THE ELASTODYNAMICS OF MOVING LOADS

PART I:

The field of a semi-infinite line load moving on the surface of an elastic solid with constant supersonic velocity

MICHAEL PAPADOPOULOS

(received 29 December 1961, revised 4 May 1962)

Abstract

When a semi-infinite line load moves lengthways, at supersonic velocity, on the plane surface of an elastic solid, the resulting velocity field is conical. There are two characteristic cones, one associated with dilatation effects and the other with shear effects. The propagation process is more complicated than the well-known case of conical flow in supersonic aerodynamics not only because of the presence of two cones of discontinuity but also because the presence of a free surface implies interaction between shear and dilatation effects. It is the interaction process at the free surface which is examined in detail in this paper.

The results of this fundamental problem may be extended by the process of superposition to more general steadily moving loads. In particular by differentiating with respect to time, the potential of a steadily moving point load is obtained explicitly.

1. Introduction

In this paper we shall consider the effect of a semi-infinite line load which is moving lengthways on the free surface of an isotropic elastic medium. Since this is a linear problem the solution may be built up from known solutions of elastodynamics. The most obvious approach is to use integral transform methods. This is a process which appears to lead to explicit, elementary solutions only at large distances from the source of energy; it obscures the structure of the associated disturbance, and, what is more important mathematically, it hides the characteristic properties of the original hyperbolic wave equations which are of direct consequence in the physical description of the problem. Another approach is to take the known field for a transient point load, then to consider a line distribution of progressively applied point loads, and then to follow this with an integration along the line of motion. This is approach is being used by Payton [1], but it seems to lead to complications of a forbidding nature and his primary results need careful analysis before they are shown to be equivalent to those given in this paper. There are no previously published references to this problem.

The classical problem of elastodynamics, that of the propagation of tremors over the surface of an elastic solid, was examined in detail by Lamb [2]. The fundamental excitation considered was that of the point — or line source, and the problem was to determine what effect the setting-up of such sources has on the solid particularly on the free surface.

This same type of problem has since been examined in various ways by various authors, e.g. Cagniard [3], Lapwood [4], Pekeris [5], Strick [6] and Craggs [7], and of these only the method described by Craggs is of any interest in the understanding of the present paper. This situation follows a recognition that when an infinite line load is set up on the surface of an elastic solid, the subsequent disturbance has the property of dynamic similarity. This property is seized on not only by Craggs, but also by Maue [8] Miles [9] and the author [10] as the basis for further investigations into problems of elastodynamics.

The analytical results of assuming dynamic similarity in a solution are very similar to those in the method of conical flows (e.g. Ward and Goldstein [11]); it is clear indeed that the field of a uniform infinite line source, suddenly set up, is the limiting field of a semi-infinite line source moving lengthways, with its velocity approaching infinity. In this paper we analyse the possibility of conical motion associated with steady rectilinear movement of velocity a (greater than both the shear velocity c_2 and the dilatation velocity c_1) of such a line load.

The analysis rest on the following results. A scalar quantity $F(r, \theta, z, t)$ satisfies the wave equation

$$\nabla^2 F(r, \theta, z, t) = \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2}, \tag{1}$$

with c a constant propagation velocity. Assume steady motion in the z-direction with constant velocity $a > c$, and introduce the new variable $\tau = (at - z)/a$. Then

$$\nabla^2 F(r, \theta, \tau) = \frac{1}{\gamma^2}\frac{\partial^2 F}{\partial \tau^2}, \tag{2}$$

with $1/\gamma^2 = 1/c^2 - 1/a^2$.

For conical motion only the variables s $(= r/\tau)$ and θ may appear, for $\tau > 0$; it follows that outside a steadily moving half-cone $s = \gamma$,

$$\frac{\partial^2 F}{\partial u^2} - \frac{\partial^2 F}{\partial \theta^2} = 0. \tag{3}$$

where $s = \gamma \sec u$, and hence F must take the form

$$F = f(u + \theta) + g(u - \theta). \tag{4}$$

The surfaces $u \pm \theta = \text{const.}$ are tangent planes to the cone $s = \gamma$, and these with their envelope, the cone itself, form families of characteristic surfaces across which $\partial F/\partial r$ may be discontinuous, but at which the tangential derivative must be continuous.

