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COMPACT ELEMENTS AND OPERATORS OF QUANTUM GROUPS

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Abstract. A locally compact group G is compact if and only if its convolution algebras contain non-zero (weakly) completely continuous elements. Dually, G is discrete if its function algebras contain non-zero completely continuous elements. We prove non-commutative versions of these results in the case of locally compact quantum groups.

2000 *Mathematics Subject Classification.* Primary 46L89, 47L10, 43A60, Secondary 22D25, 46L51, 43A22, 43A20.

1. Introduction. A classical result of Sakai [**26**] states that a locally compact group G is compact if and only if it admits (weakly) compact convolution operators. Dually, G is discrete if and only if it admits (weakly) compact multiplication operators on $L^2(G)$.

In this paper, we investigate these connections in the case of locally compact quantum groups.

It is known that for a locally compact group G the following are equivalent: (i) G is discrete, (ii) $L^\infty(G)$ contains a compact operator on $L^2(G)$, (iii) $L^1(G)$ has the

Radon–Nikodym property, (iv) the von Neumann algebra $L^\infty(G)$ is purely atomic (cf. [**7**] and [**30**]).

In the more general setting of locally compact quantum groups, (i) implies other properties, (iii) and (iv) are equivalent, and there are examples that satisfy (iii) but not (i) (cf. [**30**] and [**14**]).

The main result of [**13**] extends the implication (ii) $\Rightarrow$ (i) to the case of regular locally compact quantum group. We remove the regularity condition, and prove this result for all locally compact quantum groups.

THEOREM. *A locally compact quantum group $\mathbb{G}$ is discrete if and only if $L^\infty(\mathbb{G}) \cap K(L^2(\mathbb{G})) \neq \{0\}$, and in this case we have $C_0(\mathbb{G}) = L^\infty(\mathbb{G}) \cap K(L^2(\mathbb{G}))$.*

We use this theorem to establish an equivalence between discreteness of $\mathbb{G}$ and existence of (weakly) compact elements in $C_0(\mathbb{G})$ or $L^\infty(\mathbb{G})$. Consequently, we show G is discrete if and only if $L^\infty(\mathbb{G})$ has a finite dimensional direct summand.

THEOREM. *Let $\mathbb{G}$ be a locally compact quantum group. If the von Neumann algebra $L^\infty(\mathbb{G})$ has a decomposition $L^\infty(\mathbb{G}) \cong M \oplus N$ where M is finite dimensional C^*-algebra, then $\mathbb{G}$ is a discrete quantum group.*

The significance of this result is that it characterizes discreteness only in terms of the von Neumann algebra structure of $L^\infty(\mathbb{G})$, without referring to the quantum group structure of $\mathbb{G}$.

Dually, compactness is characterized in terms of (weak) compactness of convolution operators: for a locally compact group G the following are equivalent: (i) G is compact, (ii) there exists a non-zero measure $\mu \in M(G)$ such that the convolution map $L^1(G) \ni \nu \mapsto \mu * \nu \in L^1(G)$ is compact, (iii) there exists a non-zero measure $\mu \in M(G)$ such that the convolution map $L^1(G) \ni \nu \mapsto \mu * \nu \in L^1(G)$ is weakly compact [**26**].

We prove quantum versions of these equivalences.

THEOREM. *For a locally compact quantum group $\mathbb{G}$ the following are equivalent:*

(1) *$\mathbb{G}$ is compact;*
(2) *there exists $\mu \in M(\mathbb{G})$ such that the convolution map $\nu \mapsto \mu \star \nu$ on $L^1(\mathbb{G})$ is compact and its image contains an invertible element;*
(3) *there exists $\mu \in M(\mathbb{G})$ such that the convolution map $\nu \mapsto \mu \star \nu$ on $L^1(\mathbb{G})$ is weakly compact and its image contains an invertible element.*

In particular, if the convolution operator of a non-zero positive quantum measure $\mu \in M(\mathbb{G})^+$ is weakly compact, then $\mathbb{G}$ is compact. We are not able to remove the requirement on the image in the above theorem entirely, but we prove:

THEOREM. *A locally compact quantum group $\mathbb{G}$ is compact if and only if there is $\mu \in M(\mathbb{G})$ such that the convolution map $\nu \mapsto \mu \star \nu$ on $L^1(\mathbb{G})$ has finite rank.*

Denote by $\tilde{\mathbb{G}}$ the group of *quantum point masses* of a co-amenable locally compact quantum group $\mathbb{G}$, i.e. the spectrum of $C_0(\mathbb{G})$ (cf. [**15**]). Another generalization of Sakai's result is formulated as follows.

THEOREM. *Let $\mathbb{G}$ be a co-amenable locally compact quantum group. If there is a non-zero $\mu \in M(\mathbb{G})$ such that the convolution operator $M(\mathbb{G}) \ni \omega \mapsto \mu \star \omega \in M(\mathbb{G})$ is compact, then $\tilde{\mathbb{G}}$ is compact.*

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Weakly compact multipliers on the second dual $L^1(\mathbb{G})^{**}$ (endowed with the Arens product) have been studied in **[10]** and **[19]**, where the authors prove that with the left Arens product, $L^1(\mathbb{G})^{**}$ admits a weakly compact left multiplier if and only if G is amenable, and it admits a weakly compact left multiplier if and only if G is compact.

