangle)$.

We have $\mathfrak{foc}(C_{\overline{\mathcal{C}_1}}(\langle \bar{u} \rangle)) = C_{\overline{X_1}}(\bar{u}) \cap C_{\overline{K}}(\bar{u})'$ by the focal subgroup theorem [23, Chapter 7, Theorem 3.4]. So the claim follows from (1).

(4) *Suppose that* $n = 6$ *and* $q \equiv 3$ *or* $5 \bmod 8$. *Then we have* $i = 2$ *and hence* ${A_1}^+ \cong SL^{\varepsilon}_2(q^*) \cong {A_2}^+$. *Moreover,* $\overline{X_1} \cap {A_1}^+ = \overline{T_1}$ *and* $\overline{X_1} \cap {A_2}^+ = \overline{T_2}$.

Since $n = 6$ and $2 \leq i < n-2 = 4$, we have $i = 2$. Thus, ${A_1}^+ \cong H_1 \cong SL^{\varepsilon}_2(q^*) \cong H_2 \cong {A_2}^+$. By Proposition 6.1, we have $q \sim \varepsilon q^*$, whence $q^* \equiv 3$ or $5 \bmod 8$. Clearly, $\overline{X_1} \cap {A_1}^+ \in \mathrm{Syl}_2({A_1}^+)$ and $\overline{X_1} \cap {A_2}^+ \in \mathrm{Syl}_2({A_2}^+)$. Lemma 3.12 implies that $\overline{X_1} \cap {A_1}^+ \cong Q_8 \cong \overline{X_1} \cap {A_2}^+$. By Lemma 5.8 (iii), $\overline{T_1}$ and $\overline{T_2}$ are the only subgroups of $\mathfrak{foc}(C_{\overline{\mathcal{C}_1}}(\langle \bar{u} \rangle))$ which are isomorphic to Q_8 and strongly closed in $C_{\overline{\mathcal{C}_1}}(\langle \bar{u} \rangle)$. So, by (3), $\{\overline{X_1} \cap {A_1}^+, \overline{X_1} \cap {A_2}^+\} = \{\overline{T_1}, \overline{T_2}\}$. We have $\bar{u} \in \overline{T_1}$, and $\bar{u} \notin {A_2}^+$ by (1). It follows that $\overline{X_1} \cap {A_1}^+ = \overline{T_1}$ and $\overline{X_1} \cap {A_2}^+ = \overline{T_2}$.

(5) *Suppose that* $n = 6$ *and* $q \equiv 1$ *or* $7 \bmod 8$ *or that* $n \geq 7$. *Then we have* $i = 2$, *and hence* ${A_1}^+ \cong SL^{\varepsilon}_2(q^*)$ *and* ${A_2}^+ \cong SL^{\varepsilon}_{n-4}(q^*)$. *Moreover,* $\overline{X_1} \cap {A_1}^+ = \overline{T_1}$ *and* $\overline{X_1} \cap {A_2}^+ = \overline{T_2}$.

We begin by proving that $\overline{X_1} \cap {A_2}^+ = \overline{T_2}$. As a consequence of Lemma 5.8 (v), $C_{\overline{\mathcal{C}_1}}(\langle \bar{u} \rangle)$ has a component with Sylow group $\overline{T_2}$. Applying (2), we may conclude that $\overline{T_2} = \overline{X_1} \cap {A_1}^+$ or $\overline{X_1} \cap {A_2}^+$. Since $\bar{u} \in A_1^+$ by (1), but $\bar{u} \notin \overline{T_2}$, we have $\overline{X_1} \cap {A_2}^+ = \overline{T_2}$.

We show next that $i = 2$. Using Proposition 3.19, or using the order formulas for $|SL_{n-4}(q^*)|$ and $|SU_{n-4}(q^*)|$ given by [32, Proposition 1.1 and Corollary 11.29], we see that

$$|SL^{\varepsilon}_{n-4}(q^*)|_2 = |SL_{n-4}(q)|_2 = |\overline{T_2}| = |{A_2}^+|_2 = |H_2|_2 = |SL^{\varepsilon}_{n-2-i}(q^*)|_2.$$

Using the order formulas cited above, we may conclude that $n-2-i = n-4$, whence $i = 2$. In particular, ${A_1}^+ \cong SL_2^{\varepsilon}(q^*)$ and ${A_2}^+ \cong SL_{n-4}^{\varepsilon}(q^*)$.

It remains to prove $\overline{X_1} \cap {A_1}^+ = \overline{T_1}$. If $q \equiv 1$ or 7 mod 8, then Lemma 5.8 (v) shows that $C_{\overline{C_1}}(\langle \bar{u} \rangle)$ has a component with Sylow group $\overline{T_1}$. Since $\bar{u} \in \overline{T_1}$, but $\bar{u} \notin {A_2}^+$, we have $\overline{X_1} \cap {A_1}^+ = \overline{T_1}$ by (2).

Now suppose that $q \equiv 3$ or 5 mod 8. Then we have $q^* \equiv 3$ or 5 mod 8 since $q \sim \varepsilon q^*$. So, by Lemma 3.12, a Sylow 2-subgroup of ${A_1}^+$ is isomorphic to Q_8. In particular, $\overline{X_1} \cap {A_1}^+ \cong Q_8$. By (3), $\overline{X_1} \cap {A_1}^+$ is a subgroup of $\mathfrak{foc}(C_{\overline{C_1}}(\langle \bar{u} \rangle))$ and is strongly closed in $C_{\overline{C_1}}(\langle \bar{u} \rangle)$. Moreover, by (1), $\overline{X_1} \cap {A_1}^+$ centralizes $\overline{X_1} \cap {A_2}^+ = \overline{T_2}$. Lemma 5.8 (iv) now implies that $\overline{T_1} = \overline{X_1} \cap {A_1}^+$.

(6) $C_{\overline{K}}(\bar{u})' = {A_1}^+ \times {A_2}^+$.

We have ${A_1}^+ \cong SL_2^{\varepsilon}(q^*)$ by (4) and (5), and $\bar{u} \in Z({A_1}^+)$ by (1). It follows that $Z({A_1}^+) = \langle \bar{u} \rangle$. By (1), ${A_1}^+ \cap {A_2}^+ \leq Z({A_1}^+)$ and $\bar{u} \notin {A_1}^+ \cap {A_2}^+$. It follows that ${A_1}^+ \cap {A_2}^+ = 1$. So (1) implies that $C_{\overline{K}}(\bar{u})' = {A_1}^+ \times {A_2}^+$.

(7) *Assume that* ${A_1}^\circ$ *and* ${A_2}^\circ$ *are normal subgroups of* $C_{\overline{K}}(\bar{u})'$ *such that* $C_{\overline{K}}(\bar{u})' = {A_1}^\circ \times {A_2}^\circ$, ${A_1}^\circ \cong SL_2^{\varepsilon}(q^*)$, ${A_2}^\circ \cong SL_{n-4}^{\varepsilon}(q^*)$ *and* $\bar{u} \in {A_1}^\circ$. *Then* ${A_1}^\circ = {A_1}^+$ *and* ${A_2}^\circ = {A_2}^+$.

Let $j \in \{1, 2\}$. As a consequence of (4) and (5), ${A_j}^+$ is either quasisimple or isomorphic to $SL_2(3)$. In either case, ${A_j}^+$ is indecomposable, i.e., ${A_j}^+$ cannot be written as an internal direct product of two proper normal subgroups. Moreover, $|{A_1}^+/({A_1}^+)'|$ and $|Z({A_2}^+)|$ as well as $|{A_2}^+/({A_2}^+)'|$ and $|Z({A_1}^+)|$ are coprime. A consequence of the Krull–Remak–Schmidt theorem, namely [35, Kapitel I, Satz 12.6], implies that $\{{A_1}^+, {A_2}^+\} = \{{A_1}^\circ, {A_2}^\circ\}$. Since $\bar{u} \in {A_1}^+$ and $\bar{u} \notin {A_2}^\circ$, we have ${A_1}^+ = {A_1}^\circ$ and ${A_2}^+ = {A_2}^\circ$.

(8) *The isomorphism* $\varphi : \overline{K} \to H$ *maps* ${A_1}^+$ *to the image of*

$$\left\{ \begin{pmatrix} A & \\ & I_{n-4} \end{pmatrix} : A \in SL_2^{\varepsilon}(q^*) \right\}$$

in H *and* ${A_2}^+$ *to the image of*

$$\left\{ \begin{pmatrix} I_2 & \\ & B \end{pmatrix} : B \in SL_{n-4}^{\varepsilon}(q^*) \right\}$$

in H.

By (4) and (5), we have $i = 2$. So the claim follows from the definitions of ${A_1}^+$ and ${A_2}^+$. □

From now on, ${A_1}^+$ and ${A_2}^+$ will always have the meanings given to them by Lemma 6.4.

Lemma 6.5. *Let* $C := C_G(t)$ *and* $\overline{C} := C/O(C)$. *Then* ${A_1}^+$ *and* ${A_2}^+$ *are normal subgroups of* $C_{\overline{C}}(\bar{u})$.

Proof. We have $C_{\overline{K}}(\bar{u}) \trianglelefteq C_{\overline{C}}(\bar{u})$ as $\overline{K} \trianglelefteq \overline{C}$. Thus, $C_{\overline{K}}(\bar{u})' \trianglelefteq C_{\overline{C}}(\bar{u})$. Having this observed, the lemma is immediate from Lemma 6.4. □

Let $C := C_G(t)$ and $\overline{C} := C/O(C)$. Next, we introduce certain preimages of ${A_1}^+$ and ${A_2}^+$ in $C_C(u)$. By Corollary 2.2, we have $C_{\overline{C}}(\bar{u}) = \overline{C_C(u)}$. We may see from Proposition 2.4 that there is a bijection from the set of 2-components of $C_C(u)$ to the set of 2-components of $C_{\overline{C}}(\bar{u})$ sending each 2-component A of $C_C(u)$ to $\overline{A}$.

Suppose that $q^* \neq 3$. Then ${A_1}^+$ is a component and hence a 2-component of $C_{\overline{C}}(\bar{u})$. We use A_1 to denote the 2-component of $C_C(u)$ corresponding to ${A_1}^+$ under the bijection described above.

Suppose that $q^* \neq 3$ or $n \geq 7$. Then ${A_2}^+$ is a component and hence a 2-component of $C_{\overline{C}}(\bar{u})$. We use A_2 to denote the 2-component of $C_C(u)$ corresponding to ${A_2}^+$ under the bijection described above.

Suppose that $q^* = 3$. By Lemma 6.3 (ii), $O(C_{\overline{C}}(\bar{u})) = 1$. So the factor group $C_C(u)/(C_C(u) \cap O(C))$ is core-free, whence $O(C_C(u)) = C_C(u) \cap O(C)$. Let $O(C_C(u)) \leq A_1 \leq C_C(u)$ such that $A_1/O(C_C(u))$ corresponds to ${A_1}^+$ under the natural group isomorphism $C_C(u)/O(C_C(u)) \to C_{\overline{C}}(\bar{u})$. Furthermore, if $n = 6$, let $O(C_C(u)) \leq A_2 \leq C_C(u)$ such that $A_2/O(C_C(u))$ corresponds to ${A_2}^+$ under the natural group isomorphism $C_C(u)/O(C_C(u)) \to C_{\overline{C}}(\bar{u})$.

Lemma 6.6. *We have $T_1 \leq A_1$ and $T_2 \leq A_2$.*

Proof. Let $i \in \{1,2\}$. Set $C := C_G(t)$ and $\overline{C} := C/O(C)$.

Let $C_C(u) \cap O(C) \leq \widetilde{A_i} \leq C_C(u)$ such that $\widetilde{A_i}/(C_C(u) \cap O(C))$ corresponds to ${A_i}^+$ under the natural group isomorphism $C_C(u)/(C_C(u) \cap O(C)) \to C_{\overline{C}}(\overline{u})$. We have $T_i \leq C_C(u)$ and, by Lemma 6.4, $\overline{T_i} \leq {A_i}^+$. Thus, $T_i \leq \widetilde{A_i}$. If ${A_i}^+ \cong SL_2(3)$, then we have $A_i = \widetilde{A_i}$, and thus, $T_i \leq A_i$. Assume now that ${A_i}^+$ is a component of $C_{\overline{C}}(\overline{u})$. Then A_i is the 2-component of $C_C(u)$ associated to the 2-component $\widetilde{A_i}/(C_C(u) \cap O(C))$ of $C_C(u)/(C_C(u) \cap O(C))$. So, by Proposition 2.4, $A_i = O^{2'}(\widetilde{A_i})$, and hence, $T_i \leq A_i$. □

Lemma 6.7. *There is an element $g \in G$ such that ${T_1}^g = X_2$ and ${X_2}^g = T_1$. For each such $g \in G$, we have $u^g = t$ and $t^g = u$.*

Proof. The first statement easily follows from $\mathcal{F}_S(G) = \mathcal{F}_S(PSL_n(q))$. By Lemma 3.12, the groups T_1 and X_2 are generalized quaternion. So u is the only involution of T_1 and t is the only involution of X_2. Thus, $u^g = t$ and $t^g = u$ for any $g \in G$ with ${T_1}^g = X_2$ and ${X_2}^g = T_1$. □

With the above lemmas at hand, we can now prove the following proposition.

Proposition 6.8. *Take an element $g \in G$ such that ${T_1}^g = X_2$ and ${X_2}^g = T_1$. Set $C := C_G(t)$ and $\overline{C} := C/O(C)$. Let $L := {A_1}^g$. Then the following hold.*

(i) *$L \leq C_C(u)$.*
(ii) *$\overline{L}$ is subnormal in $\overline{C}$ and $\overline{L} \cong SL_2(q^*)$.*
(iii) *The subgroups $\overline{K}$ and $\overline{L}$ are the only subgroups of $\overline{C}$ which are components or solvable 2-components of $\overline{C}$. In particular, $\overline{K}$ and $\overline{L}$ are normal subgroups of $\overline{C}$.*

Proof. By Lemma 6.7, we have $t^g = u$ and $u^g = t$. Hence, $C_C(u)^g = C_C(u)$. As A_1 is a subgroup of $C_C(u)$, we thus have $L = {A_1}^g \leq C_C(u)$. So (i) holds.

Before proving (ii), we show that $C_{\overline{L}}(\overline{K})$ is a normal subgroup of $\overline{L}$ containing $\overline{X_2}$. Since $C_{\overline{C}}(\overline{K}) \trianglelefteq \overline{C}$, we have $C_{\overline{L}}(\overline{K}) = \overline{L} \cap C_{\overline{C}}(\overline{K}) \trianglelefteq \overline{L}$. Because of Lemma 6.6, we have $X_2 = {T_1}^g \leq {A_1}^g = L$. Thus, $\overline{X_2} \leq \overline{L}$. By the definition of X_2 and by Lemma 6.2, we have $\overline{X_2} \leq C_{\overline{C}}(\overline{K})$. Thus, $\overline{X_2} \leq C_{\overline{L}}(\overline{K})$.

Note that $\overline{X_2}$ is generalized quaternion by Lemma 3.12 and in particular nonabelian.

We now prove (ii) for the case $q^* \neq 3$. Then A_1 is a 2-component of $C_C(u)$. As g normalizes $C_C(u)$ and $L = {A_1}^g$, it follows that L is a 2-component of $C_C(u)$. So $\overline{L}$ is a 2-component of $C_{\overline{C}}(\overline{u})$. Moreover, we have $A_1/O(A_1) \cong SL_2(q^*)$ since $A_1/(A_1 \cap O(C)) \cong \overline{A_1} = {A_1}^+ \cong SL_2(q^*)$. Hence, $L/O(L)$ is isomorphic to $SL_2(q^*)$. The group $C_{\overline{L}}(\overline{K})O(\overline{L})/O(\overline{L})$ is normal in $\overline{L}/O(\overline{L})$, and it is nonabelian since $\overline{X_2} \leq C_{\overline{L}}(\overline{K})$. As $\overline{L}/O(\overline{L})$ is quasisimple, it follows that $C_{\overline{L}}(\overline{K})O(\overline{L}) = \overline{L}$. So $C_{\overline{L}}(\overline{K})$ has odd index in $\overline{L}$. Since $\overline{L}$ is a 2-component of $C_{\overline{C}}(\overline{u})$, we have $O^{2'}(\overline{L}) = \overline{L}$. It follows that $\overline{L} = C_{\overline{L}}(\overline{K}) \leq C_{\overline{C}}(\overline{K})$. Since $\overline{L}$ is subnormal in $C_{\overline{C}}(\overline{u})$ and $C_{\overline{C}}(\overline{K}) \leq C_{\overline{C}}(\overline{u})$, we have that $\overline{L}$ is subnormal in $C_{\overline{C}}(\overline{K})$. Hence, $\overline{L}$ is subnormal in $\overline{C}$. As $\overline{C}$ is core-free, we have $O(\overline{L}) = 1$. It follows that $O(L) = L \cap O(C)$ and hence $\overline{L} \cong L/O(L) \cong SL_2(q^*)$. So we have proved (ii) for the case $q^* \neq 3$.