Within the cone $s = \gamma$, F satisfies Laplace's equation in the variables v and θ, where, for $v < 0$,

$$s = \gamma \operatorname{sech} v,$$

or

$$v = \ln \left\{ \frac{\gamma}{s} \left[1 - \left(1 - \frac{s^2}{\gamma^2} \right)^{\frac{1}{2}} \right] \right\}. \tag{5}$$

Accordingly we are able to define a complex quantity W^F which is to be analytic in a certain semi-infinite strip of the complex $v + i\theta$-plane, for which we have the identities

$$F(v, \theta) = Rl\, W^F(v + i\theta), \tag{6}$$

and, on $\theta = 0$, say

$$\frac{\partial F}{\partial v} = Rl \left(\frac{dW^F}{dv} \right), \quad \frac{\partial F}{\partial \theta} = Rl \left(i \frac{dW^F}{dv} \right). \tag{7}$$

(The equations 7 are the Cauchy-Riemann conditions).

In the problem of elastodynamics, we first define scalar and vector velocity potentials. The former satisfies the wave equation (1) with the dilatation velocity c_1, while the rectangular cartesion components of the latter satisfy the same equation with the shear velocity c_2. We use subscripts 1 and 2 to indicate whether the quantities γ and the variables u and v, are associated with the dilatation or shear effects, and once having defined the potentials, we may label complex potentials W with superscripts as in equation 6. We also follow the convention that partial derivatives of the various potentials are indicated by subscripts.

2. Formulation of the Problem

We have an isotropic elastic solid, of density ρ and with Lamé constants λ and μ in which shear waves travel with velocity $c_2 [= (\mu/\rho)^{\frac{1}{2}}]$ and dilatation waves travel with velocity $c_1 [= [(\lambda + 2\mu)/\rho]^{\frac{1}{2}}]$. We take the plane $y = 0$

to represent the surface of the solid lying in the region $\gamma < 0$, and we imagine a surface load to be moving with velocity a along the z-axis. We assume infinitesimal displacements throughout, so that equations of elasticity may be used. It is convenient to express these in terms of the velocity vector q.

The equation of motion, differentiated with respect to time is

$$(\lambda + \mu)\nabla(\nabla \cdot q) + \mu\nabla^2 q = \rho\partial^2 q/\partial t^2. \tag{8}$$

Put

$$q = \nabla\phi + \nabla \times \psi \tag{9}$$

Then conditions to be satisfied by the scalar potential ϕ and the rectangular components (A, B, C) of the vector potential ψ are (see e.g. Sternberg [12]), that

$$\left(\nabla^2 - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)\phi = 0 \tag{10}$$

$$\left(\nabla^2 - \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0 \tag{11}$$

and

$$\nabla \cdot \psi = 0. \tag{12}$$

These equations, valid within the solid, must be satisfied in conjunction with conditions at the free surface where the stress components T_{yy}, T_{yx}, and T_{yz} must vanish except perhaps in the neighborhood of the line load (for $r = 0$, $\tau > 0$).

Although we have introduced four scalar potentials, we shall take B to be identically zero. For the simultaneous vanishing of T_{yy}, T_{yx} T_{yz} and $\nabla \cdot \psi$ on the free surface is not possible unless $B_y = 0$ for $y = 0$. The potential B is always linked with the other components of the vector potential through the equation $\nabla \cdot \psi = 0$ at all points of the solid. Accordingly we have either the possibility that B is zero together with the combination $A_x + C_z$ at all points of the solid, or that B is a function of y which is even about the surface and which plays no part in the interaction process at the free surface, but which is dependent on the quantities A and C through the equation

$$A_x + B_y + C_z = 0.$$

The more complicated case involving the presence of B will be considered, inter alia, in Part II.

At the free surface, therefore, the time derivative of the stress-strain relations may be written in the form

(13) $$\frac{\partial}{\partial t} T_{yy} = \lambda\nabla^2\phi + 2\mu[\phi_{yy} + A_{zy} - C_{xy}] = 0$$