We prove the following generalization to the case of locally compact quantum groups.

THEOREM. *A locally compact quantum group* $\mathbb{G}$ *is compact if and only if there is a weakly compact right multiplier (equivalently, left multiplier)* T *of* $L^1(\mathbb{G})^{**}$*, and* $m \in L^1(\mathbb{G})^{**}$ *such that* $T(m) \in L^1(\mathbb{G})$ *and* $\langle T(m), 1\rangle \neq 0$.

We also consider the canonical maps from convolution algebras into function algebras. More precisely, we study almost periodic elements of a locally compact quantum group as introduced and studied in **[27]** and **[5]**. We show that $\mathbb{G}$ is compact if and only if $\mathcal{AP}(C_0(\mathbb{G})) \cap C_0(\mathbb{G}) \neq \{0\}$. This generalizes a similar classical result **[3]**.

2. Preliminaries. In this section, we introduce our notation and terminology, and recall some results on locally compact quantum groups that we will be using throughout the paper. For more details on locally compact quantum groups the reader is referred to **[16, 17]**.

A locally compact quantum group $\mathbb{G}$ in the sense of Kustermans–Vaes is a quadruple $\mathbb{G} = (L^\infty(\mathbb{G}), \Delta, \varphi, \psi)$, where $L^\infty(\mathbb{G})$ is a von Neumann algebra, $\Delta : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})\bar{\otimes}L^\infty(\mathbb{G})$ is a co-associative co-multiplication, and φ and ψ are the left, respectively, right Haar weights. The pre-adjoint of the co-multiplication Δ induces an associative multiplication $\star : L^1(\mathbb{G})\hat{\otimes}L^1(\mathbb{G}) \to L^1(\mathbb{G})$ on the predual $L^1(\mathbb{G}) := L^\infty(\mathbb{G})_*$, which we call the *(quantum) convolution product*; here, $\hat{\otimes}$ is the operator space projective tensor product. Moreover, $(L^1(\mathbb{G}), \star)$ forms a completely contractive Banach algebra.

In the classical case of $L^\infty(G)$ or $VN(G)$ with G a locally compact group, the algebra $(L^1(\mathbb{G}), \star)$ is the usual convolution group algebra $L^1(G)$ and the Fourier algebra $A(G)$, respectively.

The convolution on $L^1(\mathbb{G})$ induces a canonical completely contractive $L^1(\mathbb{G})$-bimodule structure on $L^\infty(\mathbb{G})$ satisfying

$$x \star f = (f \otimes \mathrm{id})\Delta(x) \ \text{ and } \ f \star x = (\mathrm{id} \otimes f)\Delta(x) \qquad (x \in L^\infty(\mathbb{G}), f \in L^1(\mathbb{G})).$$

The quantum group $\mathbb{G}$ is said to be *co-amenable* if $L^1(\mathbb{G})$ has a bounded approximate identity.

Let $C_0(\mathbb{G})$ be the reduced C^*-algebra associated with $\mathbb{G}$ and let $M(C_0(\mathbb{G}))$ be the multiplier algebra of $C_0(\mathbb{G})$. Then, we have the inclusions $C_0(\mathbb{G}) \subseteq M(C_0(\mathbb{G})) \subseteq L^\infty(\mathbb{G})$, and $C_0(\mathbb{G})$ is a weak* dense C^*-subalgebra of $L^\infty(\mathbb{G})$.

A locally compact quantum group $\mathbb{G}$ is *compact* if $1 \in C_0(\mathbb{G})$, and it is *discrete* if the dual quantum group $\hat{\mathbb{G}}$ of $\mathbb{G}$ is compact, which is equivalent to $L^1(\mathbb{G})$ being unital. It is known that when $\mathbb{G}$ is discrete we have

$$\ell^\infty(\mathbb{G}) = \bigoplus_{\alpha \in I} M_{n_\alpha}, \tag{1}$$

where I is the set of all equivalence classes of irreducible unitary representations of the dual quantum group $\mathbb{G}$, and each M_{n_α} is a matrix algebra (cf. **[8]**).

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The co-multiplication Δ maps $C_0(\mathbb{G})$ into the multiplier algebra $M(C_0(\mathbb{G}) \otimes_{\min} C_0(\mathbb{G}))$. Then $M(\mathbb{G}) := C_0(\mathbb{G})^*$ is a completely contractive dual Banach algebra under the multiplication

$$\langle \mu \star \nu, x \rangle = \langle \mu \otimes \nu, \Delta(x) \rangle \qquad (\mu, \nu \in C_0(\mathbb{G})^*, x \in C_0(\mathbb{G})).$$

Similarly to the classical case $L^1(\mathbb{G})$ is canonically identified with a closed two-sided ideal in $M(\mathbb{G})$.