Assume now that $q^* = 3$. Then $O(C_C(u)) = C_C(u) \cap O(C)$, $O(C_C(u)) \leq A_1 \leq C_C(u)$, and $A_1/O(C_C(u))$ corresponds to ${A_1}^+ \cong SL_2(3)$ under the natural isomorphism $C_C(u)/O(C_C(u)) \to C_{\overline{C}}(\overline{u})$. By Lemma 6.5, ${A_1}^+$ is normal in $C_{\overline{C}}(\overline{u})$. Hence, $A_1/O(C_C(u))$ is a normal subgroup of $C_C(u)/O(C_C(u))$ isomorphic to $SL_2(3)$. Since g normalizes $C_C(u)$ and $L = {A_1}^g$, it follows that $O(C_C(u)) \leq L$ and that $L/O(C_C(u))$ is a normal subgroup of $C_C(u)/O(C_C(u))$ isomorphic to $SL_2(3)$. Since $L/O(C_C(u))$ corresponds to $\overline{L}$ under the natural isomorphism $C_C(u)/O(C_C(u)) \to C_{\overline{C}}(\overline{u})$, it follows that $\overline{L}$ is a normal subgroup of $C_{\overline{C}}(\overline{u})$ isomorphic to $SL_2(3)$. Recall that $\overline{X_2} \leq C_{\overline{L}}(\overline{K}) \trianglelefteq \overline{L}$. As $\overline{L}$ has order 24 and $\overline{X_2}$ has order 8, $C_{\overline{L}}(\overline{K})$ either equals $\overline{L}$ or has index 3 in $\overline{L}$. However, if the latter holds, then $\overline{L}C_{\overline{C}}(\overline{K})/C_{\overline{C}}(\overline{K})$ is a normal subgroup of $C_{\overline{C}}(\overline{u})/C_{\overline{C}}(\overline{K})$ of order 3, which is a contradiction to Lemma 6.3 (iii). Thus, $\overline{L} = C_{\overline{L}}(\overline{K}) \leq C_{\overline{C}}(\overline{K})$. As $\overline{L} \trianglelefteq C_{\overline{C}}(\overline{u})$ and $C_{\overline{C}}(\overline{K}) \leq C_{\overline{C}}(\overline{u})$, it follows that $\overline{L}$ is normal in $C_{\overline{C}}(\overline{K})$ and hence subnormal in $\overline{C}$. So we have proved (ii) for the case $q^* = 3$.

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We now prove (iii). We have $\overline{T} \cap \overline{K} = \overline{X_1}$ since $\overline{X_1} \leq \overline{T} \cap \overline{K}$ and $\overline{X_1} \in \mathrm{Syl}_2(\overline{K})$. Also, $\overline{T} \cap \overline{L} = \overline{X_2}$ since $|\overline{X_2}| = |SL_2(q)|_2 = |SL_2(q^*)|_2 = |\overline{L}|_2$ and $\overline{X_2} \leq \overline{L}$. As a consequence of Lemma 5.4, the fusion system $\mathcal{F}_{\overline{T}}(\overline{C})/(\overline{X_1}\,\overline{X_2})$ is nilpotent. Applying Lemma 2.18, we may conclude that $\overline{K}$ and $\overline{L}$ are the only subgroups of $\overline{C}$ which are components or solvable 2-components of $\overline{C}$. As $\overline{K}$ and $\overline{L}$ are not isomorphic, both are characteristic and hence normal in $\overline{C}$. □

Let $E = \langle u, t \rangle$. By construction, g acts on E and $A_1 \trianglelefteq C_G(E)$. Hence, the definition of L in Proposition 6.8 is independent of the choice of g. From now on, L will always have the meaning given to it by the above proposition.

6.2. 2-components of centralizers of involutions conjugate to t_i, $i \neq 2$

Having described the components and the solvable 2-components of the group $C_G(t)/O(C_G(t))$, we now turn our attention to centralizers of involutions of G not conjugate to t.

First, we recall some notation from Section 5. Let $1 \leq i < n$. If i is even, then t_i denotes the image of

$$\begin{pmatrix} I_{n-i} & \\ & -I_i \end{pmatrix}$$

in $PSL_n(q)$. We use ρ to denote an element of $\mathbb{F}_q^*$ with order $(n, q-1)$, and if ρ is a square in $\mathbb{F}_q$, then μ denotes an element of $\mathbb{F}_q^*$ with $\mu^2 = \rho$. If n is even, ρ is a square in $\mathbb{F}_q$ and i is odd, then t_i is defined to be the image of

$$\begin{pmatrix} \mu I_{n-i} & \\ & -\mu I_i \end{pmatrix} \in SL_n(q)$$

in $PSL_n(q)$. It is easy to note that t_i lies in T and hence in S whenever t_i is defined.

Let $\mathcal{S}$ denote the set of all subgroups E of $PSL_n(q)$ such that there is some elementary abelian 2-subgroup $\widetilde{E} \leq SL_n(q)$ with $E = \widetilde{E}Z(SL_n(q))/Z(SL_n(q))$. For each $3 \leq i \leq n$, we define $\mathcal{S}_i$ to be the set of all elements E of $\mathcal{S}$ such that E contains a $PSL_n(q)$-conjugate of t_j for some even $2 \leq j < i$.

Lemma 6.9. *Let $1 \leq i < n$ such that t_i is defined. Assume that $i \neq 2$ and that $i \leq \frac{n}{2}$ if n is even. Let P be a Sylow 2-subgroup of $C_{PSL_n(q)}(t_i)$ and $\mathcal{F} := \mathcal{F}_P(C_{PSL_n(q)}(t_i))$. Then the following hold.*

(i) *Assume that $i \notin \{1, n-1\}$. Then $\mathcal{F}$ has precisely two components. Denoting them in a suitable way by $\mathcal{E}_1$ and $\mathcal{E}_2$, the following hold.*
   (a) *$\mathcal{E}_1$ is isomorphic to the 2-fusion system of $SL_{n-i}(q)$.*
   (b) *$\mathcal{E}_2$ is isomorphic to the 2-fusion system of $SL_i(q)$.*
   (c) *Let Y_1 be the Sylow group of $\mathcal{E}_1$, and let Y_2 be the Sylow group of $\mathcal{E}_2$. Then Y_1Y_2 is strongly $\mathcal{F}$-closed and $\mathcal{F}/Y_1Y_2$ is nilpotent. The group Y_i, where $i \in \{1, 2\}$, contains a $PSL_n(q)$-conjugate of t. Moreover, any elementary abelian subgroup of Y_1 of rank at least 2 is contained in $\mathcal{S}_{n-i}$, and any elementary abelian subgroup of Y_2 of rank at least 2 is contained in $\mathcal{S}_i$.*

(ii) *Assume that $i = 1$ or $i = n-1$. Then $\mathcal{F}$ has a unique component. This component is isomorphic to the 2-fusion system of $SL_{n-1}(q)$. If Y is its Sylow group, then Y is strongly $\mathcal{F}$-closed and $\mathcal{F}/Y$ is nilpotent. Moreover, any elementary abelian subgroup of Y of rank at least 2 is contained in $\mathcal{S}_{n-1}$.*

Proof. Assume that $i \notin \{1, n-1\}$. By hypothesis, we have $i \neq 2$, and $i \leq \frac{n}{2}$ if n is even. It follows that $i \geq 3$ and $n - i \geq 3$. Let J_1 be the image of

$$\left\{ \begin{pmatrix} A & \\ & I_i \end{pmatrix} : A \in SL_{n-i}(q) \right\}$$

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in $PSL_n(q)$, and let J_2 be the image of

$$\left\{\begin{pmatrix} I_{n-i} & \\ & A \end{pmatrix} \ : \ A \in SL_i(q)\right\}$$

in $PSL_n(q)$. Then J_1 and J_2 are the only 2-components of $C_{PSL_n(q)}(t_i)$. Applying Proposition 2.17 and Lemma 3.21, we may conclude that $\mathcal{E}_1 := \mathcal{F}_{P\cap J_1}(J_1)$ and $\mathcal{E}_2 := \mathcal{F}_{P\cap J_2}(J_2)$ are the only components of $\mathcal{F} = \mathcal{F}_P(C_{PSL_n(q)}(t_i))$. By definition, $\mathcal{E}_1$ is isomorphic to the 2-fusion system of $SL_{n-i}(q)$, while $\mathcal{E}_2$ is isomorphic to the 2-fusion system of $SL_i(q)$. Set $Y_1 := P \cap J_1$ and $Y_2 := P \cap J_2$. Since $Y_1Y_2 \leq P \cap J_1J_2$ and since both Y_1Y_2 and $P \cap J_1J_2$ are Sylow 2-subgroups of J_1J_2, we have $Y_1Y_2 = P \cap J_1J_2$. As $J_1J_2 \trianglelefteq C_{PSL_n(q)}(t_i)$, we have that Y_1Y_2 is strongly $\mathcal{F}$-closed. By Lemma 2.11, $\mathcal{F}/Y_1Y_2$ is isomorphic to the 2-fusion system of $C_{PSL_n(q)}(t_i)/J_1J_2$. Since $C_{PSL_n(q)}(t_i)/J_1J_2$ is 2-nilpotent, it follows from [39, Theorem 1.4] that $\mathcal{F}/Y_1Y_2$ is nilpotent. As $i \geq 3 \leq n-i$, both J_1 and J_2 contain a $PSL_n(q)$-conjugate of t. Hence, Y_k has an element which is $PSL_n(q)$-conjugate to t for $k \in \{1,2\}$. For any elementary abelian 2-subgroup E of J_k, $k \in \{1,2\}$, $E \cap Z(SL_n(q)) = 1$, so E lies in $\mathcal{S}$. Moreover, any noncentral involution of J_1 is $PSL_n(q)$-conjugate to t_j for some even $2 \leq j < n-i$, and any noncentral involution of J_2 is $PSL_n(q)$-conjugate to t_j for some even $2 \leq j < i$. This implies that any elementary abelian subgroup of Y_1 of rank at least 2 is contained in $\mathcal{S}_{n-i}$ and that any elementary abelian subgroup of Y_2 of rank at least 2 is contained in $\mathcal{S}_i$. This completes the proof of (i).

We omit the proof of (ii) since it is very similar to the one of (i). □

Proposition 6.10. *Let $1 \leq i < n$ such that t_i is defined. Assume that $i \notin \{1, 2, n-1\}$ and that $i \leq \frac{n}{2}$ if n is even. Let x be an involution of S which is G-conjugate to t_i. Then $C_G(x)$ has precisely two 2-components. Denoting them in a suitable way by J_1 and J_2, the following hold.*

(i) *$J_1/O(J_1)$ is isomorphic to $SL^{\varepsilon}_{n-i}(q^*)/O(SL^{\varepsilon}_{n-i}(q^*))$, where ε and q^* are as in Proposition 6.1.*
(ii) *$J_2/O(J_2) \cong SL^{\varepsilon}_i(q^*)/O(SL^{\varepsilon}_i(q^*))$, where ε and q^* are as in Proposition 6.1.*
(iii) *Any elementary abelian 2-subgroup of J_1 of rank at least 2 is G-conjugate to a subgroup of S lying in $\mathcal{S}_{n-i}$, and any elementary abelian 2-subgroup of J_2 of rank at least 2 is G-conjugate to a subgroup of S lying in $\mathcal{S}_i$.*

Proof. Let $\mathcal{F} := \mathcal{F}_S(G) = \mathcal{F}_S(PSL_n(q))$. It suffices to prove the proposition under the assumption that $\langle x\rangle$ is fully $\mathcal{F}$-centralized, and we will assume that this is the case. So we have $C_S(x) \in \mathrm{Syl}_2(C_G(x))$ and $C_S(x) \in \mathrm{Syl}_2(C_{PSL_n(q)}(x))$. Also, $\mathcal{F}_{C_S(x)}(C_G(x)) = C_{\mathcal{F}}(\langle x\rangle) = \mathcal{F}_{C_S(x)}(C_{PSL_n(q)}(x))$.

As x is G-conjugate to t_i, we have that x is $PSL_n(q)$-conjugate to t_i. So Lemma 6.9 (i) shows together with Lemma 5.10 (i) that there exist two distinct 2-components J_1 and J_2 of $C_G(x)$ satisfying the following conditions, where $Y_1 := C_S(x) \cap J_1$ and $Y_2 := C_S(x) \cap J_2$.

(1) $\mathcal{F}_{Y_1}(J_1)$ is isomorphic to the 2-fusion system of $SL_{n-i}(q)$.
(2) $\mathcal{F}_{Y_2}(J_2)$ is isomorphic to the 2-fusion system of $SL_i(q)$.
(3) Y_1Y_2 is strongly closed in $C_S(x)$ with respect to $C_{\mathcal{F}}(\langle x\rangle)$, and $C_{\mathcal{F}}(\langle x\rangle)/Y_1Y_2$ is nilpotent.
(4) For $k \in \{1,2\}$, Y_k contains a G-conjugate of t.
(5) Any elementary abelian subgroup of Y_1 of rank at least 2 lies in $\mathcal{S}_{n-i}$, and any elementary abelian subgroup of Y_2 of rank at least 2 lies in $\mathcal{S}_i$.

By (3) and Corollary 2.19, J_1 and J_2 are the only 2-components of $C_G(x)$. It remains to show that J_1 and J_2 satisfy (i)-(iii). As $Y_k \in \mathrm{Syl}_2(J_k)$ for $k \in \{1,2\}$, (5) implies (iii).

We now prove (ii). The proof of (i) will be omitted since it is very similar to the proof of (ii).

Let s be an element of J_1 which is G-conjugate to t. Set $C := C_G(s)$, $\widehat{C} := C/O(C)$ and $\overline{C_G(x)} := C_G(x)/O(C_G(x))$.

Since $\overline{J_1}$ and $\overline{J_2}$ are distinct components of $\overline{C_G(x)}$, we have $[\overline{J_1}, \overline{J_2}] = 1$ by [37, 6.5.3]. As $\bar{s} \in \overline{J_1}$, it follows that $\overline{J_2}$ is a component of $C_{\overline{C_G(x)}}(\bar{s})$. As a consequence of Corollary 2.2 and Proposition 2.4, $C_G(x) \cap C$ has a 2-component H with $\overline{H} = \overline{J_2}$.

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By assumption, s is G-conjugate to t. So, by Proposition 6.8, $\widehat{C}$ has a unique normal subgroup K^+ isomorphic to $SL^{\varepsilon}_{n-2}(q^*)/O(SL^{\varepsilon}_{n-2}(q^*))$ and a unique normal subgroup L^+ isomorphic to $SL_2(q^*)$. Moreover, K^+ and L^+ are the only subgroups of $\widehat{C}$ which are components or solvable 2-components of $\widehat{C}$.

Clearly, $\widehat{H}$ is a 2-component of $C_{\widehat{C}}(\widehat{x})$. Lemma 2.5 implies that $\widehat{H}$ is a 2-component of $C_{K^+}(\widehat{x})$ or of $C_{L^+}(\widehat{x})$. By Corollary 3.47 (i), we even have that $\widehat{H}$ is a component of $C_{K^+}(\widehat{x})$ or $C_{L^+}(\widehat{x})$. It is easy to note that $\widehat{H}/Z(\widehat{H}) \cong H/Z^*(H) \cong \overline{J_2}/Z(\overline{J_2})$. By Corollary 3.47 (ii), we have $\widehat{H}/Z(\widehat{H}) \ncong M_{11}$, and so $\overline{J_2}/Z(\overline{J_2}) \ncong M_{11}$. Now (2) and Lemma 5.10 (ii) imply that $\overline{J_2} \cong SL_i^{\varepsilon_0}(q_0)/O(SL_i^{\varepsilon_0}(q_0))$ for some nontrivial odd prime power q_0 and some $\varepsilon_0 \in \{+,-\}$ with $q \sim \varepsilon_0 q_0$. Hence, $\widehat{H}/Z(\widehat{H}) \cong \overline{J_2}/Z(\overline{J_2}) \cong PSL_i^{\varepsilon_0}(q_0)$. Note that $\varepsilon q^* \sim q \sim \varepsilon_0 q_0$ and in particular $({q^*}^2-1)_2 = ({q_0}^2-1)_2$. Applying Corollary 3.47 (iii), we may conclude that $q_0 = q^*$ and $\varepsilon_0 = \varepsilon$. Consequently, we have $J_2/O(J_2) \cong SL_i^{\varepsilon}(q^*)/O(SL_i^{\varepsilon}(q^*))$. So we have proved (ii). □

The proof of the following proposition runs along the same lines as that of the previous result.

Proposition 6.11. *Suppose that n is odd and $i = n-1$ or that n is even, $i = 1$ and t_1 is defined. Let x be an involution of S which is G-conjugate to t_i. Then $C_G(x)$ has precisely one 2-component J. We have $J/O(J) \cong SL^{\varepsilon}_{n-1}(q^*)/O(SL^{\varepsilon}_{n-1}(q^*))$, where ε and q^* are as in Proposition 6.1. Moreover, any elementary abelian 2-subgroup of J of rank at least 2 is G-conjugate to a subgroup of S lying in $\mathcal{S}_{n-1}$.*

Proof. Let $\mathcal{F} := \mathcal{F}_S(G) = \mathcal{F}_S(PSL_n(q))$. It suffices to prove the proposition under the assumption that $\langle x\rangle$ is fully $\mathcal{F}$-centralized, and we will assume that this is the case. So we have $C_S(x) \in \mathrm{Syl}_2(C_G(x))$ and $C_S(x) \in \mathrm{Syl}_2(C_{PSL_n(q)}(x))$. Also, $\mathcal{F}_{C_S(x)}(C_G(x)) = C_{\mathcal{F}}(\langle x\rangle) = \mathcal{F}_{C_S(x)}(C_{PSL_n(q)}(x))$.

As x is G-conjugate to t_i, we have that x is $PSL_n(q)$-conjugate to t_i. Lemma 6.9 (ii) implies that $C_{\mathcal{F}}(\langle x\rangle)$ has a unique component $\mathcal{E}$ and that $\mathcal{E}$ is isomorphic to the 2-fusion system of $SL_{n-1}(q)$. Applying Lemma 5.10 (i), we may conclude that $C_G(x)$ has a unique 2-component J with $\mathcal{E} = \mathcal{F}_{C_S(x)\cap J}(J)$. By Lemma 5.10 (ii), $J/O(J) \cong SL^{\varepsilon_0}_{n-1}(q_0)/O(SL^{\varepsilon_0}_{n-1}(q_0))$ for some nontrivial odd prime power q_0 and some $\varepsilon_0 \in \{+,-\}$ with $\varepsilon_0 q_0 \sim q$.

Set $Y := C_S(x) \cap J$. By Lemma 6.9 (ii), Y is strongly closed in $C_S(x)$ with respect to $C_{\mathcal{F}}(\langle x\rangle)$ and $C_{\mathcal{F}}(\langle x\rangle)/Y$ is nilpotent. Applying Corollary 2.19, we may conclude that J is the only 2-component of $C_G(x)$. Using Lemma 6.9 (ii), we see that any elementary abelian subgroup of Y of rank at least 2 lies in $\mathcal{S}_{n-1}$. As $Y \in \mathrm{Syl}_2(J)$, it follows that any elementary abelian 2-subgroup of J of rank at least 2 is G-conjugate to a subgroup of S lying in $\mathcal{S}_{n-1}$.