(14) $$\frac{\partial}{\partial t} T_{yx} = \mu[2\phi_{yx} + A_{xz} - C_{xx} + C_{yy}] = 0,$$

and

(15) $$\frac{\partial}{\partial t} T_{yz} = \mu[2\phi_{yz} + A_{zz} - A_{yy} - C_{xz}] = 0,$$

together with the condition 12 which is that for $y \leqq 0$

(16) $$A_x + C_z = 0.$$

These conditions may be simplified. From equations 14 and 16 it follows that

(17) $$2\phi_{yx} - C_{zz} - C_{xx} + C_{yy} = 0$$

and from equations 15, 16 and 11, it follows that

(18) $$-2\phi_{zy} + 2A_{zz} - 2C_{zx} - c_2^{-2}A_{tt} = 0$$

With assumptions of conical flow such that

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial z} = -\frac{1}{a}\frac{\partial}{\partial \tau}$$

and such that only the variables $s(= r/\tau)$ and θ may appear we can, on switching to polar co-ordinates r and θ reduce the conditions 16, 13, 17 and 18 to be satisfied on $\theta = 0$ to the following:

(19) $$aA_s + sC_s = 0$$

(20) $$\frac{\partial}{\partial s}\left\{\frac{2}{a}A_\theta - \frac{2}{s}C_\theta + \left[s^2\left(\frac{1}{\gamma_2^2} - \frac{1}{a^2}\right) - 2\right]\phi_s\right\} = 0$$

(21) $$\frac{\partial}{\partial s}\left\{\frac{2}{s}\phi_\theta + \left[s^2\left(\frac{1}{\gamma_2} - \frac{1}{a^2}\right) - 2\right]C_s\right\} = 0$$

and

$$\frac{\partial}{\partial s}\left\{\frac{2}{a}\phi_\theta - s^2\left[\frac{1}{a^2} - \frac{1}{\gamma_2^2}\right]A_s - \frac{2sC_s}{a}\right\} = 0.$$

The final equation is redundant as may be seen by combining it with equation 19 to eliminate A_s. Also we see that since

$$\frac{\partial}{\partial r} = \frac{1}{\tau}\frac{\partial}{\partial s} = -\frac{1}{s}\frac{\partial}{\partial \tau}$$

we may integrate equations 20 and 21 with respect to t; since it is the stress components and not just the time derivatives which must vanish, we have finally the conditions that at $\theta = 0$

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$$
\left.\begin{aligned}
&aA_s + sC_s = 0 \\
&\frac{2}{a} A_\theta - \frac{2}{s} C_\theta + \left[s^2 \left(\frac{1}{\gamma_2^2} - \frac{1}{a^2} \right) - 2 \right] \phi_s = 0, \qquad (22)
\end{aligned}\right\}
$$

and

$$
\frac{2}{s} \phi_\theta = \left[s^2 \left(\frac{1}{\gamma_2^2} - \frac{1}{a^2} \right) - 2 \right] C_s = 0.
$$

3. Solution of the problem

The structure of the conical velocity field follows from the examination of the wave equation. The singular line load starts at a moving origin; its subsequent effects are characterized by the presence of two distinct conical fronts, the cone $r = \gamma_1 \tau$ which represents for the scalar potential the dividing surface between the region in which ϕ is of simple wave type and that in which it is harmonic, and the cone $r = \gamma_2 \tau$ which does the same for the vector components A and C. The structure is shown in figure 1, which although depicted with s and θ for polar co-ordinates is an instantaneous representation in the (r, θ) plane of any transferse section behind the tip of the moving load.

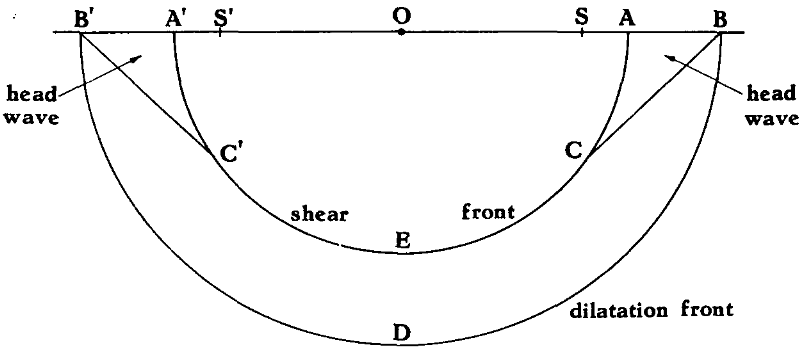

Fig. 1. A transverse section of the field showing the shear and dilatation fronts, and the head wave regions.