We denote by $\hat{\mathbb{G}}$ the dual quantum group, and by $W \in L^\infty(\mathbb{G})\bar{\otimes}L^\infty(\hat{\mathbb{G}})$ and $V \in L^\infty(\hat{\mathbb{G}})'\bar{\otimes}L^\infty(\mathbb{G})$ the left and right fundamental unitaries of $\mathbb{G}$, respectively. The *left regular representation* is a completely contractive injection defined by

$$\lambda : f \in L^1(\mathbb{G}) \mapsto (f \otimes \mathrm{id})(W) \in C_0(\hat{\mathbb{G}}) \subseteq L^\infty(\hat{\mathbb{G}})$$

Moreover, λ has a canonical (weak* continuous) extension to a completely contractive algebra homomorphism $M(\mathbb{G}) \to M(C_0(\hat{\mathbb{G}})) \subseteq L^\infty(\hat{\mathbb{G}})$, still denoted by λ, given by $\langle \lambda(\mu), \hat{f} \rangle = \langle \mu, \lambda_*(\hat{f}) \rangle$, where $\lambda_* : L^1(\hat{\mathbb{G}}) \to C_0(\mathbb{G}) \subseteq L^\infty(\mathbb{G})$ is the completely contractive injection $\hat{f} \mapsto (\mathrm{id} \otimes \hat{f})(W)$. We have $\overline{\lambda(L^1(\mathbb{G}))}^{\|\cdot\|} = C_0(\hat{\mathbb{G}})$. Similarly, the right regular representation is defined by $\rho : f \in L^1(\mathbb{G}) \mapsto (\mathrm{id} \otimes f)(V) \in L^\infty(\hat{\mathbb{G}})'$.

A locally compact quantum group $\mathbb{G}$ is *amenable* if there exists a left invariant mean on $L^\infty(\mathbb{G})$, i.e. a state $m \in L^\infty(\mathbb{G})^*$ such that $m(f \otimes \mathrm{id})\Delta = \langle 1, f \rangle\, m$ for all $f \in L^1(\mathbb{G})$. Right, and two-sided invariant means are defined similarly.

In fact, both the existence of right invariant means, and two-sided invariant means are equivalent to amenability of $\mathbb{G}$.

We define

$$LUC(\mathbb{G}) = \langle L^\infty(\mathbb{G}) \star L^1(\mathbb{G}) \rangle \quad \text{and} \quad RUC(\mathbb{G}) = \langle L^1(\mathbb{G}) \star L^\infty(\mathbb{G}) \rangle.$$

We then have [**25**] the inclusions

$$C_0(\mathbb{G}) \subseteq LUC(\mathbb{G}) \subseteq M(C_0(\mathbb{G})).$$

A bounded linear map μ on a Banach algebra A is called a right multiplier if $\mu(ab) = a\mu(b)$ for all $a, b \in A$. We denote by $RM(A)$ the algebra of right multiplier maps on A. When A is a completely contractive Banach algebra, $RM_{cb}(A)$ will denote the algebra of completely bounded maps in $RM(A)$.

The left and completely bounded left multiplier algebras of A are defined similarly, and are denoted by $LM(A)$ and $LM_{cb}(A)$, respectively. For every $a \in A$ we define the multiplication maps $\mathfrak{L}_a$ and $\mathfrak{R}_a$ on A by $\mathfrak{L}_a(x) = ax$, and $\mathfrak{R}_a(x) = xa$. Then $\mathfrak{L}_a \in LM(A)$ and $\mathfrak{R}_a \in RM(A)$.

We say that $a \in A$ is a left (respectively, right) completely finite rank element of A if $\mathfrak{L}_a$ (respectively, $\mathfrak{R}_a$) is of finite rank; $a \in A$ is a completely finite rank element if it is both left and right completely finite rank element, i.e. both the right ideal aA and the left ideal Aa are finite-dimensional. We denote by A^{lcf} and A^{rcf} the set of all left, respectively right, completely finite rank elements of A. Both A^{lcf} and A^{rcf}, as well as the set A^{cf} of all completely finite rank elements in A are closed ideals of A.

Similarly, we say that $a \in A$ is a left (respectively, right) completely continuous element of A if $\mathfrak{L}_a$ (respectively, $\mathfrak{R}_a$) is a compact operator on A; $a \in A$ is a completely continuous element if it is both left and right completely continuous element. We denote by A^{lcc} and A^{rcc} the ideals of all left, respectively right, completely continuous elements of A, and by A^{cc} the ideal of all completely continuous elements in A.

We say that $a \in A$ is a compact element if $\mathfrak{L}_a\mathfrak{R}_a$ is a compact operator. The set A^{cpt} of all compact elements in A is only a closed multiplicative semigroup in A in general. Finally, we say that a is a finite rank element if $\mathfrak{L}_a\mathfrak{R}_a$ is a finite rank operator. The set of all finite rank elements in A is denoted by A^f.

We say that a is a left (respectively, right) weakly completely continuous element of A if $\mathfrak{L}_a$ (respectively, $\mathfrak{R}_a$) is a weakly compact operator on A. The set of all left (respectively, right) weakly completely continuous elements of A is denoted by A^{lwcc} (respectively, A^{rwcc}). Again, both A^{lwcc} and A^{rwcc} are closed ideals of A. In the case of a C^*-algebra, these ideals are thus self-adjoint, and since the involution is weakly continuous it follows $A^{lwcc} = A^{rwcc}$. Therefore we shall simply write A^{wcc} to denote the set of (left, or equivalently right) weakly completely continuous elements.