It remains to show that $\varepsilon_0 = \varepsilon$ and $q_0 = q^*$. Define $s := t_i$ if $i = 1$ and $s := t_A$, where $A := \{1,\dots,n-1\}$, if $i = n-1$. Then we have $s \in C_G(t)$, and s is G-conjugate to x. Set $\overline{C_G(t)} := C_G(t)/O(C_G(t))$. Lemma 6.2 shows that $\bar{s}$ centralizes $\overline{K}$. Hence, $\overline{K}$ is a component of $C_{\overline{C_G(t)}}(\bar{s})$. As a consequence of Corollary 2.2 and Proposition 2.4, $C_G(t) \cap C_G(s)$ has a 2-component H with $\overline{H} = \overline{K}$. Set $C := C_G(s)$ and $\widehat{C} := C/O(C)$. Then $\widehat{H}$ is a 2-component of $C_{\widehat{C}}(\widehat{t})$. Since s is G-conjugate to x, $\widehat{C}$ has precisely one component J^+, and J^+ is isomorphic to $SL^{\varepsilon_0}_{n-1}(q_0)/O(SL^{\varepsilon_0}_{n-1}(q_0))$. By Lemma 2.5, $\widehat{H}$ is a 2-component of $C_{J^+}(\widehat{t})$. We see from Corollary 3.47 (i) that $\widehat{H}$ is in fact a component of $C_{J^+}(\widehat{t})$. It is easy to see that $\widehat{H}/Z(\widehat{H}) \cong H/Z^*(H) \cong \overline{K}/Z(\overline{K}) \cong PSL^{\varepsilon}_{n-2}(q^*)$. Note that $\varepsilon_0 q_0 \sim q \sim \varepsilon q^*$ and in particular $({q_0}^2-1)_2 = ({q^*}^2-1)_2$. Using this, we may deduce from Corollary 3.47 (iii) that $q_0 = q^*$ and $\varepsilon_0 = \varepsilon$. □

6.3. 2-components of centralizers of involutions conjugate to w

Recall that we assume ρ to be an element of $\mathbb{F}_q^*$ with order $(n, q-1)$. Recall moreover that if n is even and ρ is a nonsquare element of $\mathbb{F}_q$, then $\widetilde{w}$ denotes the matrix

$$\begin{pmatrix} & I_{n/2} \\ \rho I_{n/2} & \end{pmatrix}$$

and, if $\widetilde{w} \in SL_n(q)$, then w denotes its image in $PSL_n(q)$.

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Lemma 6.12. *Suppose that w is defined. Let P be a Sylow 2-subgroup of $C_{PSL_n(q)}(w)$, and let $\mathcal{F}$ denote the fusion system $\mathcal{F}_P(C_{PSL_n(q)})(w))$. Then $\mathcal{F}$ has precisely one component. This component is isomorphic to the 2-fusion system of a nontrivial quotient of $SL_{\frac{n}{2}}(q^2)$. If Y is its Sylow subgroup, then Y is strongly $\mathcal{F}$-closed, and $\mathcal{F}/Y$ is nilpotent.*

Proof. By Lemma 3.6 (i), $C_{PSL_n(q)}(w)$ has precisely one 2-component J, and J is isomorphic to a nontrivial quotient of $SL_{\frac{n}{2}}(q^2)$. Applying Proposition 2.17 and Lemma 3.21, we may conclude that $\mathcal{F}_{P\cap J}(J)$ is the only component of $\mathcal{F}$. The last statement of the lemma is given by Lemma 3.6 (ii). □

Proposition 6.13. *Suppose that w is defined. Let x be an involution of S which is $PSL_n(q)$-conjugate to w. Then $C_G(x)$ has precisely one 2-component, say J. The group $J/O(J)$ is isomorphic to a nontrivial quotient of $SL^{\varepsilon_0}_{\frac{n}{2}}(q_0)$ for some nontrivial odd prime power q_0 and some $\varepsilon_0 \in \{+,-\}$ with $q^2 \sim \varepsilon_0 q_0$.*

Proof. Let $\mathcal{F} := \mathcal{F}_S(G) = \mathcal{F}_S(PSL_n(q))$. It suffices to prove the proposition under the assumption that $\langle x\rangle$ is fully $\mathcal{F}$-centralized, and we will assume that this is the case. So we have $C_S(x) \in \mathrm{Syl}_2(C_G(x))$ and $C_S(x) \in \mathrm{Syl}_2(C_{PSL_n(q)}(x))$. Also, $\mathcal{F}_{C_S(x)}(C_G(x)) = C_{\mathcal{F}}(\langle x\rangle) = \mathcal{F}_{C_S(x)}(C_{PSL_n(q)}(x))$.

As x is $PSL_n(q)$-conjugate to w, Lemma 6.12 implies that $C_{\mathcal{F}}(\langle x\rangle)$ has precisely one component, say $\mathcal{E}$, and that $\mathcal{E}$ is isomorphic to the 2-fusion system of a nontrivial quotient of $SL_{\frac{n}{2}}(q^2)$. By Lemma 5.10 (i), $C_G(x)$ has a unique 2-component J such that $\mathcal{E} = \mathcal{F}_{C_S(x)\cap J}(J)$. Set $Y := C_S(x) \cap J$. As a consequence of Lemma 6.12, Y is strongly closed in $C_S(x)$ with respect to $C_{\mathcal{F}}(\langle x\rangle)$, and the factor system $C_{\mathcal{F}}(\langle x\rangle)/Y$ is nilpotent. So, by Corollary 2.19, J is the only 2-component of $C_G(x)$. Lemma 5.10 (iii) shows that $J/O(J)$ is isomorphic to a nontrivial quotient of $SL^{\varepsilon_0}_{\frac{n}{2}}(q_0)$ for some nontrivial odd prime power q_0 and some $\varepsilon_0 \in \{+,-\}$ with $q^2 \sim \varepsilon_0 q_0$. □

7. The components of $C_G(t)$

The goal of this section is to determine the isomorphism types of K and L. In order to do so, we will apply the signalizer functor techniques introduced by Gorenstein and Walter in [31]. In particular, we will see that L is isomorphic to $SL_2(q^*)$. This will enable us in Section 8 to prove that a certain collection of conjugates of L generates a subgroup G_0 of G which is isomorphic to a nontrivial quotient of $SL^{\varepsilon}_n(q^*)$ and normal in G. This will complete the proof of Theorem 5.2.

7.1. 3-generation of involution centralizers

For each $3 \leq i \leq n$, we define $\mathcal{U}_i$ to be the set of all subgroups U of $PSL_n(q)$ such that U has a subgroup E with $E \in \mathcal{S}_i$ and $m(E) \geq 3$. The following lemma will be important later in this section.

Lemma 7.1. *Let $1 \leq i < n$ such that t_i is defined. Suppose that $i \leq \frac{n}{2}$ if n is even. Let x be an involution of S such that x is G-conjugate to t_i and such that $\langle x\rangle$ is fully $\mathcal{F}_S(G)$-centralized. Then $C_G(x)$ is 3-generated in the sense of Definition 3.36. Moreover, if $i \geq 4$, then we have*

$$C_G(x) = \langle N_{C_G(x)}(U) \mid U \leq C_S(x), U \in \mathcal{U}_i\rangle.$$

If $i = 2$, then we have

$$C_G(x) = \langle N_{C_G(x)}(U) \mid U \leq C_S(x), U \in \mathcal{U}_{n-2}\rangle.$$

Proof. Set $C := C_G(x)$ and $\overline{C} := C/O(C)$. Recall that $L_{2'}(C)$ denotes the subgroup of C generated by the 2-components of C and that $E(\overline{C})$ denotes the product of all components of $\overline{C}$. As a consequence of Proposition 2.4, $\overline{L_{2'}(C)} = E(\overline{C})$.

First, we consider the case $(n,i) \neq (6,3)$. Then, by Propositions 6.1, 6.10 and 6.11, C has a 2-component J such that $\overline{J} \cong SL^{\varepsilon}_k(q^*)/O(SL^{\varepsilon}_k(q^*))$ for some $k \geq 4$ and such that any elementary

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abelian subgroup of $Y := C_S(x) \cap J$ of rank at least 2 lies in $\mathcal{S}_k$. If $i \geq 4$, then we may assume that $k = i$, and if $i = 2$, then $k = n - 2$.

We have $Y \in \mathrm{Syl}_2(J)$ since $C_S(x) \in \mathrm{Syl}_2(C)$ and J is subnormal in C. By Lemma 3.38, we have that $\bar{J}$ is 3-generated. So we have

$$\bar{J} = \langle N_{\bar{J}}(\bar{U}) \mid U \leq Y, m(U) \geq 3 \rangle.$$

Set $X := C_S(x) \cap L_{2'}(C)$. By the Frattini argument, $E(\bar{C}) = \bar{J} N_{E(\bar{C})}(\bar{Y})$ and $\bar{C} = E(\bar{C}) N_{\bar{C}}(\bar{X})$. It follows that

$$\bar{C} = \langle N_{\bar{C}}(\bar{U}) \mid U = X, \text{ or } U \leq Y \text{ and } m(U) \geq 3 \rangle.$$

Lemma 2.1 implies that C is generated by $O(C)$ together with the normalizers $N_C(U)$, where $U = X$, or $U \leq Y$ and $m(U) \geq 3$.

Let E denote the subgroup of S generated by t, $t_{\{n-2,n-1\}}$, $t_{\{n-3,n-2\}}$ and $t_{\{n-4,n-3\}}$. Then $E \cong E_{16}$. Since x is G-conjugate to t_i and $E \leq C_G(t_i)$, there is a subgroup E_x of $C_S(x)$ which is G-conjugate to E. By [27, Proposition 11.23], we have

$$O(C) = \langle C_{O(C)}(D) \mid D \leq E_x, D \cong E_8 \rangle.$$

As remarked above, any elementary abelian subgroup of Y of rank at least 2 lies in $\mathcal{S}_k$. So, if $U \leq Y$ and $m(U) \geq 3$, then $U \in \mathcal{U}_k$. Also, $X \in \mathcal{U}_k$. Clearly, any E_8-subgroup of E_x lies in $\mathcal{S}_k$ and hence in $\mathcal{U}_k$. We therefore have

$$C = \langle N_C(U) \mid U \leq C_S(x), U \in \mathcal{U}_k \rangle.$$

Consequently, C is 3-generated, and the last two statements of the lemma are satisfied.

Suppose now that $(n, i) = (6, 3)$. By Proposition 6.10, C has precisely two 2-components J_1 and J_2, and we have $\overline{J_1} \cong PSL_3^{\varepsilon}(q^*) \cong \overline{J_2}$. Set $Y_1 := C_S(x) \cap J_1$ and $Y_2 := C_S(x) \cap J_2$. Since $\overline{J_1}$ is 2-generated by Lemma 3.37, we have

$$\overline{J_1} = \langle N_{\overline{J_1}}(\bar{U}) \mid U \leq Y_1, m(U) \geq 2 \rangle.$$

Let y be an involution of Y_2. We have $[\overline{J_1}, \overline{J_2}] = 1$ by [37, 6.5.3], and so $\bar{y}$ centralizes $\overline{J_1}$. As $Z(\overline{J_1}) = 1$, we have $\bar{y} \notin \overline{J_1}$. Now let $U \leq Y_1$ with $m(U) \geq 2$. Then $\langle \bar{U}, \bar{y} \rangle$ has rank at least 3. Moreover, $N_{\overline{J_1}}(\bar{U})$ normalizes $\langle \bar{U}, \bar{y} \rangle$ as $\overline{J_1}$ centralizes $\bar{y}$. Thus,

$$\overline{J_1} = \langle N_{\overline{J_1}}(\bar{U}) \mid U \leq Y_1 Y_2, m(U) \geq 3 \rangle.$$

Interchanging the roles of J_1 and J_2, we also see that

$$\overline{J_2} = \langle N_{\overline{J_2}}(\bar{U}) \mid U \leq Y_1 Y_2, m(U) \geq 3 \rangle.$$

By the Frattini argument, $\bar{C} = \overline{J_1}\,\overline{J_2} N_{\bar{C}}(\overline{Y_1 Y_2})$. Lemma 2.1 implies that C is generated by $O(C)$ together with the normalizers $N_C(U)$, where $U \leq Y_1 Y_2$ and $m(U) \geq 3$. For any E_{16}-subgroup A of $C_S(x)$, we have

$$O(C) = \langle C_{O(C)}(B) \mid B \leq A, B \cong E_8 \rangle.$$

by [27, Proposition 11.23]. It follows that C is 3-generated. The proof is now complete. □

Lemma 7.2. *Suppose that w is defined. Let x be an involution of S which is $PSL_n(q)$-conjugate to w. Then $C_G(x)$ is 3-generated.*

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Proof. Set $C := C_G(x)$ and $\overline{C} := C/O(C)$. By Proposition 6.13, C has a unique 2-component J, and $\overline{J}$ is isomorphic to a nontrivial quotient of $SL^{\varepsilon_0}_{\frac{n}{2}}(q_0)$ for some nontrivial odd prime power q_0 and some $\varepsilon_0 \in \{+,-\}$ with $q^2 \sim \varepsilon_0 q_0$. Note that $q_0 \equiv \varepsilon_0 \bmod 8$.

First, we prove that $\overline{C}$ is 3-generated. Let R be a Sylow 2-subgroup of C and $Y := R \cap J$. We consider two cases.

Case 1: $n \geq 8$.

As $q_0 \equiv \varepsilon_0 \bmod 8$, by Lemma 3.38, $\overline{J}$ is 3-generated. Hence,

$$\overline{J} = \langle N_{\overline{J}}(\overline{U}) \mid U \leq Y, m(U) \geq 3 \rangle.$$

By the Frattini argument, $\overline{C} = \overline{J} N_{\overline{C}}(\overline{Y})$. So $\overline{C}$ is 3-generated.

Case 2: $n = 6$.

We have $\overline{J} \cong PSL^{\varepsilon_0}_3(q_0)$. By Lemma 3.37, $\overline{J}$ is 2-generated. Applying the Frattini argument, we may conclude that

$$\overline{C} = \langle N_{\overline{C}}(\overline{U}) \mid U \leq Y, m(U) \geq 2 \rangle.$$

Now let $U \leq Y$ with $m(U) \geq 2$. Since $\bar{x}$ is a central involution of $\overline{C}$ and $Z(\overline{J})$ is trivial, we have $\bar{x} \notin \overline{J}$ and hence $\bar{x} \notin \overline{U}$. It follows $\langle \overline{U}, \bar{x} \rangle$ has rank at least 3. Moreover, as $\bar{x}$ is central in $\overline{C}$, we have $N_{\overline{C}}(\overline{U}) \leq N_{\overline{C}}(\langle \overline{U}, \bar{x} \rangle)$. Clearly, $\langle \overline{U}, \bar{x} \rangle \leq \overline{R}$. It follows that

$$\overline{C} = \langle N_{\overline{C}}(\overline{U}) \mid U \leq R, m(U) \geq 3 \rangle.$$

Hence, $\overline{C}$ is 3-generated.

Applying Lemma 2.1, we may conclude that C is generated by $O(C)$ together with the normalizers $N_C(U)$, where $U \leq R$ and $m(U) \geq 3$. By Lemma 3.6 (iii), R has an elementary abelian 2-subgroup of rank 4, say A. By [27, Proposition 11.23], we have

$$O(C) = \langle C_{O(C)}(B) \mid B \leq A, B \cong E_8 \rangle.$$

So C is 3-generated. □

Corollary 7.3. *Let x be an involution of S. Then $C_G(x)$ is 3-generated.*

Proof. As a consequence of Proposition 3.5, x is G-conjugate to t_i for some $1 \leq i < n$ such that t_i is defined or $PSL_n(q)$-conjugate to w (if defined). So the statement follows from Lemmas 7.1 and 7.2. □

7.2. *The case* $q^* = 3$

Recall that our goal is to determine the isomorphism types of K and L. First, we will deal with the case $q^* = 3$. We will prove that, in this case, $O(C_G(t)) = 1$.

Lemma 7.4. *Let x be an involution of S, and let J be a 2-component of $C_G(x)$. Let $1 \leq i < n$ such that t_i is defined. Suppose that $q^* = 3$ and that x is G-conjugate to t_i. Then $J/O(J)$ is locally balanced.*

Proof. By Propositions 6.8 (iii), 6.10 and 6.11, we have $J/O(J) \cong SL^{\varepsilon}_k(3)$ for some $3 \leq k < n$. So $J/O(J)$ is locally balanced by Lemma 3.48. □

Lemma 7.5. *Let P and Q be subgroups of S.*

(i) *If $P \in \mathcal{S}$ and $m(P) \leq 2$, then there is a subgroup $\overline{P}$ of S such that $P < \overline{P}$, $\overline{P} \in \mathcal{S}$ and $m(\overline{P}) = 3$.*
(ii) *If P and Q are elements of $\mathcal{S}$ of rank at least 3, then there exist some $m \geq 1$ and a sequence*

$$P = P_1, \ldots, P_m = Q,$$

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where P_i, $1 \leq i \leq m$, is a subgroup of S of rank at least 2 lying in $\mathcal{S}$ and where

$$P_i \subseteq P_{i+1} \text{ or } P_{i+1} \subseteq P_i$$

for all $1 \leq i < m$.

Proof. Suppose that $P \in \mathcal{S}$ and $m(P) \leq 2$. Let $\widetilde{S}$ be a Sylow 2-subgroup of $SL_n(q)$ such that S is the image of $\widetilde{S}$ in $PSL_n(q)$. Note that $\widetilde{S}$ is unique. Since P is an element of $\mathcal{S}$, there exists some elementary abelian 2-subgroup $\widetilde{P}$ of $SL_n(q)$ such that P is the image of $\widetilde{P}$ in $PSL_n(q)$. Clearly, $\widetilde{P} \leq \widetilde{S}$. We have $m(\widetilde{P}) \leq 3$ as $m(P) \leq 2$. By Corollary 3.35, $\widetilde{P}$ is contained in an E_{16}-subgroup of $\widetilde{S}$. This implies (i).