Outside the larger region BDB' we expect no potential ϕ either for physical reasons, or by noting that zero initial state for $\tau = 0$ mans zero field not only as $s \to \infty$ but thence on all characteristic surfaces which approach infinity either directly or by reflection at the free surface in a shear-free region. The same reasoning applies to the vector components A and C, but only outside the region $BCEC'B'$. The scalar potential ϕ being harmonic does not vanish within BDB', in particular on the interface AB. Interaction imposed by the conditions 22 means A and C do not vanish either on AB, or by virtue of the simplewave structure, within ABC.

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This is known as the head wave region, where

$$A = A(u_2 - \theta), \quad C = C(u_2 - \theta) \tag{23}$$

and where, as in equation 3, $s = \gamma_2 \sec u_2$.

Within the smaller semi-circle $s = \gamma_2$ in the (s, θ) plane, A and C are harmonic functions, so that we have complex potentials $W^A(v_2 + i\theta)$, $W^c(v_2 + i\theta)$, and within the larger semi-circle $s = \gamma_1$, we have a complex potential $W^\phi(v_1 + i\theta)$, each potential being analytic within a strip $v < 0$, $0 > \theta > -\pi$ in the appropriate complex plane.

It is the virtue of the shear field structure in the head wave region that the functional behavior given in equation 23 gives an explicit relation between normal and tangential derivatives of ϕ or, alternatively determines the argument of the complex derivative dW^ϕ/dv on the appropriate segment of the boundary of the strip of regularity. We have, on combining 22 and 23, and applying the Cauchy-Riemann relations 7, the result that with $M = \gamma_2/\gamma_1 < 1$

$$Rl \left\{ \frac{dW^\phi}{dv_1} \left[\left(\frac{s^2}{\gamma_2^2} - \frac{s^2}{a^2} - 2 \right)^2 \frac{dv_1}{ds} - 4i \frac{ds}{du_2} \left(\frac{a^2 + s^2}{a^2 s^2} \right) \right] \right\} = 0 \tag{24}$$

for $0 > v_1 > -\operatorname{arc\,sech} M$.

Rather than consider the strip $v_1 < 0$, $0 > \theta > -\pi$ we use the conformal mapping $\zeta_1 = \xi_1 + i\eta_1 = \operatorname{sech}(v_1 + i\theta)$ to open out the strip into the lower half ζ_1-plane, in which the real axis corresponds to the whole of the boundary $OBDB'$ of the region in which ϕ is harmonic. Points on the section OA of the interface have the co-ordinate $\xi_1 = s/\gamma_1$ on the segment $0 < \xi_1 < 1$ of the real axis.

The derivative $dW^\phi/d\zeta_1$, must satisfy the following conditions:

i) It must be regular in the lower half ζ-plane with singularities restricted to the real axis, at the points $\zeta_1 = 0, \pm 1, \pm M$ which correspond either to singularities of load, or to points where $dW^\phi/d\zeta_1$ has sudden changes in argument due to changes in ϕ, A and C from hyperbolic to elliptic régimes and also at any points S, S' where simple poles associated with surface waves may appear.

ii) It must be imaginary on the real axis when $|\zeta_1| > 1$ because $\partial\phi/\partial\theta$ vanishes on both sides of the characteristic surface $s = \gamma_1$.

iii) On the segment $M < \xi_1 < 1$ we may write (with $\gamma_1/a = L < 1$)

$$\frac{dW^\phi}{d\zeta_1} = iF(\zeta_1) \left\{ \left[\frac{\zeta_1^2}{M^2}(1 - L^2M^2) - 2 \right]^2 - 4i(1 + L^2\zeta_1^2)(1 - \zeta_1^2)^{\frac{1}{2}} \left(\frac{\zeta_1^2}{M^2} - 1 \right)^{\frac{1}{2}} \right\}^{-1} \tag{25}$$

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where for $M < \xi_1 < 1$, $F(\xi_1)$ is to be a real function. This result follows from equation 24.

iv) The point at infinity in the ζ_1-plane corresponds to the point D which is an ordinary point, where velocities, displacements and stresses are bounded. These conditions imply bounded potentials at infinity.

v) At the origin the magnitude and nature of the singularity is determined by the nature of the moving load. This will be discussed later.

The problem is now one of finding a function $F(\zeta_1)$ on the segment $M < \xi_1 < 1$ which when continued analytically into the whole of the lower half ζ_1-plane satisfies all the conditions of the problem including the interaction relations on the segment $0 < \xi_1 < M$ corresponding to the section OA of the interface.