If $A = B(H)$, then $a \in A^{cpt}$ if and only if $a \in K(H)$. For a C^*-algebra A, it is known that $A^{cpt} = A^{wcc}$ (cf. [**33**]). Also, there is a faithful representation π of A on a Hilbert space H such that, for $a \in A$, we have $a \in A^{cpt}$ if and only if $\pi(a) \in K(H)$ (cf. [**32**]).

For a locally compact quantum group $\mathbb{G}$, $M(\mathbb{G})$ is embedded in $RM_{cb}(L^1(\mathbb{G}))$ via $\mu \mapsto \mathfrak{R}_\mu$, where $\mathfrak{R}_\mu : f \in L^1(\mathbb{G}) \mapsto f \star \mu \in L^1(\mathbb{G})$.

We denote by $L^2(\mathbb{G})$ the GNS Hilbert space of the right Haar weight, which is canonically identified with the GNS Hilbert space of the left Haar weight. Then, the right fundamental unitary V of $\mathbb{G}$ induces on $B(L^2(\mathbb{G}))$ a co-associative co-multiplication

$$\Delta^r : B(L^2(\mathbb{G})) \to B(L^2(\mathbb{G}))\bar{\otimes}B(L^2(\mathbb{G})) \quad ; \quad x \mapsto V(x \otimes 1)V^*,$$

such that the restriction of Δ^r to $L^\infty(\mathbb{G})$ is just the co-multiplication Δ on $L^\infty(\mathbb{G})$. The pre-adjoint of Δ^r defines on $T(L^2(\mathbb{G})) := B(L^2(\mathbb{G}))_*$ an associative completely contractive multiplication

$$\triangleright : T(L^2(\mathbb{G}))\hat{\otimes}T(L^2(\mathbb{G})) \to T(L^2(\mathbb{G})), \omega \otimes \gamma \mapsto \omega \triangleright \gamma = (\omega \otimes \gamma) \circ \Delta^r.$$

Analogously, the left fundamental unitary W of $\mathbb{G}$ induces the co-multiplication

$$\Delta^l : B(L^2(\mathbb{G})) \to B(L^2(\mathbb{G}))\bar{\otimes}B(L^2(\mathbb{G})) \quad ; \quad x \mapsto W^*(1 \otimes x)W,$$

such that the restriction of Δ^l to $L^\infty(\mathbb{G})$ is just the co-multiplication Δ on $L^\infty(\mathbb{G})$. Similarly, the pre-adjoint of Δ^l defines on $T(L^2(\mathbb{G}))$ the multiplication

$$\triangleleft : T(L^2(\mathbb{G}))\hat{\otimes}T(L^2(\mathbb{G})) \to T(L^2(\mathbb{G})), \omega \otimes \gamma \mapsto \omega \triangleleft \gamma = (\omega \otimes \gamma) \circ \Delta^l.$$

For $\omega \in T(L^2(\mathbb{G}))$ we use $\mathfrak{R}^\triangleright_\omega$, $\mathfrak{R}^\triangleleft_\omega$, $\mathfrak{L}^\triangleright_\omega$, and $\mathfrak{R}^\triangleleft_\omega$ to distinguish the multiplication operators in the corresponding product.

3. Compact elements and discreteness. A locally compact group G is discrete if and only if $L^\infty(G)$ contains a compact operator (on $L^2(G)$). In the quantum setting, if $\mathbb{G}$ is a discrete quantum group, then $c_0(\mathbb{G}) \subset K(\ell^2(\mathbb{G}))$. But the converse has only been partially proved. In [**11**] it was shown that if $C_0(\mathbb{G}) \subseteq K(L^2(\mathbb{G}))$ then $\mathbb{G}$ is discrete. Then the second author proved in [**13**] that if $\mathbb{G}$ is a regular locally compact quantum group, and $L^\infty(\mathbb{G}) \cap K(L^2(\mathbb{G})) \neq \{0\}$, then $\mathbb{G}$ is discrete.

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We remove the (restrictive) regularity condition and prove the above result for general locally compact quantum groups.

THEOREM 3.1. *A locally compact quantum group $\mathbb{G}$ is discrete if and only if $L^\infty(\mathbb{G}) \cap K(L^2(\mathbb{G})) \neq \{0\}$, and in this case, $C_0(\mathbb{G}) = L^\infty(\mathbb{G}) \cap K(L^2(\mathbb{G}))$.*

Proof. The forward implication is obvious. For the converse, suppose $L^\infty(\mathbb{G}) \cap K(L^2(\mathbb{G})) \neq \{0\}$. Since then $L^2(\mathbb{G})$ is the standard Hilbert space of $L^\infty(\mathbb{G})$, it follows that $L^\infty(\mathbb{G})' \cap K(L^2(\mathbb{G})) \neq \{0\}$. Take a non-zero $x \in L^\infty(\mathbb{G})' \cap K(L^2(\mathbb{G}))$, and choose $\omega \in T(L^2(\mathbb{G}))$ such that $\omega(x) \neq 0$. Then using [**11**, Theorem 3.1], we obtain

$$(id \otimes \omega)(\hat{W}^*(1 \otimes x)\hat{W}) = \omega \,\hat{\triangleleft}\, x = \omega(x)1 \in C_0(\hat{\mathbb{G}}).$$