We now prove (ii). Suppose that P and Q are elements of $\mathcal{S}$ of rank at least 3. There are elementary abelian subgroups $\widetilde{P}$ and $\widetilde{Q}$ of $SL_n(q)$ such that P is the image of $\widetilde{P}$ in $PSL_n(q)$ and such that Q is the image of $\widetilde{Q}$ in $PSL_n(q)$. Clearly, $\widetilde{P}, \widetilde{Q} \leq \widetilde{S}$. Also, $m(\widetilde{P}), m(\widetilde{Q}) \geq 3$. Since $\widetilde{S}$ is 3-connected by Corollary 3.34, there exist some $m \geq 1$ and a sequence

$$\widetilde{P} = \widetilde{P}_1, \ldots, \widetilde{P}_n = \widetilde{Q},$$

where $\widetilde{P}_i$ $(1 \leq i \leq m)$ is an elementary abelian subgroup of $\widetilde{S}$ of rank at least 3 and where

$$\widetilde{P}_i \subseteq \widetilde{P}_{i+1} \text{ or } \widetilde{P}_{i+1} \subseteq \widetilde{P}_i$$

for all $1 \leq i < m$. Let P_i, $1 \leq i \leq m$, denote the image of $\widetilde{P}_i$ in S. Then the sequence

$$P = P_1, \ldots, P_m = Q$$

has the desired properties. □

Lemma 7.6. *Suppose that $q^* = 3$. For each elementary abelian subgroup E of S of rank at least 2, let*

$$W_E := \langle O(C_G(x)) \mid x \in E^{\#} \rangle.$$

Let P and Q be subgroups of S with $P, Q \in \mathcal{S}$ and $m(P), m(Q) \geq 3$. Then $W_P = W_Q$.

Proof. By Lemma 7.5 (ii), there exist some $m \geq 1$ and a sequence

$$P = P_1, \ldots, P_m = Q,$$

where P_i, $1 \leq i \leq m$, is a subgroup of S of rank at least 2 lying in $\mathcal{S}$ and where

$$P_i \subseteq P_{i+1} \text{ or } P_{i+1} \subseteq P_i$$

for all $1 \leq i < m$. By Lemma 7.5 (i), there is a subgroup $\overline{P_i}$ of S such that $\overline{P_i} \in \mathcal{S}$, $m(\overline{P_i}) \geq 3$ and $P_i \leq \overline{P_i}$ for each $1 \leq i \leq m$.

Let $1 \leq i \leq m$, and let x be an involution of $\overline{P_i}$. Also, let J be a 2-component of $C_G(x)$. As $\overline{P_i} \in \mathcal{S}$, we have that x is G-conjugate to t_j for some even $2 \leq j < n$. Therefore, by Lemma 7.4, $J/O(J)$ is locally balanced. Applying [31, Corollary 5.6], we may conclude that G is balanced with respect to $\overline{P_i}$.

Let $1 \leq i < m$. We have $m(P_i \cap P_{i+1}) \geq 2$ since $P_i \subseteq P_{i+1}$ or $P_{i+1} \subseteq P_i$ and $m(P_i), m(P_{i+1}) \geq 2$. Hence, $m(\overline{P_i} \cap \overline{P_{i+1}}) \geq 2$. Proposition 2.8 (ii) implies

$$W_{P_i} = W_{\overline{P_i}} = W_{\overline{P_i} \cap \overline{P_{i+1}}} = W_{\overline{P_{i+1}}} = W_{P_{i+1}}.$$

Consequently, $W_P = W_Q$, as wanted. □

Proposition 7.7. *Suppose that $q^* = 3$. Let x be an involution of S which is G-conjugate to t_i for some even $2 \leq i < n$. Then we have $O(C_G(x)) = 1$. In particular, $O(C_G(t)) = 1$.*

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Proof. We follow the pattern of the proof of [31, Theorem 9.1]. Let E be the subgroup of S consisting of all t_A, where $A \subseteq \{1, \ldots, n\}$ has even order. For each elementary abelian 2-subgroup A of G of rank at least 2, let

$$W_A := \langle O(C_G(y)) \mid y \in A^{\#} \rangle.$$

Set $W_0 := W_E$ and $M := N_G(W_0)$. We accomplish the proof step by step.

(1) $N_G(S) \leq M$.

Let $g \in N_G(S)$. Clearly, $E \in \mathcal{S}$, and it is easy to note E^g still lies in $\mathcal{S}$. Lemma 7.6 implies that $W_0 = W_{E^g}$. On the other hand, we have $(W_0)^g = W_{E^g}$ by Proposition 2.8 (i). So we have $(W_0)^g = W_0$ and hence $g \in M$.

(2) *Let y be an involution of S such that y is G-conjugate to t_j for some even $2 \leq j < n$. Then y is M-conjugate to t_j.*

We have $\langle y \rangle \in \mathcal{S}$. By Lemma 7.5 (i), there is a subgroup A of S with $\langle y \rangle \leq A$, $A \in \mathcal{S}$ and $m(A) = 3$. As a consequence of Lemma 3.22, there is an element g of G with $A^g \leq E$. By Lemma 7.6 and Proposition 2.8 (i), we have $(W_0)^g = (W_A)^g = W_{A^g} = W_0$. Thus, $g \in M$.

We have $y^g \in E$, and y^g is G-conjugate and hence $PSL_n(q)$-conjugate to t_j. So we have $y^g = t_B$ for some $B \subseteq \{1, \ldots, n\}$ with $|B| = j$. From Lemma 3.23 (i), we see that $y^g = t_B$ is $N_{PSL_n(q)}(E)$-conjugate and hence $N_G(E)$-conjugate to t_j. As $N_G(E) \leq M$, it follows that y^g is M-conjugate to t_j. Hence, y is M-conjugate to t_j.

(3) *Let y be an involution of S such that y is G-conjugate to t_j for some even $2 \leq j < n$. Then $C_G(y) \leq M$.*

Because of (2), we may assume that $\langle y \rangle$ is fully $\mathcal{F}_S(G)$-centralized. Then, by Lemma 7.1, $C_G(y)$ is generated by the normalizers $N_{C_G(y)}(U)$, where U is a subgroup of $C_S(y)$ such that there exists $B \leq U$ with $B \in \mathcal{S}$ and $m(B) \geq 3$. It suffices to show that each such normalizer lies in M.

Let U and B be as above, and let $g \in N_{C_G(y)}(U)$. By Lemma 7.6 and Proposition 2.8 (i), we have $(W_0)^g = (W_B)^g = W_{B^g} = W_0$. Thus, $g \in M$ and hence $N_{C_G(y)}(U) \leq M$, as required.

(4) *Let y be an involution of S. Then $C_G(y) \leq M$.*

We can see from Lemmas 3.14 and 3.15 that $Z(S)$ has an involution s which is G-conjugate to t_j for some even $2 \leq j < n$. Let P be a Sylow 2-subgroup of $C_G(y)$ with $s \in P$. By (1), $s \in M$ and hence $s \in P \cap M$. Now let $r \in N_P(P \cap M)$. Then $s^r \in P \cap M$. As a consequence of (1) and (2), s^r and s are M-conjugate to t_j. Therefore, there is some $m \in M$ with $s^r = s^m$. We have $rm^{-1} \in C_G(s)$, and so $rm^{-1} \in M$ by (3). Hence, $r \in M$. Consequently, $N_P(P \cap M) = P \cap M$. It follows that $P = P \cap M$.

Let $U \leq P$ with $m(U) \geq 3$, and let $g \in N_{C_G(y)}(U)$. By Lemma 2.3, any E_8-subgroup of S has an involution which is the image of an involution of $SL_n(q)$. Since $m(U) \geq 3$, it follows that U has an element u which is G-conjugate to t_k for some even $2 \leq k < n$. By the preceding paragraph, $u, u^g \in U \leq P \leq M$. As a consequence of (1) and (2), u and u^g are M-conjugate to t_k. So there is some $m \in M$ with $u^g = u^m$. Hence, $gm^{-1} \in C_G(u)$. From (3), we see that $C_G(u) \leq M$, and so $gm^{-1} \in M$. Thus, $g \in M$ and hence $N_{C_G(y)}(U) \leq M$. Since $C_G(y)$ is 3-generated by Corollary 7.3, it follows that $C_G(y) \leq M$.

(5) $M = G$.

Assume that $M \neq G$. By [27, Proposition 17.11], we may deduce from (1) and (4) that M is strongly embedded in G, i.e., $M \cap M^g$ has odd order for any $g \in G \setminus M$. Applying [50, Chapter 6, 4.4], it follows that G has only one conjugacy class of involutions. On the other hand, we see from Proposition 3.5 that G has at least two conjugacy classes of involutions. This contradiction shows that $M = G$.

(6) *Conclusion.*

Let $y \in E^{\#}$, and let J be a 2-component of $C_G(y)$. By Lemma 7.4, $J/O(J)$ is locally balanced. So, by [31, Corollary 5.6], G is balanced with respect to E. Proposition 2.8 (ii) implies that W_0 has odd order. By (5), we have $M = G$ and hence $W_0 \trianglelefteq G$. As $O(G) = 1$ by Hypothesis 5.1, it follows that $W_0 = 1$. So we have $O(C_G(y)) = 1$ for all $y \in E^{\#}$, and the statement of the proposition follows. □

Proposition 7.7 implies that if $q^* = 3$, then $K \cong SL^{\varepsilon}_{n-2}(3)$ and $L \cong SL_2(3)$. Our next goal is to find the isomorphism types of K and L for the case $q^* \neq 3$.

In general, $O(C_{PSL_n(q)}(t))$ is not trivial. So, if q^* is not assumed to be 3, we have no chance to prove that $O(C_G(t)) = 1$. However, we will be able to show that

$$\Delta_G(F) = \bigcap_{a \in F^\#} O(C_G(a)) = 1$$

for any Klein four subgroup F of G consisting of elements of the form t_A, where $A \subseteq \{1, \ldots, n\}$ has even order. This will later enable us to determine the isomorphism types of K and L for the case $q^* \neq 3$.

7.3. 2-balance of G

In this subsection, we prove that G is 2-balanced when $q^* \neq 3$.

Lemma 7.8. *Set $C := C_G(t)$ and $\overline{C} := C/O(C)$. Let F be a Klein four subgroup of C. Then $[\Delta_{\overline{C}}(\overline{F}), \overline{K}] = 1$.*

Proof. We closely follow arguments found in the proof of [31, Theorem 5.2].

First, we consider the case that F has a nontrivial element y such that $\overline{y}$ centralizes $\overline{K}$. Then $\overline{K}$ normalizes $O(C_{\overline{C}}(\overline{y}))$ and, as $\overline{K} \trianglelefteq \overline{C}$, $O(C_{\overline{C}}(\overline{y}))$ also normalizes $\overline{K}$. It follows that

$$[\overline{K}, O(C_{\overline{C}}(\overline{y}))] \leq \overline{K} \cap O(C_{\overline{C}}(\overline{y})).$$

Hence, $[\overline{K}, O(C_{\overline{C}}(\overline{y}))]$ is a subgroup of $\overline{K}$ with odd order. By [37, 1.5.5], $\overline{K}$ normalizes $[\overline{K}, O(C_{\overline{C}}(\overline{y}))]$. It follows that

$$[\overline{K}, O(C_{\overline{C}}(\overline{y}))] \leq O(\overline{K}).$$

As $O(\overline{K}) = 1$, this implies that $O(C_{\overline{C}}(\overline{y}))$ centralizes $\overline{K}$. By definition of $\Delta_{\overline{C}}(\overline{F})$, we have $\Delta_{\overline{C}}(\overline{F}) \leq O(C_{\overline{C}}(\overline{y}))$. Consequently, $\Delta_{\overline{C}}(\overline{F})$ centralizes $\overline{K}$.

Now we treat the case that $C_{\overline{F}}(\overline{K}) = 1$. For each subgroup or element X of C, let $\widehat{X}$ denote the image of $\overline{X}$ in $\overline{C}/C_{\overline{C}}(\overline{K})$. Since $C_{\overline{F}}(\overline{K}) = 1$, we have $\widehat{F} \cong \overline{F}$, and so $\widehat{F}$ is a Klein four subgroup of $\widehat{C}$. As $\overline{K} \cong SL^{\varepsilon}_{n-2}(q^*)/O(SL^{\varepsilon}_{n-2}(q^*))$, we have that $\overline{K}$ is locally 2-balanced (see Lemma 3.49). Using this together with the fact that the group $\widehat{C} = \overline{C}/C_{\overline{C}}(\overline{K})$ is isomorphic to a subgroup of $\mathrm{Aut}(\overline{K})$ containing $\mathrm{Inn}(\overline{K})$, we may conclude that $\Delta_{\widehat{C}}(\widehat{F}) = 1$. By [31, Proposition 3.11], if X is a finite group, B a 2-subgroup of X and $N \trianglelefteq X$, then the image of $O(C_X(B))$ in X/N lies in $O(C_{X/N}(BN/N))$. Thus, if y is an involution of F, then the image of $O(C_{\overline{C}}(\overline{y}))$ in $\widehat{C}$ lies in $O(C_{\widehat{C}}(\widehat{y}))$. It follows that the image of $\Delta_{\overline{C}}(\overline{F})$ in $\widehat{C}$ is contained in $\Delta_{\widehat{C}}(\widehat{F}) = 1$. Hence, $\Delta_{\overline{C}}(\overline{F}) \leq C_{\overline{C}}(\overline{K})$. □

Lemma 7.9. *Let $C := C_G(t)$ and $\overline{C} := C/O(C)$. Then $C_{\overline{C}}(\overline{K}) \cap C_{\overline{C}}(\overline{L})$ is a 2-group.*

Proof. For convenience, we denote $C_{\overline{C}}(\overline{K}) \cap C_{\overline{C}}(\overline{L})$ by $C_{\overline{C}}(\overline{K}, \overline{L})$. Since $\overline{C}$ is core-free, we have that $C_{\overline{C}}(\overline{K}, \overline{L})$ is core-free. So it is enough to prove that $C_{\overline{C}}(\overline{K}, \overline{L})$ is 2-nilpotent. By [39, Theorem 1.4], it suffices to show that $C_{\overline{C}}(\overline{K}, \overline{L})$ has a nilpotent 2-fusion system.

Let X denote the subgroup of T consisting of all elements of T of the form

$$\begin{pmatrix} A & \\ & B \end{pmatrix} Z(SL_n(q))$$

with $A \in W \cap Z(GL_{n-2}(q))$, $B \in V \cap Z(GL_2(q))$ and $\det(A)\det(B) = 1$.

Let $A \in W$ and $B \in V$ with $\det(A)\det(B) = 1$ and

$$m := \begin{pmatrix} A & \\ & B \end{pmatrix} Z(SL_n(q)) \in T.$$

Assume that $\overline{m}$ centralizes $\overline{K}$ and $\overline{L}$. Then we have $A \in Z(GL_{n-2}(q))$ by Lemma 6.2. Since $\overline{m}$ centralizes $\overline{L}$, $\overline{m}$ also centralizes $\overline{X_2}$. Thus, m centralizes X_2, and so B centralizes $V \cap SL_2(q)$. Lemma 3.17 implies that $B \in Z(GL_2(q))$. So we have $m \in X$. Conversely, if $A \in Z(GL_{n-2}(q))$ and $B \in Z(GL_2(q))$, then $\overline{m} \in C_{\overline{C}}(\overline{K}, \overline{L})$ as a consequence of Lemmas 6.2 and 3.44. It follows that $\overline{T} \cap C_{\overline{C}}(\overline{K}, \overline{L}) = \overline{X}$.

Let $\mathcal{F} := \mathcal{F}_S(PSL_n(q)) = \mathcal{F}_S(G)$. Since X is central in $C_{PSL_n(q)}(t)$, the only subsystem of $C_{\mathcal{F}}(\langle t \rangle)$ on X is the nilpotent fusion system on X. It follows that $\mathcal{F}_{\overline{X}}(C_{\overline{C}}(\overline{K}, \overline{L}))$ is nilpotent. So $C_{\overline{C}}(\overline{K}, \overline{L})$ has a nilpotent 2-fusion system, as required. □

In the following lemma, A_1 and A_2 have the meanings given to them after Lemma 6.5.

Lemma 7.10. *Set $C := C_G(t)$. Suppose that $q^* \neq 3$. Then A_1, A_2 and L are the only 2-components of $C_C(u)$. Moreover, the following hold:*

(i) *A_1 is the only 2-component of $C_C(u)$ containing u.*
(ii) *A_2 is the only 2-component of $C_C(u)$ containing neither u nor t.*
(iii) *L is the only 2-component of $C_C(u)$ containing t.*

Proof. By definition, A_1 and A_2 are 2-components of $C_C(u)$. Also, it is clear from the definition of L (see Proposition 6.8) that L is a 2-component of $C_C(u)$.

Set $\overline{C} := C/O(C)$. As a consequence of Lemma 6.4, $\overline{A_1}$ and $\overline{A_2}$ are the only 2-components of $C_{\overline{K}}(\overline{u})$. Moreover, $\overline{L}$ is a component of $C_{\overline{C}}(\overline{u})$. So Lemma 2.5 shows that $\overline{A_1}$, $\overline{A_2}$ and $\overline{L}$ are the only 2-components of $C_{\overline{C}}(\overline{u})$. As we have observed after Lemma 6.5, there is a bijection from the set of 2-components of $C_C(u)$ to the set of 2-components of $C_{\overline{C}}(\overline{u})$ sending each 2-component A of $C_C(u)$ to $\overline{A}$. Therefore, A_1, A_2 and L are the only 2-components of $C_C(u)$.

It remains to prove (i), (ii) and (iii). We have $T_1 \leq A_1$ by Lemma 6.6 and thus $u \in A_1$. From the definition of L, it is clear that $t \in L$. Moreover, $u \notin L$ since $\overline{t}$ is the only involution of $\overline{L}$. Similarly, $t \notin A_1$. Also, it is easy to see from Lemma 6.4 that u and t cannot be elements of A_2. □

Lemma 7.11. *Suppose that $q^* \neq 3$. Let F be a Klein four subgroup of T. Then we have $\Delta_G(F) \cap C_G(t) \leq O(C_G(t))$.*

Proof. Set $C := C_G(t)$, $D := \Delta_G(F) \cap C$ and $\overline{C} := C/O(C)$. We are going to show that $\overline{D}$ is trivial.