In the region $A'EA$, of course, A and C are harmonic functions. Hence, we introduce the mapping $\zeta_2 = \operatorname{sech}(v_2 + i\theta)$ which opens out the strip of regularity so that every point in the physical region $s < \gamma_2$, $0 > \theta > -\pi$ has a corresponding point in the lower half of the complex ζ_2-plane. For the fourth quadrant interaction between ϕ and A or C occurs only at the interface OA for which $0 < \zeta_2 < 1$ and where we have the identity $M\zeta_2 = \zeta_1$.

The interaction conditions 22 become for $0 < \zeta_2 < 1$

$$(26) \quad 0 = Rl\left\{\frac{dW^A}{d\zeta_2} + LM\zeta_2\frac{dW^C}{d\zeta_2}\right\} = Rl\{iP_1(\zeta_2)\}, \quad \text{say,}$$

$$(27) \quad \begin{aligned} 0 &= Rl\left\{[\zeta_2^2(1-L^2M^2)-2]\frac{dW^\phi}{d\zeta_1} + 2i(1-\zeta_2^2)^{\frac{1}{2}}\left[L\zeta_2\frac{dW^A}{d\zeta_2} - \frac{M}{1}\frac{dW^C}{d\zeta_2}\right]\right\} \\ &= Rl[i(1+L^2M^2\zeta_2^2)P_2(\zeta_2)], \quad \text{say,} \end{aligned}$$

$$(28) \quad \begin{aligned} 0 &= Rl\left\{2iM(1-M^2\zeta_2^2)^{\frac{1}{2}}\frac{dW^\phi}{d\zeta_1} + [\zeta_2^2(1-L^2M^2)-2]\frac{dW^C}{d\zeta_2}\right\} \\ &= Rl\{iP_3(\zeta_2)\}, \quad \text{say.} \end{aligned}$$

The real functions P_1, P_2, P_3 defined above are, for $0 < \zeta_2 < 1$ three linear combinations of the three complex derivatives. From these equations we may write that for $0 < \zeta_1 < 1$

$$(29) \quad \frac{dW^\phi}{d\zeta_1} = \frac{\left[\frac{\zeta_1^2}{M^2}(1-L^2M^2)-2\right]\left\{(1+L^2\zeta_1^2)P_2 i - 2\left(1-\frac{\zeta_1^2}{M^2}\right)^{\frac{1}{2}}\frac{L\zeta_1}{M}P_1\right\} - \frac{2P_3}{M}(1+L^2\zeta_1^2)\left(1-\frac{\zeta_1^2}{M^2}\right)^{\frac{1}{2}}}{R(\zeta_1)}$$

where

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$$R(M\zeta_2) = [\zeta_2^2(1-L^2M^2)-2]^2-4(1-M^2\zeta_2^2)^{\frac{1}{2}}(1-\zeta_2^2)^{\frac{1}{2}}(1+L^2M^2\zeta_2^2).$$

And in order that both 29 and 25 should represent the continuation of the same expressions around the branch points at $\zeta_1 = +M$ and at $\zeta_1 = +1$ and in order to satisfy condition *ii* above we can assert that P_1, P_2, P_3 must be real functions not only as defined for $0 < \zeta_1 < M$ but also for $\zeta_1 > M$. Similarly we have the result that for $0 < \zeta_2 < 1$

$$(30)\quad \frac{dW^C}{d\zeta_2} = \frac{-2iM(1-M^2\zeta_2^2)^{\frac{1}{2}}[iP_2(1+L^2M^2\zeta_2^2)+2L\zeta_2P_1(1-\zeta_2^2)+iP_3[\zeta_2^2(1-L^2M^2)-2]}{R(M\zeta_2)}.$$

We have an immediate reason for putting $P_1 \equiv 0$. The tangential derivatives $\partial C/\partial\theta$ and $\partial A/\partial\theta$ are constant on the tangent planes in the head wave region ABC and are continuous across the arc AC. There is thus a linkage between the tangential derivatives of A and C just within the elliptic region and, from the first and third of equations 22, the normal derivative of ϕ on AB. The knowledge that P_1, P_2 and P_3 are real functions of ζ_2 on AB enables us to make this linkage without any further analysis, and it is then found that P_1 must vanish. This simplifies equations 29 and 30 somewhat.