Thus $1 \in C_0(\hat{\mathbb{G}})$, which yields compactness of the dual quantum group $\hat{\mathbb{G}}$. Hence, $\mathbb{G}$ is discrete. □

It is a well-known classical result that a locally compact group G is discrete if and only if $L^1(G)$ has the Radon-Nikodym property (RNP) (recall a Banach space X is said to have RNP if for each finite measure space (Ω, S, μ) and each bounded linear operator $T : L^1(\Omega, S, \mu) \to X$, there is a bounded μ-measurable function $\phi : \Omega \to X$ such that $Tf = \int_\Omega f\phi\, d\mu$ for all $f \in (L^1(\Omega, S, \mu))$.

This is not true in the general locally compact quantum group setting (cf. [**30**]). Now Theorem 3.1 also allows to remove regularity condition in [**13**, Lemma 6 and Corollary 7] to obtain equivalent conditions that distinguish the RNP from discreteness in the quantum setting.

COROLLARY 3.2. *Let $\mathbb{G}$ be a locally compact quantum group such that $L^\infty(\mathbb{G}) = l^\infty - \bigoplus_{i\in I} B(H_i)$. Then, the following conditions are equivalent:*

(1) *$dim(H_i) < \infty$ for all $i \in I$;*
(2) *$dim(H_i) < \infty$ for some $i \in I$;*
(3) *$dim(H_i) = 1$ for some $i \in I$;*
(4) *$C_0(\mathbb{G}) = c_0 - \bigoplus_{i\in I} K(H_i)$;*
(5) *$\mathbb{G}$ is discrete.*

Next, we characterize discreteness of a quantum group $\mathbb{G}$ in terms of existence of compact elements in $L^\infty(\mathbb{G})$. As mentioned in the previous section, for a Hilbert space H, it is known that $a \in B(H)$ is a compact element if and only if $a \in K(H)$. Moreover if A is an irreducible C^*-subalgebra of $B(H)$, then $a \in A$ is a compact element if and only if $a \in K(H)$ (cf. [**33**]). The irreducibility condition cannot be removed in general (see for example [**28**]).

Clearly $L^\infty(\mathbb{G}) \subseteq B(L^2(\mathbb{G}))$ is not irreducible unless $\mathbb{G}$ is trivial. Therefore, in light of Theorem 3.1 it is natural to ask whether discreteness of $\mathbb{G}$ can be detected via existence of compact elements in $L^\infty(\mathbb{G})$.

THEOREM 3.3. *Let $\mathbb{G}$ be a locally compact quantum group. Then the following are equivalent:*

(1) $C_0(\mathbb{G}) = C_0(\mathbb{G})^{cpt}$
(2) $C_0(\mathbb{G}) \subseteq L^\infty(\mathbb{G})^{cpt}$
(3) $L^\infty(\mathbb{G})^{lcc} \neq \{0\}$
(4) $L^\infty(\mathbb{G})^{lcf} \neq \{0\}$
(5) *$\mathbb{G}$ is discrete.*

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Proof. The implications (2) $\Rightarrow$ (1), and (4) $\Rightarrow$ (3) are obvious, and (5) $\Rightarrow$ (4) follows from (1).
The equivalence (1) $\Leftrightarrow$ (5) is a direct consequence of [**23**, Theorem 4.4] and [**22**, Proposition 1.14.3].
(5) $\Rightarrow$ (2): By [**6**, Lemma 2.6] for every $a \in C_0(\mathbb{G})$ and $f \in L^1(\mathbb{G})$ the set

$$\{(aga^*) \star f \ : \ g \in L^1(\mathbb{G}) \text{ and } \|g\| \leq 1\}$$

is relatively compact in $L^1(\mathbb{G})$. Since $\mathbb{G}$ is discrete, $L^1(\mathbb{G})$ has an identity, and therefore it follows that the set $\{aga^* : g \in L^1(\mathbb{G}) \textit{ and } \|g\| \leq 1\}$ is relatively compact in $L^1(\mathbb{G})$ for all $a \in C_0(\mathbb{G})$. This implies that the map $L^1(\mathbb{G}) \ni g \mapsto aga^* \in L^1(\mathbb{G})$ is compact, whence so is the adjoint map $x \mapsto a^*xa$ on $L^\infty(\mathbb{G})$. Thus, by [**28**, Lemma 2.3.12] we conclude $a \in L^\infty(\mathbb{G})^{wcc} = L^\infty(\mathbb{G})^{cpt}$.
(3) $\Rightarrow$ (4): Replacing by a^*a, we may assume that a is hermitian. By [**20**, Theorem 2.1.1] the spectral radius, the norm of a, and the norm of the linear map $x \mapsto ax$ coincide. Therefore, the map $x \mapsto ax$ has a non-zero eigenvalue and a non-zero eigenvector b. Now [**4**, Lemma 4.1] implies that the linear map $x \mapsto bx$ is of finite rank.
(4) $\Rightarrow$ (5): Let $0 \neq a \in L^\infty(\mathbb{G})^{lcf}$. The set of square integrable elements $\mathcal{N}_\varphi = \{x \in L^\infty(\mathbb{G}) : \varphi(x^*x) < \infty\}$ is a left ideal of $L^\infty(\mathbb{G})$ and is dense in $L^2(\mathbb{G})$. In particular, $x \mapsto ax$ maps $\mathcal{N}_\varphi$ into a finite dimensional subspace of $\mathcal{N}_\varphi$. Thus the operator a has finite rank as an operator on the GNS Hilbert space $L^2(\mathbb{G})$. By Theorem 3.1, $\mathbb{G}$ is discrete. □