A direct calculation shows that $D \leq \Delta_C(F)$. For each $a \in F^\#$, we have $\overline{O(C_C(a))} \leq O(C_{\overline{C}}(\overline{a}))$ as a consequence of Corollary 2.2. Therefore, we have $\overline{\Delta_C(F)} \leq \Delta_{\overline{C}}(\overline{F})$, and hence, $\overline{D} \leq \Delta_{\overline{C}}(\overline{F})$. Lemma 7.8 implies that $[\overline{D}, \overline{K}] = 1$. In particular, $\overline{D} \leq C_{\overline{C}}(\overline{u}) = \overline{C_C(u)}$. Fix a subgroup D_0 of $C_C(u)$ with $\overline{D_0} = \overline{D}$. Also, let $g \in G$ with $u^g = t$ and $t^g = u$ (such an element exists by Lemma 6.7). Note that $(D_0)^g \leq (C_C(u))^g = C_C(u)$.

We accomplish the proof step by step.

(1) *A_1, A_2 and L are normal subgroups of $C_C(u)$.*

This is immediate from Lemma 7.10.

(2) *There is a group isomorphism* $\mathrm{Aut}(\overline{A_1}) \to \mathrm{Aut}(\overline{L})$ *which maps* $\mathrm{Inn}(\overline{A_1})$ *to* $\mathrm{Inn}(\overline{L})$ *and* $\mathrm{Aut}_{\overline{(D_0)^g}}(\overline{A_1})$ *to* $\mathrm{Aut}_{\overline{D}}(\overline{L})$.

Let $\mathrm{Aut}_{D_0}(L/O(L))$ denote the image of $\mathrm{Aut}_{D_0}(L)$ under the natural group homomorphism $\mathrm{Aut}(L) \to \mathrm{Aut}(L/O(L))$. Also, let $\mathrm{Aut}_{(D_0)^g}(A_1/O(A_1))$ denote the image of $\mathrm{Aut}_{(D_0)^g}(A_1)$ under the natural group homomorphism $\mathrm{Aut}(A_1) \to \mathrm{Aut}(A_1/O(A_1))$.

From Lemma 7.10, it is clear that $(A_1)^{g^{-1}} = L$. The group isomorphism $c_{g^{-1}}|_{A_1,L}$ induces a group isomorphism $A_1/O(A_1) \to L/O(L)$, and this group isomorphism induces a group isomorphism

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$\mathrm{Aut}(A_1/O(A_1)) \to \mathrm{Aut}(L/O(L))$. By a direct calculation, the group isomorphism just mentioned maps $\mathrm{Aut}_{(D_0)^g}(A_1/O(A_1))$ to $\mathrm{Aut}_{D_0}(L/O(L))$ and $\mathrm{Inn}(A_1/O(A_1))$ to $\mathrm{Inn}(L/O(L))$.

We have $A_1/(A_1 \cap O(C)) \cong \overline{A_1} \cong SL_2(q^*)$. As $SL_2(q^*)$ is core-free, it follows that $A_1 \cap O(C) = O(A_1)$. So the natural group homomorphism $A_1 \to \overline{A_1}$ induces a group isomorphism $A_1/O(A_1) \to \overline{A_1}$. This group isomorphism induces a group isomorphism $\mathrm{Aut}(A_1/O(A_1)) \to \mathrm{Aut}(\overline{A_1})$. By a direct calculation, the group isomorphism just mentioned maps $\mathrm{Aut}_{(D_0)^g}(A_1/O(A_1))$ to $\mathrm{Aut}_{\overline{(D_0)^g}}(\overline{A_1})$ and $\mathrm{Inn}(A_1/O(A_1))$ to $\mathrm{Inn}(\overline{A_1})$. In a very similar way, we obtain an isomorphism $\mathrm{Aut}(L/O(L)) \to \mathrm{Aut}(\overline{L})$ which maps $\mathrm{Aut}_{D_0}(L/O(L))$ to $\mathrm{Aut}_{\overline{D_0}}(\overline{L}) = \mathrm{Aut}_{\overline{D}}(\overline{L})$ and $\mathrm{Inn}(L/O(L))$ to $\mathrm{Inn}(\overline{L})$.

As a consequence of the preceding observations, there is a group isomorphism $\mathrm{Aut}(\overline{A_1}) \to \mathrm{Aut}(\overline{L})$ which maps $\mathrm{Inn}(\overline{A_1})$ to $\mathrm{Inn}(\overline{L})$ and $\mathrm{Aut}_{\overline{(D_0)^g}}(\overline{A_1})$ to $\mathrm{Aut}_{\overline{D}}(\overline{L})$, as asserted.

(3) $\mathrm{Aut}_{\overline{(D_0)^g}}(\overline{A_1}) \le \mathrm{Inn}(\overline{A_1})$.

As observed above, $\overline{D_0} = \overline{D}$ centralizes $\overline{K}$. In particular, $\overline{D}$ centralizes $\overline{A_2}$. This implies that $[D_0, A_2] \le O(C)$. As D_0 normalizes A_2 by (1), we also have that $[D_0, A_2] \le A_2$. Consequently, $[D_0, A_2] \le O(A_2)$. Because of Lemma 7.10, we have $(A_2)^g = A_2$. It follows that $[(D_0)^g, A_2] \le O(A_2)$. This easily implies $[\overline{(D_0)^g}, \overline{A_2}] \le O(\overline{A_2})$. As $\overline{A_2} \cong SL^{\varepsilon}_{n-4}(q^*)$ by Lemma 6.4, we have $O(\overline{A_2}) \le Z(\overline{A_2})$. It follows that $[\overline{A_2}, \overline{(D_0)^g}, \overline{A_2}] = [\overline{(D_0)^g}, \overline{A_2}, \overline{A_2}] \le [Z(\overline{A_2}), \overline{A_2}] = 1$. The three subgroups lemma [37, 1.5.6] implies $[\overline{A_2}, \overline{(D_0)^g}] = [\overline{A_2}, \overline{A_2}, \overline{(D_0)^g}] = 1$. Hence, $\overline{(D_0)^g}$ centralizes $\overline{A_2}$. By (1), $\overline{(D_0)^g}$ normalizes $\overline{A_1}$. Moreover, $\mathrm{Aut}_{\overline{(D_0)^g}}(\overline{K})$ has odd order since $\overline{(D_0)^g}$ has odd order. The assertion now follows from Lemmas 6.4 (iii), 3.50 and 3.51.

(4) $\overline{D} \le \bigcap_{y \in F^\#} O(C_{\overline{L}}(\overline{y}))$.

As a consequence of (2) and (3), we have $\mathrm{Aut}_{\overline{D}}(\overline{L}) \le \mathrm{Inn}(\overline{L})$. This implies $\overline{D} \le \overline{L} C_{\overline{C}}(\overline{L})$. By [37, 6.5.3], $\overline{L} \le C_{\overline{C}}(\overline{K})$. As observed above, $[\overline{D}, \overline{K}] = 1$ and hence $\overline{D} \le C_{\overline{C}}(\overline{K})$. It follows that $\overline{D}$ is a subgroup of $\overline{L}(C_{\overline{C}}(\overline{L}) \cap C_{\overline{C}}(\overline{K}))$. By Lemma 7.9, $C_{\overline{C}}(\overline{L}) \cap C_{\overline{C}}(\overline{K})$ is a 2-group. As $\overline{D}$ has odd order and $\overline{L} \trianglelefteq \overline{C}$, this implies that $\overline{D} \le \overline{L}$. Now we see that

$$\begin{aligned}
\overline{D} &\le \overline{L} \cap \Delta_{\overline{C}}(\overline{F}) \\
&= \bigcap_{y \in F^\#} (\overline{L} \cap O(C_{\overline{C}}(\overline{y}))) \\
&= \bigcap_{y \in F^\#} (C_{\overline{L}}(\overline{y}) \cap O(C_{\overline{C}}(\overline{y}))) \\
&= \bigcap_{y \in F^\#} O(C_{\overline{L}}(\overline{y})).
\end{aligned}$$

(5) *Conclusion.*

As F is a Klein four subgroup of T, we have $F = \langle y_1, y_2 \rangle$ for two commuting involutions y_1 and y_2 of T. For $i \in \{1, 2\}$, we have

$$y_i = \begin{pmatrix} A_i & \\ & B_i \end{pmatrix} Z(SL_n(q))$$

for some $A_i \in W$ and $B_i \in V$ with $\det(A_i)\det(B_i) = 1$. Let $y_3 := y_1 y_2$, $A_3 := A_1 A_2$ and $B_3 := B_1 B_2$. As y_1, y_2, y_3 are involutions, we have $(B_i)^2 \in Z(GL_2(q))$ for each $i \in \{1, 2, 3\}$.

It is easy to note that $\overline{X_2} \in \mathrm{Syl}_2(\overline{L})$. If $B \in V \cap SL_2(q)$ and

$$y := \begin{pmatrix} I_{n-2} & \\ & B \end{pmatrix} Z(SL_n(q)) \in X_2,$$

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then

$$y^{y_i} = \begin{pmatrix} I_{n-2} & \\ & B^{B_i} \end{pmatrix} Z(SL_n(q))$$

for each $i \in \{1, 2, 3\}$. Applying Lemma 3.52, we deduce that

$$\bigcap_{y \in F^\#} O(C_{\overline{L}}(\overline{y})) = 1.$$

So we have $\overline{D} = 1$ by (4). This completes the proof. □

Lemma 7.12. *Suppose that $q^* \neq 3$. Then G is 2-balanced.*

Proof. Let F be a Klein four subgroup of G, and let a be an involution of G centralizing F. We have to show that $\Delta_G(F) \cap C_G(a) \leq O(C_G(a))$.

Assume that a is G-conjugate to t. Then there is some $g \in G$ with $a^g = t$ and $F^g \leq T$. By Lemma 7.11, we have $\Delta_G(F^g) \cap C_G(t) \leq O(C_G(t))$. Clearly, $\Delta_G(F)^g = \Delta_G(F^g)$. It follows that $\Delta_G(F) \cap C_G(a) \leq O(C_G(a))$.

Assume now that a is not G-conjugate to t. Let J be a 2-component of $C_G(a)$. By Propositions 6.10, 6.11 and 6.13, either $J/O(J) \cong SL_k^{\varepsilon}(q^*)/O(SL_k^{\varepsilon}(q^*))$ for some $k \geq 3$, or $J/O(J)$ is isomorphic to a nontrivial quotient of $SL_{\frac{n}{2}}^{\varepsilon_0}(q_0)$ for some nontrivial odd prime power q_0 and some $\varepsilon_0 \in \{+, -\}$. So $J/O(J)$ is locally 2-balanced by Lemma 3.49. Applying [31, Theorem 5.2], we may conclude that $\Delta_{C_G(a)}(F) \leq O(C_G(a))$. A direct calculation shows that $\Delta_G(F) \cap C_G(a) \leq \Delta_{C_G(a)}(F)$. Hence, $\Delta_G(F) \cap C_G(a) \leq O(C_G(a))$. □

7.4. *The case $q^* \neq 3$: triviality of $\Delta_G(F)$*

Lemma 7.13. *Suppose that $q^* \neq 3$. Assume moreover that $q \equiv 1 \mod 4$ or $n \geq 7$. Then we have $\Delta_G(F) = 1$ for each Klein four subgroup F of S.*

Proof. We follow the pattern of the proof of [31, Theorem 9.1].

For each elementary abelian 2-subgroup A of G of rank at least 3, we define

$$W_A := \langle \Delta_G(F) \mid F \leq A, m(F) = 2 \rangle.$$

Let P and Q be elementary abelian subgroups of S of rank at least 3. We claim that $W_P = W_Q$. By Corollary 3.34 (iii), S is 3-connected. So there exist a natural number $m \geq 1$ and a sequence

$$P = P_1, \ldots, P_m = Q$$

such that P_i, $1 \leq i \leq m$, is an elementary abelian subgroup of S of rank at least 3 and such that

$$P_i \subseteq P_{i+1} \text{ or } P_{i+1} \subseteq P_i$$

for all $1 \leq i < m$. By Lemma 7.12, G is 2-balanced. Proposition 2.8 (ii) implies that $W_{P_i} = W_{P_{i+1}}$ for all $1 \leq i < m$. Therefore, $W_P = W_Q$, as asserted.

We use W_0 to denote W_P, where P is an elementary abelian subgroup of S of rank at least 3. Let $M := N_G(W_0)$. We accomplish the proof step by step.

(1) $N_G(S) \leq M$.

Let $g \in N_G(S)$. Take an elementary abelian subgroup P of S with $m(P) \geq 3$. By Proposition 2.8 (i), we have $(W_0)^g = (W_P)^g = W_{P^g} = W_0$. Thus, $g \in M$.

(2) *Let x be an involution of S. Then $C_G(x) \leq M$.*

By Corollary 3.35, there is an elementary abelian subgroup P of S with $x \in P$ and $m(P) = 4$. Clearly, $P \le C_G(x)$. Let R be a Sylow 2-subgroup of $C_G(x)$ containing P. By Corollary 7.3, $C_G(x)$ is 3-generated. Hence, $C_G(x)$ is generated by the normalizers $N_{C_G(x)}(U)$, where $U \le R$ and $m(U) \ge 3$. It suffices to show that each such normalizer lies in M.

So let U be a subgroup of R with $m(U) \ge 3$, and let $g \in N_{C_G(x)}(U)$. Let Q be an elementary abelian subgroup of U with $m(Q) = 3$, and let $h \in G$ with $R^h \le S$. Then $W_{Q^h} = W_{Q^{gh}} = W_{P^h} = W_0$. Proposition 2.8 (i) implies that $W_Q = W_{Q^g} = W_P = W_0$. Applying Proposition 2.8 (i) again, it follows that $(W_0)^g = (W_Q)^g = W_{Q^g} = W_0$. Hence, $g \in M$ and thus $N_{C_G(x)}(U) \le M$.

(3) $M = G$.

Assume that $M \ne G$. By [27, Proposition 17.11]; we may deduce from (1) and (2) that M is strongly embedded in G, i.e., $M \cap M^g$ has odd order for any $g \in G \setminus M$. Applying [50, Chapter 6, 4.4], it follows that G has only one conjugacy class of involutions. On the other hand, we see from Proposition 3.5 that G has at least two conjugacy classes of involutions. This contradiction shows that $M = G$.

(4) *Conclusion.*

Let F be a Klein four subgroup of S. By Corollary 3.35, there is an elementary abelian subgroup P of S with $F \le P$ and $m(P) = 4$. Clearly, $\Delta_G(F) \le W_P$. Since G is 2-balanced, W_P has odd order by Proposition 2.8 (ii). Since $W_P = W_0$, we have $W_P \trianglelefteq G$ by (3). As $O(G) = 1$ by Hypothesis 5.1, it follows that $W_P = 1$. Hence, $\Delta_G(F) = 1$. □

Next, we deal with the case that $n = 6$, $q \equiv 3 \bmod 4$ and $q^* \ne 3$. We show that, in this case, $\Delta_G(F) = 1$ for each Klein four subgroup F of S consisting of elements of the form t_A, where $A \subseteq \{1, \dots, n\}$ has even order. We need the following lemma.

Lemma 7.14. *Suppose that $q^* \ne 3$. Set $\ell := n - 4$. Let E be the subgroup of T consisting of all t_A, where $A \subseteq \{1, \dots, n\}$ has even order. Let E_1 denote the subgroup of X_1 consisting of all t_A, where A is a subset of $\{1, \dots, n-2\}$ of even order. Then we may choose elements $m_1, \dots, m_\ell \in N_K(E_1)$ and an E_8-subgroup E_0 of E with*

$$K = \langle O(K), L_{2'}(C_K(E_0)), L_{2'}(C_K(E_0))^{m_1}, \dots, L_{2'}(C_K(E_0))^{m_\ell} \rangle.$$

Proof. Set $C := C_G(t)$ and $\overline{C} := C/O(C)$. Let $H := SL^{\varepsilon}_{n-2}(q^*)/O(SL^{\varepsilon}_{n-2}(q^*))$. Let $\widetilde{D}$ be the subgroup of $SL^{\varepsilon}_{n-2}(q^*)$ consisting of all diagonal matrices in $SL^{\varepsilon}_{n-2}(q^*)$ with diagonal entries in $\{1, -1\}$, and let D denote the image of $\widetilde{D}$ in H. Denote by H_1 the image of

$$\left\{ \begin{pmatrix} A & \\ & I_{n-4} \end{pmatrix} \ : \ A \in SL^{\varepsilon}_2(q^*) \right\}$$

in H.

We claim that there is a group isomorphism $\psi : \overline{K} \to H$ which maps $\overline{E_1}$ to D and $\overline{A_1}$ to H_1. By Lemma 6.4 (iii), there is a group isomorphism $\varphi : \overline{K} \to H$ under which $\overline{A_1}$ corresponds to H_1. Since $\bar{u}$ is the only involution of $\overline{A_1}$, we have that $\bar{u}^{\varphi}$ is the image of $\mathrm{diag}(-1, -1, 1, \dots, 1) \in SL^{\varepsilon}_{n-2}(q^*)$ in H. Clearly, $\overline{E_1}$ is elementary abelian of order 2^{n-3}. Using Lemma 3.22, we conclude that $\overline{E_1}^{\varphi}$ is H-conjugate to D. So there is some $\alpha \in \mathrm{Inn}(H)$ mapping $\overline{E_1}^{\varphi}$ to D. We may assume that α centralizes $\bar{u}^{\varphi}$. Then $H_1{}^{\alpha} = H_1$, and the isomorphism $\psi := \varphi\alpha$ maps $\overline{E_1}$ to D and $\overline{A_1}$ to H_1, as desired.