The corresponding expression for A is given by

$$(31)\quad \frac{dW^A}{d\zeta_2} = -LM\zeta_2\left\{\frac{iP_3[\zeta_2^2(1-L^2M^2)-2]+2MP_2(1-M^2\zeta_2^2)^{\frac{1}{2}}[1+L^2M^2\zeta_2^2]}{R(M\zeta_2)}\right\}.$$

The functions P_2 and P_3 may be determined by considering the behavior at the origin and at infinity. At the origin, for example, the functions P_2 and P_3 must be of (positive or negative) integral order in ζ. This follows because our analysis making P_2 and P_3 real for positive ζ is applicable with the same result to the negative ζ-axis. Thus the behavior at the origin can only be responsible for jumps, in the arguments of the complex derivatives, of $\pm n\pi$, where n is any integer.

In effect, therefore, since the functions P_2 and P_3 are real on the real axis, with singularities at the origin and at infinity, we may write both P_2 and P_3 in a real Laurent expansion of powers of ζ. The highest power of ζ is determined by the need to have bounded displacements as $\zeta \to \infty$. And although any one of the individual terms in the two series is associated with a conical field due to some particular form of singular load, yet we shall restrict interest to the case when $P_2 = A_0/\zeta_2$ and $P_3 = MA_1/\zeta_2$ with A_0 and A_1 constant. These give velocities which can be linked with a singular load in the vicinity of the z-axis, preserving bounded displacements and velocities as $\zeta \to \infty$.

The explicit formulae to be examined are

$$
\left.
\begin{aligned}
&\frac{dW^{\phi}}{d\zeta_1} = \frac{(1+L^2\zeta_1^2)M}{R(\zeta_1)\zeta_1}\left\{iA_0\left[\frac{\zeta_1^2}{M^2}(1-L^2M^2)-2\right]-A_1\left(1-\frac{\zeta_1^2}{M^2}\right)^{\frac{1}{2}}\right\}\\
(32)\quad &\frac{dW^{C}}{d\zeta_2} = \frac{\{2(1-M^2\zeta_2^2)^{\frac{1}{2}}(1+L^2M^2\zeta_2^2)A_0+iA_1[\zeta_2^2(1-L^2M^2)-2]\}}{\zeta_2 R(M\zeta_2)}\\
&\text{and}\\
&\frac{dW^{A}}{d\zeta_2} = -LM\zeta_2\frac{dW^{C}}{d\zeta_2}.
\end{aligned}
\right\}
$$

These particular forms of complex derivative are displayed because they may be associated with simple strain singularities. The method of investigation of their nature is not of great interest, because it involves first the expansion of the expression 32 in power series of ζ, as $\zeta \to 0$, then the derivation of the potentials ϕ, A and C for small values of $s = r/\tau$, and finally the calculation of the local stresses and velocities in the vicinity of the axis $r = 0$.

It is found that the constants A_0 and A, are associated respectively with rotational and dilatational singularities which are restricted to the semi-infinite segment $z < at$ of the line $r = 0$. The imposed singular load is one which produces a volume flux P and a circulation C about the z-axis in its immediate vicinity. Thus

$$
A_0 = -\frac{2C}{\pi} = -\frac{2}{\pi}\lim_{r\to 0}\int_{-\pi}^{0} rU_\theta\, d\theta
$$

$$
A_1 = +\frac{2P}{\pi} = \frac{2}{\pi}\lim_{r\to 0}\int_{-\pi}^{0} rU_r d\theta,
$$

U_r and U_θ being the radial and transverse velocity components.

The formulae of Equation 32 may be used to find the space and time derivatives of the various potentials ane thence the velocity and stress components everywhere. Thus

$$
(33)\quad \frac{\partial\phi}{\partial\theta} = Rl\left[i\frac{dW^{\phi}}{d\zeta_1}\zeta_1(1-\zeta_1^2)^{\frac{1}{2}}\right]
$$

$$
= Rl\left\{\frac{iM(1+L^2\zeta_1^2)(1-\zeta_1^2)^{\frac{1}{2}}\left\{iA_0\left[\frac{\zeta_1^2}{M^2}(1-L^2M^2)-2\right]-A_1\left(1-\frac{\zeta_1^2}{M^2}\right)^{\frac{1}{2}}\right\}}{R(\zeta_1)}\right\},
$$