It follows from Theorem 3.3 that if $\mathbb{G}$ is discrete then $\ell^\infty(\mathbb{G})$ contains compact elements. But the converse is not true in general as shown in the following example.

EXAMPLE 3.4. Let $\mathbb{G}$ be a locally compact quantum group such that $L^\infty(\mathbb{G}) = B(H)$ for some (infinite dimensional) Hilbert space (e.g. the non-semiregular example of [**1**]). Then $L^\infty(\mathbb{G})^{cpt} = K(H)$, but since $L^\infty(\mathbb{G})$ is a factor, $\mathbb{G}$ is not discrete.

REMARK 1. A very interesting observation is that since the C^*-algebras $\langle C_0(\mathbb{G})C_0(\hat{\mathbb{G}})\rangle$ and $\langle L^\infty(\mathbb{G})L^\infty(\hat{\mathbb{G}})\rangle$ act irreducibly on $L^2(\mathbb{G})$, they contain non-zero compact elements if and only if $\mathbb{G}$ is a semi-regular locally compact quantum group.

THEOREM 3.5. *Let $\mathbb{G}$ be a locally compact quantum group. If the von Neumann algebra $L^\infty(\mathbb{G})$ has a decomposition $L^\infty(\mathbb{G}) \cong M \oplus N$ where M is finite dimensional C^*-algebra, then $\mathbb{G}$ is a discrete quantum group.*

Proof. Without loss of generality, we may assume $L^\infty(\mathbb{G}) = B(H) \oplus N$, where H is a finite dimensional Hilbert space and N is a von Neumann subalgebra of $L^\infty(\mathbb{G})$. Let 1_H be the identity map in $B(H)$. It is clear that the map $L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) : b \mapsto 1_H b$ is of finite rank. The conclusion follows from Theorem 3.3. □

COROLLARY 3.6. *Let $\mathbb{G}$ be a locally compact quantum group. Then*

(1) *$L^\infty(\mathbb{G})$ has a non-zero compact element if and only if the type I summand of the von Neumann algebra $L^\infty(\mathbb{G})$ is non-zero and contains a type I factor.*
(2) *$L^1(\mathbb{G})$ has the* RNP *if and only if the set of compact element of $L^\infty(\mathbb{G})$ is weak* dense in $L^\infty(\mathbb{G})$.*

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Proof. (1) By results of Section 4 of [**9**] there is a central projection $p \in L^\infty(\mathbb{G})$ such that

$$L^\infty(\mathbb{G}) = L^\infty(\mathbb{G})p \otimes L^\infty(\mathbb{G})(1-p), \tag{2}$$

where $L^\infty(\mathbb{G})p$ is the part of the factor decomposition of $L^\infty(\mathbb{G})$ that consists of a direct sum of type I factors and contains all compact elements of $L^\infty(\mathbb{G})$. Hence, the equivalence in (1) holds.
(2) It follows from the decomposition (2) that the compact elements of $L^\infty(\mathbb{G})$ are weak* dense in $L^\infty(\mathbb{G})$ if and only if $p = 1$, which is equivalent to $L^\infty(\mathbb{G})$ being a direct sum of type I factors. By [**14**, Theorem 4.4.5] the latter is equivalent to $L^1(\mathbb{G})$ having the RNP. □

4. Compact convolution operators. In this section, we consider the question of existence of compact, and weakly compact convolution operators. In the group setting, by a classical result of Sakai [**26**], a locally compact group G is compact if and only if there is a non-zero element in the measure algebra $\mu \in M(G)$ such that the convolution operator $L^1(G) \ni f \mapsto \mu \star f \in L^1(G)$ is (weakly) compact.

The main goal of this section is to investigate quantum versions of Sakai's result.

4.1. Convolution operators on $L^1(\mathbb{G})$. It follows from [**23**, Theorem 3.8] that a locally compact quantum group $\mathbb{G}$ is compact if and only if the convolution product on $L^1(\mathbb{G})$ is weakly compact. Next result strengthens one direction of this equivalence.

Recall that for $\mu \in L^1(\mathbb{G})$, $\mathfrak{R}_\mu$ denotes the convolution operator $L^1(\mathbb{G}) \ni \omega \mapsto \omega \star \mu \in L^1(\mathbb{G})$.