Using Lemma 3.39, we can find elements $x_1, \dots, x_\ell \in N_H(D)$ such that $H = \langle H_1, H_1{}^{x_1}, \dots, H_1{}^{x_\ell} \rangle$. Therefore, K has elements $m_1, \dots, m_\ell$ such that

$$\overline{K} = \langle \overline{A_1}, \overline{A_1}^{\overline{m_1}}, \dots, \overline{A_1}^{\overline{m_\ell}} \rangle$$

and $\overline{m_1}, \dots, \overline{m_\ell} \in N_{\overline{K}}(\overline{E_1})$. From Lemma 2.1, we see that $N_{\overline{K}}(\overline{E_1}) = \overline{N_K(E_1)}$. So we may assume $m_i \in N_K(E_1)$ for $i \in \{1, \dots, \ell\}$. Let $E_0 := \langle u, t_{\{3,4\}}, t_{\{4,5\}} \rangle$. By Lemma 6.5, we have $\overline{A_1} \trianglelefteq C_{\overline{C}}(\bar{u})$. In particular, $\overline{E_0}$ normalizes $\overline{A_1}$. Moreover, $\overline{E_0}$ centralizes $\overline{T_1}$. We have $\overline{A_1} \cong SL_2(q^*)$ and $\overline{T_1} \in \mathrm{Syl}_2(\overline{A_1})$

(see Lemma 6.4). Applying Lemma 3.44, we conclude that $\overline{A_1} \leq C_{\overline{K}}(\overline{E_0})$. As $\overline{A_1} \trianglelefteq C_{\overline{K}}(\overline{u})$ and $\overline{A_1} \leq C_{\overline{K}}(\overline{E_0}) \leq C_{\overline{K}}(\overline{u})$, we even have that $\overline{A_1}$ is a component of $C_{\overline{K}}(\overline{E_0})$. It follows that

$$\overline{K} = \langle L_{2'}(C_{\overline{K}}(\overline{E_0})), L_{2'}(C_{\overline{K}}(\overline{E_0}))^{\overline{m_1}}, \dots, L_{2'}(C_{\overline{K}}(\overline{E_0}))^{\overline{m_\ell}} \rangle.$$

Let $k \in K$ such that $\overline{k} \in C_{\overline{K}}(\overline{E_0})$. As $K \trianglelefteq C$, we have $[k, E_0] \leq O(C) \cap K = O(K)$. Thus, $kO(K) \in C_{C/O(K)}(E_0O(K)/O(K))$. By Lemma 2.1, there is an element $z \in C_C(E_0)$ such that $kO(K) = zO(K)$. Observing that $z \in C_K(E_0)$ and that $\overline{k} = \overline{z}$, we may conclude that $C_{\overline{K}}(\overline{E_0}) = \overline{C_K(E_0)}$. If $1 \leq i \leq \ell$, then $L_{2'}(C_{\overline{K}}(\overline{E_0}))^{\overline{m_i}} = L_{2'}(\overline{C_K(E_0)})^{\overline{m_i}} = \overline{L_{2'}(C_K(E_0))}^{\overline{m_i}} = \overline{L_{2'}(C_K(E_0))^{m_i}}$, where the second equality follows from Proposition 2.4. It follows that

$$K = \langle O(K), L_{2'}(C_K(E_0)), L_{2'}(C_K(E_0))^{m_1}, \dots, L_{2'}(C_K(E_0))^{m_\ell} \rangle.$$

This completes the proof. □

Lemma 7.15. *Suppose that $n = 6$, $q \equiv 3 \bmod 4$ and $q^* \neq 3$. Let E denote the subgroup of S consisting of all t_A, where A is a subset of $\{1, \dots, n\}$ of even order. Then $\Delta_G(F) = 1$ for any Klein four subgroup F of E.*

Proof. We follow the pattern of the proof of [31, Theorem 9.1].

Set $W_0 := \langle \Delta_G(F) \mid F \leq E, m(F) = 2 \rangle$ and $M := N_G(W_0)$. Since T is the image of

$$\left\{ \begin{pmatrix} A & \\ & B \end{pmatrix} \ : \ A \in W, B \in V, \det(A)\det(B) = 1 \right\}$$

in $PSL_n(q)$, we have $T \in \mathrm{Syl}_2(PSL_n(q))$ by Lemma 3.15. Hence, $S = T$ and thus $t \in Z(S)$. By choice of W (see Section 5), we have

$$W = \left\{ \begin{pmatrix} A & \\ & B \end{pmatrix} \ : \ A, B \in V \right\} \cdot \left\langle \begin{pmatrix} & I_2 \\ I_2 & \end{pmatrix} \right\rangle$$

We accomplish the proof step by step.

(1) *For each subgroup E_0 of E with order at least 8, we have $N_G(E_0) \leq M$.*

Clearly, $E \cong E_{16}$. Therefore, the statement follows from the 2-balance of G (see Lemma 7.12) and Proposition 2.8 (ii).

(2) $N_G(S) \leq M$.

First, we prove $S \leq M$. By (1), we have $E \leq M$. As $q \equiv 3 \bmod 4$ and $S = T$, any element of S can be written as a product of an element of E and an element of S induced by a matrix of the form

$$\begin{pmatrix} A & \\ & B \end{pmatrix}$$

with $A \in W \cap SL_4(q)$ and $B \in V \cap SL_2(q)$. So, in order to prove that $S \leq M$, it suffices to show that each element of S induced by a matrix of this form lies in M. If $B \in V \cap SL_2(q)$, then the image of

$$\begin{pmatrix} I_4 & \\ & B \end{pmatrix}$$

in S centralizes the group $\langle t_{\{1,2\}}, t_{\{2,3\}}, t_{\{3,4\}} \rangle \cong E_8$. So it is contained in M by (1). Hence, in order to prove that $S \leq M$, it suffices to show that if $A \in W \cap SL_4(q)$, then the image of

$$\begin{pmatrix} A & \\ & I_2 \end{pmatrix}$$

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in S lies in M. So assume that $A \in W \cap SL_4(q)$. By the structure of W, there are elements M_1, M_2 of V such that $\det(M_1) = \det(M_2)$ and

$$A = \begin{pmatrix} M_1 & \\ & M_2 \end{pmatrix} \text{ or } A = \begin{pmatrix} M_1 & \\ & M_2 \end{pmatrix}\begin{pmatrix} & I_2 \\ I_2 & \end{pmatrix}.$$

The image of

$$\begin{pmatrix} M_1 & & \\ & M_2 & \\ & & I_2 \end{pmatrix}$$

in S can be written as a product of an element of E and an element of S induced by a matrix of the form

$$\begin{pmatrix} \widetilde{M_1} & & \\ & \widetilde{M_2} & \\ & & I_2 \end{pmatrix}$$

with $\widetilde{M_1}, \widetilde{M_2} \in V \cap SL_2(q)$. The images of

$$\begin{pmatrix} \widetilde{M_1} & \\ & I_4 \end{pmatrix} \text{ and } \begin{pmatrix} I_2 & & \\ & \widetilde{M_2} & \\ & & I_2 \end{pmatrix}$$

in S centralize the groups $\langle t_{\{3,4\}}, t_{\{4,5\}}, t_{\{5,6\}} \rangle$ and $\langle t_{\{1,2\}}, t_{\{2,5\}}, t_{\{5,6\}} \rangle$, respectively. So they are elements of M. It follows that the image of

$$\begin{pmatrix} M_1 & & \\ & M_2 & \\ & & I_2 \end{pmatrix}$$

in S lies in M. The image of the block matrix

$$\begin{pmatrix} & I_2 & \\ I_2 & & \\ & & I_2 \end{pmatrix}$$

in S normalizes E and is thus contained in M. It follows that the image of

$$\begin{pmatrix} A & \\ & I_2 \end{pmatrix}$$

in S lies in M. Consequently, $S \leq M$.

By Lemma 3.24, $\mathrm{Aut}_{PSL_n(q)}(S) = \mathrm{Inn}(S)$. As $\mathcal{F}_S(G) = \mathcal{F}_S(PSL_n(q))$, it follows that $\mathrm{Aut}_G(S) = \mathrm{Inn}(S)$, and so $N_G(S) = SC_G(S)$. We have seen above that $S \leq M$, and we have $C_G(S) \leq M$ by (1). Hence, $N_G(S) \leq M$.

(3) $C_G(t) \leq M$.

Let E_1 be the subgroup of X_1 consisting of all t_A, where A is a subset of $\{1, \ldots, n-2\}$ of even order. As a consequence of Lemma 7.14, there is an E_8-subgroup E_0 of E such that $K = \langle O(K), C_K(E_0), N_K(E_1) \rangle$. By (1), $C_K(E_0)$ and $N_K(E_1)$ are subgroups of M. By [27, Proposition 11.23], we have

$$O(K) = \langle C_{O(K)}(B) \mid B \leq E, m(B) = 3 \rangle.$$

Therefore, $O(K) \leq M$ by (1). Consequently, $K \leq M$. By the Frattini argument,

$$C_G(t) = KN_{C_G(t)}(X_1).$$

So it suffices to show that $N_{C_G(t)}(X_1) \leq M$. Since $\mathcal{F}_S(G) = \mathcal{F}_S(PSL_n(q))$, we may conclude from Lemma 5.7 that $\mathrm{Aut}_{C_G(t)}(X_1)$ is a 2-group. Hence, $N_{C_G(t)}(X_1)/C_{C_G(t)}(X_1)$ is a 2-group. As $X_1 \trianglelefteq T = S \in \mathrm{Syl}_2(C_G(t))$, it follows that $N_{C_G(t)}(X_1) = SC_{C_G(t)}(X_1)$. We have $S \leq M$ by (2), and $C_{C_G(t)}(X_1) \leq C_G(E_1) \leq M$ by (1). Consequently, $N_{C_G(t)}(X_1) \leq M$, as required.

(4) *Let x be an involution of S which is G-conjugate to t. Then x is M-conjugate to t.*

It is easy to see that if an element of T is $PSL_n(q)$-conjugate to t, then it is $C_{PSL_n(q)}(t)$-conjugate to an element of E. As $\mathcal{F}_S(G) = \mathcal{F}_S(PSL_n(q))$ and $S = T$, it follows that x is $C_G(t)$-conjugate and hence M-conjugate to an element y of E. From Lemma 3.23, we see that if an element of E is $PSL_n(q)$-conjugate to t, then it is $N_{PSL_n(q)}(E)$-conjugate to t. So, as $\mathcal{F}_S(G) = \mathcal{F}_S(PSL_n(q))$, we have that y is $N_G(E)$-conjugate to t. By (1), $N_G(E) \leq M$, and so x is M-conjugate to t.

(5) *Let x be an involution of S. Then $C_G(x) \leq M$.*

Let R be a Sylow 2-subgroup of $C_G(x)$ with $C_S(x) \leq R$. We have $t \in Z(S) \leq C_S(x)$ and $t \in M$. Thus, $t \in R \cap M$. Let $r \in N_R(R \cap M)$. Then $y := t^r \in R \cap M$. As a consequence of (4), y is M-conjugate to t. So there is an element m of M such that $t^r = y = t^m$. We have $rm^{-1} \in C_G(t) \leq M$ by (3), and so $r \in R \cap M$. Hence, $N_R(R \cap M) = R \cap M$, and thus, $R = R \cap M$.

By Corollary 7.3, $C_G(x)$ is 3-generated. Therefore, $C_G(x)$ is generated by the normalizers $N_{C_G(x)}(U)$, where $U \leq R$ and $m(U) \geq 3$. It suffices to show that each such normalizer lies in M.

So let $U \leq R$ with $m(U) \geq 3$, and let $g \in N_{C_G(x)}(U)$. Take an elementary abelian subgroup Q of U of rank 3. Lemma 2.3 shows that any E_8-subgroup of S has an involution which is the image of an involution of $SL_n(q)$. This implies that Q has an element s which is G-conjugate to t. Since $s, s^g \in U \leq R \leq M$, we see from (4) that s and s^g are M-conjugate to t. So there are elements $m, m' \in M$ such that $s = t^m$ and $s^g = t^{m'}$. We have $t^{m'} = s^g = (t^m)^g = t^{mg}$. Thus, $mgm'^{-1} \in C_G(t) \leq M$, and hence, $g \in M$. It follows that $N_{C_G(x)}(U) \leq M$.

(6) $M = G$.

Assume that $M \neq G$. By [27, Proposition 17.11], we may deduce from (2) and (5) that M is strongly embedded in G, i.e., $M \cap M^g$ has odd order for any $g \in G \setminus M$. Applying [50, Chapter 6, 4.4], it follows that G has only one conjugacy class of involutions. On the other hand, we see from Proposition 3.5 that G has precisely two conjugacy classes of involutions. This contradiction shows that $M = G$.

(7) *Conclusion.*

Let F be a Klein four subgroup of E. Clearly, $\Delta_G(F) \leq W_0$. By (6), we have $W_0 \trianglelefteq G$. Since G is 2-balanced, W_0 has odd order by Proposition 2.8 (ii). As $O(G) = 1$ by Hypothesis 5.1, it follows that $W_0 = 1$. Hence, $\Delta_G(F) = 1$. □

7.5. Quasisimplicity of the 2-components of $C_G(t)$

In this subsection, we determine the isomorphism types of K and L.

Lemma 7.16. *Let x and y be two commuting involutions of G. Set $C := C_G(x)$ and $\overline{C} := C/O(C)$. Then any 2-component of $C_{\overline{C}}(\overline{y})$ is a component of $C_{\overline{C}}(\overline{y})$.*

Proof. By [31, Corollary 3.2], $L_{2'}(C_{\overline{C}}(\overline{y})) = L_{2'}(C_{E(\overline{C})}(\overline{y}))$. We know from Section 6 that $E(\overline{C})$ is a K-group, i.e., the composition factors of $E(\overline{C})$ are known finite simple groups. Applying [25, Theorem 3.5], we conclude that $L_{2'}(C_{E(\overline{C})}(\overline{y})) = E(C_{E(\overline{C})}(\overline{y}))$. Therefore, any 2-component of $C_{E(\overline{C})}(\overline{y})$ is a component of $C_{E(\overline{C})}(\overline{y})$. So any 2-component of $C_{\overline{C}}(\overline{y})$ is a component of $C_{\overline{C}}(\overline{y})$.

Instead of using [25, Theorem 3.5], the lemma could be proved directly by using Corollary 3.47 (i) and the results of Section 6. □

Proposition 7.17. *K is isomorphic to a quotient of $SL^{\varepsilon}_{n-2}(q^*)$ by a central subgroup of odd order.*

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Proof. The proof is inspired from the proof of [31, Theorem 10.1].

For $q^* = 3$, the proposition follows from Proposition 7.7. From now on, we assume that $q^* \neq 3$.

Set $C := C_G(t)$. Let E denote the subgroup of T consisting of all t_A, where $A \subseteq \{1, \dots, n\}$ has even order. We assume $m_1, \dots, m_\ell$, where $\ell := n - 4$, to be elements of K and E_0 to be an E_8-subgroup of E with

$$K = \langle O(K), L_{2'}(C_K(E_0)), L_{2'}(C_K(E_0))^{m_1}, \dots, L_{2'}(C_K(E_0))^{m_\ell} \rangle.$$

Such elements $m_1, \dots, m_\ell$ and such a subgroup E_0 exist by Lemma 7.14.

The proof will be accomplished step by step.

(1) *Let f be an involution of E_0. Then $L_{2'}(C_K(E_0)) \leq L_{2'}(C_C(f))$.*

As $K \trianglelefteq C$, we have $C_K(E_0) \trianglelefteq C_C(E_0)$. This implies $L_{2'}(C_K(E_0)) \leq L_{2'}(C_C(E_0))$. By [31, Theorem 3.1], we have $L_{2'}(C_{C_C(f)}(E_0)) \leq L_{2'}(C_C(f))$. Clearly, $C_{C_C(f)}(E_0) = C_C(E_0)$. It follows that $L_{2'}(C_K(E_0)) \leq L_{2'}(C_C(E_0)) \leq L_{2'}(C_C(f))$.

(2) *Let F be a Klein four subgroup of E_0. Set $D := [C_{O(K)}(F), L_{2'}(C_K(E_0))]$. Then $D = 1$.*

Clearly, $L_{2'}(C_K(E_0))$ normalizes $C_{O(K)}(F)$. Also, $O^{2'}(L_{2'}(C_K(E_0))) = L_{2'}(C_K(E_0))$, and $C_{O(K)}(F)$ is a $2'$-group. Applying [27, Proposition 4.3 (i)], we conclude that $D = [D, L_{2'}(C_K(E_0))]$.

Now let f be an involution of F. We are going to show that $D \leq O(C_G(f))$. Set $M := L_{2'}(C_C(f))$. By (1), $L_{2'}(C_K(E_0)) \leq M$. Also, $D \leq C_C(F) \leq C_C(f)$ and $M \trianglelefteq C_C(f)$. It follows that $D = [D, L_{2'}(C_K(E_0))] \leq [C_C(f), M] \leq M$.

Let $\overline{C_G(f)} := C_G(f)/O(C_G(f))$. By Corollary 2.2, $C_{\overline{C_G(f)}}(\bar{t}) = \overline{C_C(f)}$. As a consequence of Proposition 2.4, $L_{2'}(C_{\overline{C_G(f)}}(\bar{t})) = \overline{M}$. Lemma 7.16 implies that $\overline{M} = L(C_{\overline{C_G(f)}}(\bar{t}))$. It easily follows that $O(\overline{M})$ is central in $\overline{M}$.

From the definition of D, it is clear that $D \leq O(K)$. So we have $D \leq M \cap O(K) \leq O(M)$. It follows that $\overline{D} \leq \overline{O(M)} \leq O(\overline{M}) \leq Z(\overline{M})$. In particular, $\overline{L_{2'}(C_K(E_0))}$ centralizes $\overline{D}$. Thus, $D = [D, L_{2'}(C_K(E_0))] \leq O(C_G(f))$.

Since f was arbitrarily chosen, it follows that $D \leq \Delta_G(F)$. By Lemmas 7.13 and 7.15, we have $\Delta_G(F) = 1$. Consequently, $D = 1$, as wanted.

(3) $O(K) \leq Z(K)$.

By [27, Proposition 11.23], we have

$$O(K) = \langle C_{O(K)}(F) : F \leq E_0, m(F) = 2 \rangle.$$

Because of (2), it follows that $O(K)$ centralizes $L_{2'}(C_K(E_0))$. By choice of E_0, we have

$$K = \langle O(K), L_{2'}(C_K(E_0)), L_{2'}(C_K(E_0))^{m_1}, \dots, L_{2'}(C_K(E_0))^{m_\ell} \rangle$$

for some $m_1, \dots, m_\ell \in K$. It follows that $K = O(K)C_K(O(K))$. Therefore, $C_K(O(K))$ has odd index in K. We have $O^{2'}(K) = K$ since K is a 2-component of C. It follows that $K = C_K(O(K))$. Consequently, $O(K) \leq Z(K)$.