and

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$$(34)\quad r\frac{\partial\phi}{\partial\tau} = -\frac{r}{s}\frac{\partial\phi}{\partial\tau} = \frac{\partial\phi}{\partial s} = Rl\left[\frac{dW^{\phi}}{d\zeta_1}\frac{\partial\zeta_1}{\partial v_1}\frac{dv_1}{ds}\right]$$

$$= Rl\left\{\frac{M(1+L^2\zeta_1^2)(1-\zeta_1^2)^{\frac{1}{2}}\left\{iA_0\left[\frac{\zeta_1^2}{M^2}(1-L^2M^2)-2\right]-A_1\left(1-\frac{\zeta_1^2}{M^2}\right)^{\frac{1}{2}}\right\}}{R(\zeta_1)S(1-s^2/c_1^2)^{\frac{1}{2}}}\right\},$$

for points inside the dilatation cone for which

$$\zeta_1 = \frac{r}{\gamma_1\tau\cos\theta + i\sin\theta\,(\gamma_1^2\tau^2 - r^2)^{\frac{1}{2}}}$$

while

$$(35)\quad \frac{\partial}{\partial\theta}(Ai+Ck) = Rl\left\{i\zeta_2(1-\zeta_2^2)^{\frac{1}{2}}[-LM\zeta_2 i + k]\frac{dW^C}{d\zeta_2}\right\}$$

$$= Rl\left\{\frac{i(1-\zeta_2^2)^{\frac{1}{2}}\{2(1-M^2\zeta_2^2)^{\frac{1}{2}}(1+L^2M^2\zeta_2^2)A_0 + iA_1[\zeta_2^2(1-L^2M^2)-2]\}}{R(M\zeta_2)}\right\}$$

and

$$(36)\quad \tau\frac{\partial}{\partial r}(Ai+Ck) = -\frac{\tau}{s}\frac{\partial}{\partial\tau}(Ai+Ck) = Rl\left\{(-LM\zeta_2 i+k)\frac{dW^C}{d\zeta_2}\frac{\partial\zeta_2}{\partial v_2}\frac{dv_2}{ds}\right\}$$

$$= Rl\left\{\frac{(1-\zeta_2^2)^{\frac{1}{2}}\{2(1-M^2\zeta_2^2)^{\frac{1}{2}}(1+L^2M^2\zeta_2^2)A_0 + iA_1[\zeta_2^2(1-L^2M^2)-2]\}[-LM\zeta_2 i+k]}{s(1-s^2/c_2^2)^{\frac{1}{2}}R(M\zeta_2)}\right\}$$

for points inside the shear cone for which

$$\zeta_2 = \frac{r}{\gamma_2\tau\cos\theta + i\sin\theta(\gamma_2^2\tau^2 - r^2)^{\frac{1}{2}}}.$$

There remains the formula for the shear field in the head wave region ABC. This may be found from the equation 22 which relates C_s and ϕ_θ by first determining $C_s = -aA_s/s$ on the free surface on the segment AB, where $\zeta_1 = s/\gamma_1$, and then continuing C or A into the interior of the region by using the functional form 23. For C this gives the result that

$$\frac{\partial C}{\partial u_2} = -\frac{\partial C}{\partial\theta} =$$

$$\frac{-2M\left[\frac{s^2}{\gamma_2^2}(1-L^2M^2)-2\right]^2\left(1+\frac{s^2}{a^2}\right)\left(1-\frac{s^2}{\gamma_1^2}\right)^{\frac{1}{2}}\left\{A_0\left[\frac{s^2}{\gamma_2^2}(1-L^2M^2)-2\right]+A_1\left(\frac{s^2}{\gamma_2^2}-1\right)^{\frac{1}{2}}\right\}}{\left\{\left[\frac{s^2}{\gamma_2^2}(1-L^2M^2)-2\right]^4+16\left(1-\frac{s^2}{\gamma_1^2}\right)\left(\frac{s^2}{\gamma_2^2}-1\right)\left(1+\frac{s^2}{a^2}\right)^2\right\}s^2\left(\frac{s^2}{\gamma_2^2}-1\right)^{\frac{1}{2}}}$$

on the surface AB with $\gamma_1 \geqq s = \gamma_2\sec u_2 \geqq \gamma_2$, with the same expression with $s = \sec(u_2-\theta)$ providing the field inside the region ABC.