PROPOSITION 4.1. *A locally compact quantum group $\mathbb{G}$ is compact if and only if the convolution operator $\mathfrak{R}_\mu$ is compact for every $\mu \in L^1(\mathbb{G})$.*

Proof. We only need to prove the forward implication. By [**6**, Lemma 2.6], for any $a \in C_0(\mathbb{G})$ and $\mu \in L^1(\mathbb{G})$, the set $\{(a\omega a^*) \star \mu : \omega \in L^1(\mathbb{G}),\ \|\omega\| \leq 1\}$ is relatively compact in $L^1(\mathbb{G})$. Since $\mathbb{G}$ is compact, $1 \in C_0(\mathbb{G})$. Setting $a = 1$ yields the result. □

But our aim is to conclude our results from the assumption of existence of a single (weakly) compact convolution operator.

The following observation shows that we can restrict our attention to multipliers arising from elements of $L^1(\mathbb{G})$. We denote the set of weakly compact right multipliers of $L^1(\mathbb{G})$ by $RM^{wc}(L^1(\mathbb{G})) \neq \{0\}$.

LEMMA 4.2. *If $RM^{wc}(L^1(\mathbb{G})) \neq \{0\}$, then $L^1(\mathbb{G}) \cap RM^{wc}(L^1(\mathbb{G})) \neq \{0\}$.*

Proof. Let $T \in RM(L^1(\mathbb{G}))$, and $\omega \in L^1(\mathbb{G})$. Then for every $\rho \in L^1(\mathbb{G})$, we have

$$T \circ \mathfrak{R}_\omega(\rho) = T(\rho \star \omega) = \rho \star T(\omega) = \mathfrak{R}_{T(\omega)}(\rho)$$

This shows $T \circ \mathfrak{R}_\omega = \mathfrak{R}_{T(\omega)}$. Thus, if $T(\omega) \neq 0$, then $L^1(\mathbb{G}) \cap RM^{wc}(L^1(\mathbb{G})) \neq \{0\}$. □

THEOREM 4.3. *A locally compact quantum group $\mathbb{G}$ is compact if and only if there is $\mu \in M(\mathbb{G})$, $\mu \neq 0$, such that the convolution operator $\mathfrak{R}_\mu$ is of finite rank.*

Proof. If $\mathbb{G}$ is compact with Haar state φ, then $\mu \star \varphi = \mu(1)\varphi$ for all $\omega \in L^1(\mathbb{G})$, which means $\mathfrak{R}_\varphi$ has rank one.

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For the converse, note that since the regular representation λ is an algebra homomorphism, $\lambda(L^1(\mathbb{G}))\lambda(\mu) = \lambda(L^1(\mathbb{G}) \star \mu)$ is a non-zero finite dimensional subspace of $\lambda(L^1(\mathbb{G}))$. The latter is weak* dense in $L^\infty(\hat{\mathbb{G}})$, which implies $\lambda(\mu) \in L^\infty(\hat{\mathbb{G}})^{rcf}$. Hence $\hat{\mathbb{G}}$ is discrete by Theorem 3.3, and equivalently $\mathbb{G}$ is compact. □

The following equivalences follow immediately from Theorem 4.3.

COROLLARY 4.4. *Let $\mathbb{G}$ be a locally compact quantum group. Then the following are equivalent:*

(1) *$\mathbb{G}$ is a compact quantum group;*
(2) *$L^1(\mathbb{G})$ has a non-zero finite-dimensional (one sided) ideal;*
(3) *$L^1(\mathbb{G})$ has a non-zero one-dimensional (one sided) ideal.*

THEOREM 4.5. *Let $\mathbb{G}$ be a locally compact quantum group and $\mu \in L^1(\mathbb{G})$. The convolution operator $\mathfrak{R}_\mu$ is weakly compact if and only if* $\text{image}(\mathfrak{R}_\mu^*) \subseteq C_0(\mathbb{G})$.

Proof. Suppose $\mathfrak{R}_\mu$ is weakly compact, then $\mathfrak{R}_\mu^*$ is weak*–weak continuous. Since $C_0(\mathbb{G})$ is weak* dense in $L^\infty(\mathbb{G})$, the range of $\mathfrak{R}_\mu^*$ is contained in the weak closure of the range of its restriction to $C_0(\mathbb{G})$, which is contained in $C_0(\mathbb{G})$.