(4) *Conclusion.*

Applying [27, Lemma 4.11], we deduce from (3) that K is a component of C. Therefore, K is quasisimple. We have

$$K/Z(K) \cong (K/O(K))/Z(K/O(K)) \cong PSL^{\varepsilon}_{n-2}(q^*).$$

Applying Lemmas 3.1 and 3.2, we conclude that $K \cong SL^{\varepsilon}_{n-2}(q^*)/Z$ for some central subgroup Z of $SL^{\varepsilon}_{n-2}(q^*)$. Using Proposition 3.19 or using the order formulas for $|SL^{\varepsilon}_{n-2}(q^*)|$ and $|SL_{n-2}(q)|$ given

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by [32, Proposition 1.1 and Corollary 11.29], we see that

$$|SL^{\varepsilon}_{n-2}(q^*)|_2 = |SL_{n-2}(q)|_2 = |X_1| = |K|_2 = |SL^{\varepsilon}_{n-2}(q^*)/Z|_2.$$

Thus, Z has odd order. □

Proposition 7.18. *We have $L \cong SL_2(q^*)$ and $L \trianglelefteq C_G(t)$. Moreover, L is the only normal subgroup of $C_G(t)$ which is isomorphic to $SL_2(q^*)$.*

Proof. For $q^* = 3$, this follows from Propositions 7.7 and 6.8.

Assume now that $q^* \neq 3$. Let $\widetilde{K} := KO(C_G(t))$. By the last statement in Proposition 2.4, $K = O^{2'}(\widetilde{K})$. Let $i \in \{1, 2\}$. Since A_i is a 2-component of $C_{C_G(t)}(u)$, we have $A_i = O^{2'}(A_i)$. Also, $A_i \leq \widetilde{K}$, and so $A_i \leq O^{2'}(\widetilde{K}) = K$. It follows that A_i is a 2-component of $C_K(u)$.

By Proposition 7.17, we have $K \cong SL^{\varepsilon}_{n-2}(q^*)/Z$ for some central subgroup Z of $SL^{\varepsilon}_{n-2}(q^*)$ with odd order. It is easy to see that if m is a noncentral involution of $SL^{\varepsilon}_{n-2}(q^*)/Z$ and J is a 2-component of its centralizer in $SL^{\varepsilon}_{n-2}(q^*)/Z$, then $J \cong SL^{\varepsilon}_k(q^*)$ for some $k \geq 2$. Since u is a noncentral involution of K and $A_1/O(A_1) \cong SL_2(q^*)$, it follows that $A_1 \cong SL_2(q^*)$. By definition of L (see Proposition 6.8), L is isomorphic to A_1. So we have $L \cong SL_2(q^*)$.

Let L_0 be the 2-component of $C_G(t)$ associated to $LO(C_G(t))/O(C_G(t))$. By [37, 6.5.2], we have $[L_0, K] = 1$. Hence, $L_0 \leq C_{C_G(t)}(u)$. So L_0 is a 2-component of $C_{C_G(t)}(u)$. Clearly, $A_1 \neq L_0 \neq A_2$. Lemma 7.10 implies that $L_0 = L$. From Proposition 6.8 (iii), we see that $L = L_0 \trianglelefteq C_G(t)$.

Proposition 6.8 (iii) also shows that K and L are the only 2-components of $C_G(t)$. So L is the only normal subgroup of $C_G(t)$ isomorphic to $SL_2(q^*)$. □

8. The subgroup G_0

Let A be a subset of $\{1, \ldots, n\}$ with order 2. Then t_A is G-conjugate to t. Proposition 7.18 implies that $C_G(t_A)$ has a unique normal subgroup isomorphic to $SL_2(q^*)$. We denote this subgroup by L_A, and we define G_0 to be the subgroup of G generated by the groups L_A, where $A = \{i, i+1\}$ for some $1 \leq i < n$. We are going to prove that $G_0 \trianglelefteq G$ and that G_0 is isomorphic to a nontrivial quotient of $SL^{\varepsilon}_n(q^*)$. This will complete the proof of Theorem 5.2.

By Proposition 7.17, K is isomorphic to a quotient of $SL^{\varepsilon}_{n-2}(q^*)$ by a central subgroup of odd order. By the proof of Proposition 7.18, A_1 and A_2 are 2-components of $C_K(u)$ if $q^* \neq 3$.

Lemma 8.1. *Let $Z \leq Z(SL^{\varepsilon}_{n-2}(q^*))$ with $K \cong H := SL^{\varepsilon}_{n-2}(q^*)/Z$. Let H_1 be the image of*

$$\left\{ \begin{pmatrix} A & \\ & I_{n-4} \end{pmatrix} \; : \; A \in SL^{\varepsilon}_2(q^*) \right\}$$

in H and H_2 the image of

$$\left\{ \begin{pmatrix} I_2 & \\ & A \end{pmatrix} \; : \; A \in SL^{\varepsilon}_{n-4}(q^*) \right\}$$

in H. Then there is a group isomorphism $\varphi : K \longrightarrow H$ which maps A_1 to H_1 and A_2 to H_2.

Proof. For $q^* = 3$, this follows from Proposition 7.7 and Lemma 6.4 (iii).

Assume now that $q^* \neq 3$. Let $\varphi : K \longrightarrow H$ be a group isomorphism. For each even natural number k with $2 \leq k < n-2$, let h_k be the image of

$$\begin{pmatrix} -I_k & \\ & I_{n-2-k} \end{pmatrix}$$

in H. Since Z has odd order by Proposition 7.17, we have that any involution of H is the image of an involution of $SL^{\varepsilon}_{n-2}(q^*)$. Applying Lemmas 3.3 (i) and 3.4 (ii), we conclude that each noncentral involution of H is conjugate to h_k for some even $2 \leq k < n-2$. As u is a noncentral involution of K, we may assume that $u^\varphi = h_k$ for some even $2 \leq k < n-2$.

Let $\widetilde{H_1}$ be the image of

$$\left\{ \begin{pmatrix} A & \\ & I_{n-2-k} \end{pmatrix} : A \in SL_k^\varepsilon(q^*) \right\}$$

in H and $\widetilde{H_2}$ be the image of

$$\left\{ \begin{pmatrix} I_k & \\ & A \end{pmatrix} : A \in SL_{n-2-k}^\varepsilon(q^*) \right\}$$

in H. The 2-components of $C_H(h_k)$ are precisely the quasisimple members of $\{\widetilde{H_1}, \widetilde{H_2}\}$. Also, $h_k \in \widetilde{H_1}$, but $h_k \notin \widetilde{H_2}$. On the other hand, A_1 and A_2 are the 2-components of $C_K(u)$, and we have $u \in A_1$. This implies $(A_1)^\varphi = \widetilde{H_1}$ and $(A_2)^\varphi = \widetilde{H_2}$. Since $A_1 \cong L \cong SL_2(q^*)$, we have $k = 2$, and hence, $\widetilde{H_1} = H_1$ and $\widetilde{H_2} = H_2$. □

Lemma 8.2. *Let $1 \leq i < j < n$. Set $A := \{i, i+1\}$ and $B := \{j, j+1\}$. Then:*

(i) *If $i+1 < j$, then $[L_A, L_B] = 1$.*

(ii) *Suppose that $j = i+1$. Then there is a group isomorphism from $\langle L_A, L_B\rangle$ to $SL_3^\varepsilon(q^*)$ under which L_A corresponds to the subgroup*

$$\left\{ \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{M}} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right) : M \in SL_2^\varepsilon(q^*) \right\}$$

of $SL_3^\varepsilon(q^*)$ and under which L_B corresponds to the subgroup

$$\left\{ \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{M}} \\ 0 & & \end{array}\right) : M \in SL_2^\varepsilon(q^*) \right\}$$

of $SL_3^\varepsilon(q^*)$.

(iii) *Suppose that $1 \leq i \leq n-3$ and that $j = i+1$. Set $k := i+2$ and $C := \{k, k+1\}$. Then $\langle L_A, L_B, L_C\rangle$ is isomorphic to $SL_4^\varepsilon(q^*)$.*

Proof. To prove (i), (ii) and (iii), we first introduce some notation and make some preliminary observations. Let H, H_1, H_2 and φ be as in Lemma 8.1. For each $D \subseteq \{1, \ldots, n-2\}$ of even order, let h_D be the image of the matrix $\operatorname{diag}(d_1, \ldots, d_{n-2}) \in SL_{n-2}^\varepsilon(q^*)$ in H, where $d_\ell = -1$ if $\ell \in D$ and $d_\ell = 1$ if $\ell \in \{1, \ldots, n-2\} \setminus D$. We have $u^\varphi = h_{\{1,2\}}$ as u and $h_{\{1,2\}}$ are the unique involutions of A_1 and $H_1 = (A_1)^\varphi$, respectively.

Let J be the subgroup of H consisting of all h_D, where $D \subseteq \{1, \ldots, n-2\}$ has even order, and let E_1 denote the subgroup of X_1 consisting of all t_D, where $D \subseteq \{1, \ldots, n-2\}$ has even order. Then $(E_1)^\varphi$ is an elementary abelian 2-subgroup of H of rank $n-3$. As a consequence of Lemma 3.22, there is an element $h \in H$ such that $(E_1^\varphi)^h = J$. Then $(h_{\{1,2\}})^h = (u^\varphi)^h \in (E_1^\varphi)^h = J$. Lemma 3.23 (i) shows that $(h_{\{1,2\}})^h$ is $N_H(J)$-conjugate to $h_{\{1,2\}}$. Therefore, we can assume that h centralizes $h_{\{1,2\}}$. Then $(H_1)^h = H_1$ and $(H_2)^h = H_2$. Upon replacing φ by φc_h, we may thus assume that $(E_1)^\varphi = J$.

We have $C_H(h_{\{1,2\}})' = H_1 \times H_2$, and $H_1 \cong SL_2^\varepsilon(q^*)$ and $H_2 \cong SL_{n-4}^\varepsilon(q^*)$ are indecomposable. Also, $(|H_1/H_1'|, |Z(H_2)|) = 1 = (|H_2/H_2'|, |Z(H_1)|)$. So, by a consequence of the Krull–Remark–Schmidt theorem [35, Kapitel I, Satz 12.6], $C_H(h_{\{1,2\}})' = H_1 \times H_2$ is the only decomposition of $C_H(h_{\{1,2\}})'$ into a direct product of indecomposable groups. This implies that H_1 is the only normal subgroup of $C_H(h_{\{1,2\}})$ which contains $h_{\{1,2\}}$ and is isomorphic to $SL_2^\varepsilon(q^*)$. For each $D \subseteq \{1, \ldots, n-2\}$ of order 2, h_D and $h_{\{1,2\}}$ are conjugate, and so $C_H(h_D)$ has a unique normal subgroup H_D with $h_D \in H_D$ and

$H_D \cong SL_2^{\varepsilon}(q^*)$. Note that the groups $H_{\{1,2\}}, H_{\{2,3\}}, \dots, H_{\{n-3,n-2\}}$ are the $SL_2^{\varepsilon}(q^*)$-subgroups of H corresponding to the 2×2-blocks along the main diagonal.

Now let $D_0 \subseteq \{1, \dots, n-2\}$ with order 2. Then $(t_{D_0})^{\varphi} \in (E_1)^{\varphi} = J$, and $(t_{D_0})^{\varphi}$ is conjugate to $u^{\varphi} = h_{\{1,2\}}$. Thus, $(t_{D_0})^{\varphi} = h_D$ for some $D \subseteq \{1, \dots, n-2\}$ of order 2. We claim that $L_{D_0} \le K$ and $(L_{D_0})^{\varphi} = H_D$. To see this, let $k \in K$ with $t_{D_0} = u^k = (t_{\{1,2\}})^k$. Then $L_{D_0} = (L_{\{1,2\}})^k = (A_1)^k \le K$, where the last equality follows from the definition of L (see Proposition 6.8) and the definition of $L_{\{1,2\}}$. Since $L_{D_0} \trianglelefteq C_K(t_{D_0})$, $L_{D_0} \cong SL_2^{\varepsilon}(q^*)$ and $t_{D_0} \in L_{D_0}$, the previous paragraph implies that $(L_{D_0})^{\varphi} = H_D$, as claimed.

We are now ready to prove (i), (ii) and (iii). To prove (i), suppose that $i + 1 < j$. As $\mathcal{F}_S(G) = \mathcal{F}_S(PSL_n(q))$, we see from Lemma 3.23 (i) that there is some $g \in G$ with $(t_A)^g = t_{\{1,2\}}$ and $(t_B)^g = t_{\{3,4\}}$. As $[L_A, L_B]^g = [(L_A)^g, (L_B)^g] = [L_{\{1,2\}}, L_{\{3,4\}}]$, we may assume that $A = \{1,2\}$ and $B = \{3,4\}$. Then $(L_A)^{\varphi} = (A_1)^{\varphi} = H_1$. Also, $(t_B)^{\varphi} \in (A_2)^{\varphi} = H_2$, and so $(t_B)^{\varphi} = h_D$ for some $D \subseteq \{3, 4, \dots, n-2\}$ with $|D| = 2$. By the previous paragraph, $[L_A, L_B]^{\varphi} = [(L_A)^{\varphi}, (L_B)^{\varphi}] = [H_1, H_D] = 1$, and so $[L_A, L_B] = 1$, whence (i) holds.

Assume now that $j = i + 1$. As $\mathcal{F}_S(G) = \mathcal{F}_S(PSL_n(q))$, we see from Lemma 3.23 (i) that there is some $g \in G$ with $(t_A)^g = t_{\{1,2\}}$ and $(t_B)^g = t_{\{2,3\}}$. As $\langle L_A, L_B\rangle^g = \langle (L_A)^g, (L_B)^g\rangle = \langle L_{\{1,2\}}, L_{\{2,3\}}\rangle$, we may assume that $A = \{1,2\}$ and $B = \{2,3\}$. Let $D \subseteq \{1, \dots, n-2\}$ with $(t_B)^{\varphi} = h_D$. We have $|D| = 2$ by paragraph four and $h_D \notin (H_1 \cup H_2)$ since $t_B \notin (A_1 \cup A_2)$. Thus, $D = \{k, \ell\}$ for some $k \in \{1,2\}$ and some $\ell \in \{3, 4, \dots, n-2\}$. Because of Lemma 3.23 (i), we may assume that $k = 2$ and $\ell = 3$. Since $(L_A)^{\varphi} = H_1 = H_{\{1,2\}}$ and $(L_B)^{\varphi} = H_{\{2,3\}}$, we have proved (ii).

Assume now that the hypotheses of (iii) are satisfied. Arguing as in the proof of (ii), we may assume that $A = \{1,2\}, B = \{2,3\}, C = \{3,4\}$ and $(t_B)^{\varphi} = h_{\{2,3\}}$. Let $D \subseteq \{1, \dots, n-2\}$ with $(t_C)^{\varphi} = h_D$. By paragraph four, we have $|D| = 2$. Also, $h_D \in (A_2)^{\varphi} = H_2$, so $D \cap \{1,2\} = \emptyset$. We claim that $D \cap \{2,3\} = \{3\}$. Assume not. Then $D \cap \{1,2,3\} = \emptyset$, and Lemma 3.23 (i) shows that there is an element of $N_H(J)$ which interchanges $h_{\{1,2\}}$ and $h_{\{2,3\}}$ and fixes h_D. So there is an element of $N_K(E_1)$ which interchanges u and $t_{\{2,3\}}$ and fixes $t_{\{3,4\}}$. Having in mind that $\mathcal{F}_S(G) = \mathcal{F}_S(PSL_n(q))$, we see from Lemma 3.23 (ii) that $N_K(E_1)$ has no such element. This contradiction shows that $D \cap \{2,3\} = \{3\}$. By Lemma 3.23 (i), we may assume that $D = \{3,4\}$. Now $\langle L_A, L_B, L_C\rangle^{\varphi} = \langle H_{\{1,2\}}, H_{\{2,3\}}, H_{\{3,4\}}\rangle \cong SL_4^{\varepsilon}(q^*)$, and the proof of (iii) is complete. □

Proposition 8.3. *G_0 is isomorphic to a nontrivial quotient of $SL_n^{\varepsilon}(q^*)$.*

Proof. Assume that $\varepsilon = +$. By Lemma 8.2, the groups $L_{\{1,2\}}, \dots, L_{\{n-1,n\}}$ form a weak Curtis–Tits system in G of type $SL_n(q^*)$ (in the sense of [29, p. 9]). Applying a version of the Curtis–Tits theorem, namely [29, Chapter 13, Theorem 1.4], we conclude that G_0 is isomorphic to a quotient of $SL_n(q^*)$.

Assume now that $\varepsilon = -$. Then Lemma 8.2 shows that G_0 has a weak Phan system of rank $n - 1$ over $\mathbb{F}_{q^{*2}}$ (in the sense of [13, p. 288]). If $q^* \neq 3$, then [13, Theorem 1.2] implies that G_0 is isomorphic to a quotient of $SU_n(q^*)$. If $q^* = 3$, the same follows from [13, Theorem 1.3] and Lemma 8.2 (iii). □

Lemma 8.4. *Let R be a Sylow 2-subgroup of G_0. Then $R \in \mathrm{Syl}_2(G)$ and $\mathcal{F}_R(G_0) = \mathcal{F}_R(G)$.*

Proof. Since $q \sim \varepsilon q^*$, we have that the 2-fusion system of $PSL_n^{\varepsilon}(q^*)$ is isomorphic to the 2-fusion system of $PSL_n(q)$ (see Proposition 3.20). Clearly, $G_0/Z(G_0) \cong PSL_n^{\varepsilon}(q^*)$. So the 2-fusion system of $G_0/Z(G_0)$ is isomorphic to the 2-fusion system of G. It easily follows that $|G_0|_2 = |G_0/Z(G_0)|_2 = |G|_2$, and Lemma 2.11 shows that the 2-fusion system of G_0 is isomorphic to that of $G_0/Z(G_0)$ and hence to that of G. This completes the proof. □

Lemma 8.5. *The following hold.*

(i) *If $q^* \neq 3$, then $O^{2'}(O^2(C_G(t))) = KL$.*
(ii) *If $q^* = 3$, then $O^2(C_G(t)) = KL$.*

Proof. Set $C := C_G(t)$.