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Notice the singular nature of the derivatives given here for points just outside the arc AC. This implies a logarithmic singularity in the shear potentials just outside the shear cone in the head wave region such that $C = O[\ln(\operatorname{arcsec} r/\gamma_2 \tau)] = A$ for $r = \gamma_2 \tau + 0$. Notice also that although the process for evaluating the derivatives of the potentials and hence the velocity components does not involve any but the simplest arithmetical processes, the calculation of these quantities is particularly easy on the free surface. Indeed, one result of particular interest is that apart from the surface wave associated with the vanishing of the function $R(\zeta_1)$ (this will be mentioned later) the source of pure dilatation (with $C \equiv A_0 \equiv 0$) provides only horizontal velocities and displacements on the segment $A'A$ of the surface while on the contrary, the source of pure circulation provides only vertical velocities and displacements on the same segment. On the other hand both types of source cause both horizontal and vertical displacements on the segments AB and $A'B'$.

4. Extension of Results

We have derived certain expressions for complex potentials which are associated with moving line loads of step function dependence in the variable $(at - z)/a$. It is clear enough that simple superposition is possible so that for example, we may easily calculate the effect of a longitudinally moving line load of finite length.

To obtain a more fundamental result we may differentiate throughout with respect to τ. The result of this operation gives explicitly the potentials associated with a steadily moving point load. For example, within the moving dilatation cone $r = \gamma_1(at - z)/a$ the scalar potential ϕ corresponding to a moving point source of strength P and which creates a circulation C, is

$$(37) \qquad \frac{-\gamma_1}{a[\gamma_1^2(at-z)^2 - a^2t^2]^{\frac{1}{2}}} Rl\{\zeta_1(1-\zeta_2^2)dW^{\phi}/d\zeta_1$$

where $dW^{\phi}/d\zeta_1$ is the expression given explicitly in Equations 32 and 32a. The shear potentials both outside and inside the shear cone may be found similarly.

More general superposition is feasible by use of the convolution theorem once the results for point loads are established.

5. The Rayleigh Wave

It is obvious that in the limit $L = 0$ the problem is reduced to the well-known two dimensional case of a suddenly applied infinite uniform line load.

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In the expressions for complex potentials given above this limit is a valid one, with the transverse component A of the vector potential vanishing, and with the potentials ϕ and C taking more recognizable forms.

In this strictly two dimensional case the zeros of the function $R_0(M\zeta_2)$, where

$$R_0(M\zeta_2) = (\zeta_2^2 - 2)^2 - 4(1 - \zeta_2^2)^{\frac{1}{2}}(1 - M^2\zeta_2^2)^{\frac{1}{2}}$$

and where $M = c_2/c_1$, define poles $[\zeta_2 = \pm V_R/c_2]$ of the complex derivatives of the potentials W^ϕ and W^c, and hence singularities of the velocity and stress field travelling with the specific Rayleigh velocity V_R. (V_R takes values lying in the range $0.874 \cdots < V_R/c_2 < 0.965 \cdots$ according to Rayleigh [13]).

The corresponding expression $R(M\zeta_2)$ defined in equation 29 is easily shown to have its zeros at

$$\zeta_2 = \pm \gamma_R/\gamma_2$$

where

$$\frac{1}{\gamma_R^2} = \frac{1}{V_R^2} - \frac{1}{a^2}.$$

The quantity γ^R thus defines the transverse velocity of the usual Rayleigh waves, which having an actual velocity of propagation V_R, also form a wedge singularity on the surface within the shear cone.

The residue terms which appear in conjunction with the Rayleigh zeros are interesting in that the pure dilatation source provides a residue only for the vertical velocity and displacement. Its effect is to provide a Rayleigh wave of step function type in the displacement which just counteracts or nullifies the vertical displacement which is produced in the head wave region by such a source. This, of course, ensures that the vertical displacement returns to zero after the passage of the Rayleigh wave and stays that way, whereas the horizontal component which should approach zero long after the disturbing load has passed by does so in a more gradual manner. For the source of circulation, the same sort of results are available, the horizontal displacement being the one which returns to zero with the passage of the Rayleigh wave, while the vertical displacement vanishes more smoothly. These properties are of course only true for the particular type of source under consideration. For a moving load of variable strength, they will not be true, nor are they true if the source of energy is a line force.

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https://doi.org/10.1017/S1446788700027658 Published online by Cambridge University Press