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Assume that $q^* \neq 3$. Then KL is perfect. This implies that $KL = O^{2'}(O^2(KL)) \leq O^{2'}(O^2(C))$. Since $T \cap KL = (T \cap K)(T \cap L) = X_1X_2$, Lemmas 5.4 and 2.11 show that C/KL has a nilpotent 2-fusion system. So C/KL is 2-nilpotent by [39, Theorem 1.4]. This implies $O^{2'}(O^2(C)) \leq KL$.

We assume now that $q^* = 3$. Then $KL = O^2(KL)$ since K is perfect and $L \cong SL_2(3)$. Thus, $KL \leq O^2(C)$. In order to prove equality, it suffices to show that C/KL is a 2-group. By Proposition 7.7 and Lemma 6.3 (i), $C/KC_C(K)$ is a 2-group. By [37, 6.5.2], we have $L \leq C_C(K)$. It is enough to show that $C_C(K)/L$ is a 2-group.

We have $O^2(C_C(K)) \cap T \leq O^2(C_C(X_1)) \cap T = X_2$ by Lemma 5.6 and the hyperfocal subgroup theorem [18, Theorem 1.33]. On the other hand, $X_2 \leq L = O^2(L) \leq O^2(C_C(K))$. Consequently, $X_2 = O^2(C_C(K)) \cap T \in \mathrm{Syl}_2(O^2(C_C(K)))$. Set $U := C_{O^2(C_C(K))}(X_2)$. We have $X_2 \trianglelefteq C$ since X_2 is the unique Sylow 2-subgroup of $L \cong SL_2(3)$. So we have $U \trianglelefteq C$. Hence, $Z(X_2) = X_2 \cap U \in \mathrm{Syl}_2(U)$. Applying [37, 7.2.2], we conclude that U is 2-nilpotent. We have $O(U) = 1$ since $U \trianglelefteq C$ and $O(C) = 1$ by Proposition 7.7. It follows that $U = Z(X_2)$.

Clearly, $O^2(C_C(K))/U$ is isomorphic to a subgroup of $\mathrm{Aut}(X_2)$. We have $|O^2(C_C(K))/U|_2 = 4$ since $Q_8 \cong X_2 \in \mathrm{Syl}_2(O^2(C_C(K)))$ and $U = Z(X_2)$. Also, $|O^2(C_C(K))/U| \geq 12$ since $L \leq O^2(C_C(K))$. As $\mathrm{Aut}(X_2) \cong \mathrm{Aut}(Q_8) \cong S_4$ by [37, 5.3.3], it follows that $|O^2(C_C(K))/U| = 12$. This implies $O^2(C_C(K)) = L$. So $C_C(K)/L$ is a 2-group, as required. □

Lemma 8.6. *We have $KL \leq G_0$.*

Proof. We have $t \in X_2 \leq L = L_{\{n-1,n\}} \leq G_0$. Let $R \in \mathrm{Syl}_2(G_0)$ with $t \in R$ such that $\langle t\rangle$ is fully centralized in $\mathcal{G} := \mathcal{F}_R(G_0)$. By Lemma 8.4, $R \in \mathrm{Syl}_2(G)$ and $\mathcal{G} = \mathcal{F}_R(G)$. Therefore, $C_R(t) \in \mathrm{Syl}_2(C_G(t))$ and $C_{\mathcal{G}}(\langle t\rangle) = \mathcal{F}_{C_R(t)}(C_G(t))$. Also, $T = C_S(t) \in \mathrm{Syl}_2(C_G(t))$ and $C_{\mathcal{F}_S(G)}(\langle t\rangle) = \mathcal{F}_T(C_G(t))$. So, by Lemma 5.3, $C_{\mathcal{G}}(\langle t\rangle)$ has a component isomorphic to the 2-fusion system of $SL_{n-2}(q)$.

Let $Z \leq Z(SL_n^\varepsilon(q^*))$ with $G_0 \cong SL_n^\varepsilon(q^*)/Z$. By the proof of Lemma 8.4, $Z(G_0)$ has odd order.

Let $\widetilde{x}$ be an element of $SL_n^\varepsilon(q^*)$ such that $x := \widetilde{x}Z$ is an involution of $SL_n^\varepsilon(q^*)/Z$. Set $C := C_{SL_n^\varepsilon(q^*)/Z}(x)$. Noticing that the 2-components of C are precisely the images of the 2-components of $C_{SL_n^\varepsilon(q^*)}(\widetilde{x})$ in $SL_n^\varepsilon(q^*)/Z$, one can see from Lemmas 3.3 and 3.4 that one of the following holds:

(1) $q^* \neq 3$, $O^{2'}(O^2(C)) = K_0L_0$, where K_0 and L_0 are subnormal subgroups of C such that $K_0 \cong SL_{n-i}^\varepsilon(q^*)$ and $L_0 \cong SL_i^\varepsilon(q^*)$ for some $1 \leq i < n$. Moreover, the 2-components of C are precisely the quasisimple members of $\{K_0, L_0\}$.

(2) $q^* = 3$, $O^2(C) = K_0L_0$, where K_0 and L_0 are subnormal subgroups of C such that $K_0 \cong SL_{n-i}^\varepsilon(q^*)$ and $L_0 \cong SL_i^\varepsilon(q^*)$ for some $1 \leq i < n$. Moreover, the 2-components of C are precisely the quasisimple members of $\{K_0, L_0\}$.

(3) C has precisely one 2-component, and this 2-component is isomorphic to a nontrivial quotient of $SL_{n/2}((q^*)^2)$.

As seen above, $C_{\mathcal{G}}(\langle t\rangle) = \mathcal{F}_{C_R(t)}(C_{G_0}(t))$ has a component isomorphic to the 2-fusion system of $SL_{n-2}(q)$. By Proposition 2.17, this component is induced by a 2-component of $C_{G_0}(t)$. In view of the preceding observations, we can conclude that $C_{G_0}(t)$ has subgroups K_0 and L_0 with $K_0 \cong SL_{n-2}^\varepsilon(q^*)$ and $L_0 \cong SL_2(q^*)$ such that $O^{2'}(O^2(C_{G_0}(t))) = K_0L_0$ if $q^* \neq 3$ and $O^2(C_{G_0}(t)) = K_0L_0$ if $q^* = 3$.

Clearly, $O^{2'}(O^2(C_{G_0}(t))) \leq O^{2'}(O^2(C_G(t)))$ and $O^2(C_{G_0}(t)) \leq O^2(C_G(t))$. Lemma 8.5 implies that $K_0L_0 \leq KL$. If n is odd, then it is easy to see that $|K_0L_0| = |K_0||L_0| \geq |K||L| = |KL|$. If n is even, then one can easily see that $|K_0L_0| = \frac{1}{2}|K_0||L_0| \geq \frac{1}{2}|K||L| = |KL|$. Consequently, $K_0L_0 \leq KL$ and $|K_0L_0| \geq |KL|$. It follows that $KL = K_0L_0 \leq G_0$. □

Corollary 8.7. *Let x be an involution of G_0 which is G-conjugate to t. Let L_0 be the unique normal $SL_2(q^*)$-subgroup of $C_G(x)$, and let K_0 be the component of $C_G(x)$ different from L_0. Then we have $K_0L_0 \leq G_0$.*

Proof. Since $t \in G_0$, we see from Lemma 8.4 that there is some $g \in G_0$ with $x = t^g$. Clearly, $(K_0L_0) = (KL)^g$, and so $K_0L_0 \leq G_0$ by Lemma 8.6. □

Lemma 8.8. *We have $N_G(S) \leq N_G(G_0)$.*

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Proof. Set $M := N_G(G_0)$. Let $s \in N_S(S \cap M)$, and let $1 \le i \le n-1$. We have $t_{\{i,i+1\}} \in S \cap L_{\{i,i+1\}} \le S \cap G_0 \le S \cap M$, and hence, $(t_{\{i,i+1\}})^s \in S \cap M \le M$. Since G_0 has odd index in M by Lemma 8.4, we even have $(t_{\{i,i+1\}})^s \in G_0$. Corollary 8.7 implies that $(L_{\{i,i+1\}})^s \le G_0$. So we have $s \in M$ by the definition of G_0. Thus, $N_S(S \cap M) = S \cap M$ and hence $S \le M$. We have $C_G(S) \le C_G(t_{\{i,i+1\}}) \le N_G(L_{\{i,i+1\}})$ for all $1 \le i \le n-1$. Thus, $C_G(S) \le M$. Using Lemma 3.24, we conclude that $N_G(S) = SC_G(S) \le M$. □

Lemma 8.9. *If x is an involution of S, then $C_G(x) \le N_G(G_0)$.*

Proof. Set $M := N_G(G_0)$.

We begin by proving that $C_G(t) \le M$. We have $K \le G_0 \le M$ by Lemma 8.6 and $C_G(t) = KN_{C_G(t)}(X_1)$ by the Frattini argument. Also, $N_{C_G(t)}(X_1) = TC_{C_G(t)}(X_1)$ as a consequence of Lemma 5.7, and $T \le M$ by Lemma 8.8. So it suffices to show that $C_{C_G(t)}(X_1) \le M$.

Let $z \in C_{C_G(t)}(X_1)$. In order to prove $z \in M$, it is enough to show that $(L_{\{i,i+1\}})^z \le G_0$ for all $1 \le i < n$. If $1 \le i < n$ and $i \ne n-2$, we have $z \in C_G(t_{\{i,i+1\}})$ and hence $(L_{\{i,i+1\}})^z = L_{\{i,i+1\}} \le G_0$. It remains to show that $(L_{\{n-2,n-1\}})^z \le G_0$. Since $\mathcal{F}_S(G) = \mathcal{F}_S(PSL_n(q))$, there is some $g \in G$ with $t^g = u$, $u^g = t$ and $(t_{\{2,3\}})^g = t_{\{n-2,n-1\}}$ (see Lemma 3.23 (i)). From the definition of L (Proposition 6.8), we see that $L_{\{1,2\}} = A_1 \le K$. Since $u = t_{\{1,2\}}$ and $t_{\{2,3\}}$ are K-conjugate by Lemma 3.23 (i), we thus have $L_{\{2,3\}} \le K \le L_{2'}(C_G(t))$. Hence, $L_{\{n-2,n-1\}} = (L_{\{2,3\}})^g \le L_{2'}(C_G(t))^g = L_{2'}(C_G(u))$. Since z centralizes u, it follows that $(L_{\{n-2,n-1\}})^z \le L_{2'}(C_G(u))$. From Corollary 8.7, we see that $L_{2'}(C_G(u)) \le G_0$. So we have $(L_{\{n-2,n-1\}})^z \le G_0$, and it follows that $C_{C_G(t)}(X_1) \le M$. Consequently, $C_G(t) \le M$.

Since G_0 has odd index in M by Lemma 8.4, we see from Lemma 8.8 that $S \le G_0$. Also, $\mathcal{F}_S(G_0) = \mathcal{F}_S(G)$ by Lemma 8.4. As $C_G(t) \le M$, it follows that $C_G(x) \le M$ for any involution x of S which is G-conjugate to t.

Assume now that x is an involution of S which is G-conjugate to t_i for some even natural number i with $4 \le i < n$ such that $i \le \frac{n}{2}$ if n is even. We are going to show that $C_G(x) \le M$. Arguing by induction over i and using the preceding observations, we may assume that, for each even $2 \le j < i$ and each involution y of S which is G-conjugate to t_j, we have $C_G(y) \le M$. Furthermore, we may assume that $\langle x \rangle$ is fully $\mathcal{F}_S(G)$-centralized since $\mathcal{F}_S(G) = \mathcal{F}_S(G_0)$.

As a consequence of Lemma 7.1, $C_G(x)$ is generated by the normalizers $N_{C_G(x)}(U)$, where U is a subgroup of $C_S(x)$ containing a G-conjugate of t_j for some even $2 \le j < i$. We show that each such normalizer is contained in M. Thus, let U be a subgroup of $C_S(x)$, and let y be an element of U which is G-conjugate to t_j for some even $2 \le j < i$. Also, let $g \in N_{C_G(x)}(U)$. Then $y^g \in U \le C_S(x) \le S$. Since $\mathcal{F}_S(G_0) = \mathcal{F}_S(G)$, we have that y and y^g are G_0-conjugate. Hence, there is some $m \in G_0$ with $y^g = y^m$. We have $mg^{-1} \in C_G(y) \le M$. This implies $g \in M$ since $m \in G_0 \le M$. So we have $N_{C_G(x)}(U) \le M$ and hence $C_G(x) \le M$.

Assume now that x is an arbitrary involution of S. We are going to prove that $C_G(x) \le M$. Since $\mathcal{F}_S(G) = \mathcal{F}_S(G_0)$, we may assume that $\langle x \rangle$ is fully $\mathcal{F}_S(G)$-centralized. By Corollary 7.3, $C_G(x)$ is 3-generated. Therefore, $C_G(x)$ is generated by the normalizers $N_{C_G(x)}(U)$, where $U \le C_S(x)$ and $m(U) \ge 3$. Take some $U \le C_S(x)$ with $m(U) \ge 3$. By Lemma 2.3, any E_8-subgroup of S has an involution which is the image of an involution of $SL_n(q)$. It follows that U has an element y which is G-conjugate to t_k for some even $2 \le k < n$. By the preceding observations, $C_G(y) \le M$. Arguing as above, we can conclude that $N_{C_G(x)}(U) \le M$. It follows that $C_G(x) \le M$. □

Proposition 8.10. *We have $G_0 \trianglelefteq G$.*

Proof. Suppose that $M := N_G(G_0)$ is a proper subgroup of G. By [27, Proposition 17.11], we may deduce from Lemmas 8.8 and 8.9 that M is strongly embedded in G. Therefore, by [50, Chapter 6, 4.4], G has only one conjugacy class of involutions. On the other hand, we see from Proposition 3.5 that G has at least two conjugacy classes of involutions. This contradiction shows that $M = G$. Hence, $G_0 \trianglelefteq G$. □

With Propositions 8.3 and 8.10, we have completed the proof of Theorem 5.2.

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9. Proofs of the main results

Proof of Theorem A. By Section 4, Theorem A is true for $n \leq 5$.

Suppose now that $n \geq 6$. Let q be a nontrivial odd prime power, and let G be a finite simple group satisfying $(\mathcal{CK})$.

Recall that a natural number $k \geq 6$ is said to satisfy $P(k)$ if whenever q_0 is a nontrivial odd prime power and H is a finite simple group satisfying $(\mathcal{CK})$ and realizing the 2-fusion system of $PSL_k(q_0)$, we have $H \cong PSL_k^{\varepsilon}(q^*)$ for some nontrivial odd prime power q^* and some $\varepsilon \in \{+, -\}$ with $\varepsilon q^* \sim q_0$. Theorem 5.2 shows that $P(k)$ is satisfied for all natural numbers $k \geq 6$.

Therefore, if the 2-fusion system of G is isomorphic to the 2-fusion system of $PSL_n(q)$, then condition (i) of Theorem A is satisfied.

Conversely, if one of the conditions (i), (ii), (iii) of Theorem A is satisfied, then this can only be condition (i), and Proposition 3.20 implies that the 2-fusion system of G is isomorphic to the 2-fusion system of $PSL_n(q)$. □

Proof of Theorem B. Let q be a nontrivial odd prime power, and let $n \geq 2$ be a natural number, where $q \equiv 1$ or $7 \bmod 8$ if $n = 2$. Let G be a finite simple group and $S \in \mathrm{Syl}_2(G)$. Suppose that $\mathcal{F}_S(G)$ has a normal subsystem $\mathcal{E}$ on a subgroup T of S such that $\mathcal{E}$ is isomorphic to the 2-fusion system of $PSL_n(q)$ and such that $C_S(\mathcal{E}) = 1$. We have to show that $\mathcal{F}_S(G)$ is isomorphic to the 2-fusion system of $PSL_n(q)$.

By Lemma 3.21, $PSL_n(q)$ is not a Goldschmidt group. Applying [9, Theorem 5.6.18], we conclude that $\mathcal{E}$ is simple. We see from [15, Theorem B] that $\mathcal{E}$ is tamely realized by some finite simple group of Lie type K.

By Theorem A, we have $K \cong PSL_n^{\varepsilon}(q^*)$ for some nontrivial odd prime power q^* and some $\varepsilon \in \{+, -\}$ with $\varepsilon q^* \sim q$.

By Propositions 3.40 and 3.42, we have that $\mathrm{Out}(K)$ is 2-nilpotent. Now Proposition 2.20 implies that $\mathcal{F}_S(G)$ is tamely realized by a subgroup L of $\mathrm{Aut}(K)$ containing $\mathrm{Inn}(K)$ such that the index of $\mathrm{Inn}(K)$ in L is odd. By Lemma 3.57, the 2-fusion system of L is isomorphic to the 2-fusion system of $\mathrm{Inn}(K) \cong K$ and hence isomorphic to the 2-fusion system of $PSL_n(q)$. So $\mathcal{F}_S(G)$ is isomorphic to the 2-fusion system of $PSL_n(q)$. □

Proof of Corollary C. Let q be a nontrivial odd prime power, and let $n \geq 2$ be a natural number, where $q \equiv 1$ or $7 \bmod 8$ if $n = 2$. Let G be a finite simple group, and let S be a Sylow 2-subgroup of G. Suppose that $F^*(\mathcal{F}_S(G))$ is isomorphic to the 2-fusion system of $PSL_n(q)$.

We have $F^*(\mathcal{F}_S(G)) \trianglelefteq \mathcal{F}_S(G)$ and $C_S(F^*(\mathcal{F}_S(G))) = Z(F^*(\mathcal{F}_S(G))) = 1$. So Theorem B implies that $\mathcal{F}_S(G)$ is isomorphic to the 2-fusion system of $PSL_n(q)$. □

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