Conversely, suppose that the range of $\mathfrak{R}_\mu^*$ is contained in $C_0(\mathbb{G})$. Let (x_α) be a net in $C_0(\mathbb{G})$ with weak*$-\lim x_\alpha = x \in L^\infty(\mathbb{G})$. Now suppose $\tilde{\nu} \in L^\infty(\mathbb{G})^*$, and $\nu \in M(\mathbb{G})$ is the restriction of $\tilde{\nu}$ on $C_0(\mathbb{G})$. Using the fact that $\nu \star \mu \in L^1(\mathbb{G})$, we get

$$\begin{aligned} \langle \mu \star x_\alpha , \tilde{\nu} \rangle &= \langle \mu \star x_\alpha , \nu \rangle = \langle x_\alpha , \nu \star \mu \rangle \\ &\to \langle x , \nu \star \mu \rangle = \langle \mu \star x , \nu \rangle = \langle \mu \star x , \tilde{\nu} \rangle, \end{aligned}$$

which shows $\mathfrak{R}_\mu^*$ is weak*–weak continuous, and hence is weakly compact. □

COROLLARY 4.6. *A locally compact quantum group $\mathbb{G}$ is compact if and only if there is a non-zero $\mu \in L^1(\mathbb{G})$ such that $\mathfrak{R}_\mu$ is weakly compact and* $\text{image}(\mathfrak{R}_\mu^*)$ *contains an invertible element of $L^\infty(\mathbb{G})$.*

In particular, we have:

COROLLARY 4.7. *A locally compact quantum group $\mathbb{G}$ is compact if and only if there is $\mu \in L^1(\mathbb{G})$ with $\mu(1) \neq 0$ (e.g. μ is positive), such that $\mathfrak{R}_\mu$ is weakly compact.*

Proof. The forward implication is obvious. For the converse, note that $\mathfrak{R}_\mu(1) = \mu(1)1$. Hence $1 \in \text{image}(\mathfrak{R}_\mu^*)$. So, $\mathbb{G}$ is compact by Corollary 4.6. □

Denote by $\mathcal{G}(\mathbb{G})$ the intrinsic group of $\mathbb{G}$, that is the group of unitaries $u \in L^\infty(\mathbb{G})$ with $\Delta(u) = u \otimes u$.

COROLLARY 4.8. *A locally compact quantum group $\mathbb{G}$ is compact if and only if there is $\mu \in L^1(\mathbb{G})$ such that $\mathfrak{R}_\mu$ is weakly compact, and $\mu(u) \neq 0$ for some $u \in \mathcal{G}(\mathbb{G})$.*

Proof. For $u \in \mathcal{G}(\mathbb{G})$ we have $(\mathfrak{R}_\mu)^*(u) = (\text{id} \otimes \mu)\Delta(u) = \mu(u)u$. Therefore, if $\mu(u) \neq 0$ for some $u \in \mathcal{G}(\mathbb{G})$, *image*$(\mathfrak{R}_\mu^*)$ contains u. Hence, $\mathbb{G}$ is compact by Corollary 4.6. □

4.2. Convolution operators on $T(L^2(\mathbb{G}))$. We continue our study of (weakly) compact convolution operators in this section at the level of $T(L^2(\mathbb{G}))$.

We first consider convolution operators $\mathfrak{R}_\omega^\triangleleft$, $\omega \in T(L^2(\mathbb{G}))$.

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In the classical case, for a locally compact group G, every weakly compact convolution operator is compact (cf. [**26**]). For general locally compact quantum group, we do not know whether this holds. We have the following partial result for positive functionals.

PROPOSITION 4.9. *Let $0 \neq \omega \in T(L^2(\mathbb{G}))$ be positive and let $\mathfrak{R}_\omega^{\triangleleft}$ be weakly compact. Then, $\mathfrak{R}_\omega^{\triangleleft}$ is compact.*

Proof. Let $(\omega_n)_{n=1}^\infty \subseteq T(L^2(\mathbb{G}))_1^+$ be a sequence of positive linear functional in the unit ball of $T(L^2(\mathbb{G}))$. Since $\mathfrak{R}_\omega^{\triangleleft}$ is weakly compact, $(\omega_n \triangleleft \omega)$ has a weakly convergent subsequence, say $(\omega_{n_k} \triangleleft \omega)$. By [**29**, Corollary III.5.11], this subsequence is norm convergent.

Now, since every element of the unit ball of $T(L^2(\mathbb{G}))$ can be written as a linear combination of four positive elements of the unit ball, it follows that for every sequence $(\omega_n)_{n=1}^\infty \subseteq T(L^2(\mathbb{G}))_1$, $(\omega_n \triangleleft \omega)$ has a norm convergent subsequence. Hence $\mathfrak{R}_\omega^{\triangleleft}$ is a compact operator. □

PROPOSITION 4.10. *Let $\mathbb{G}$ be a locally compact quantum group, and $\omega \in T(L^2(\mathbb{G}))^+$. If $\mathfrak{R}_\omega^{\triangleleft}$ is weakly compact, then $(\omega \otimes \mathrm{id})W \in K(L^2(\mathbb{G}))$.*

Proof. Let $y = (\omega \otimes \mathrm{id})W$, and for the sake of contradiction, suppose that $y \notin K(L^2(\mathbb{G}))$. In particular, y is not a compact element of $B(L^2(\mathbb{G}))$. By [**28**, Lemma 2.3.12], the map $B(L^2(\mathbb{G})) \ni x \mapsto xy \in B(L^2(\mathbb{G}))$ is not weakly compact. Equivalently, the preadjoint $T(L^2(\mathbb{G})) \ni \omega \mapsto y\omega \in T(L^2(\mathbb{G}))$ is not weakly compact.

Now [**29**, Theorem III.5.4] implies that there exists a decreasing sequence of projections $(p_m)_{m=1}^\infty \subseteq B(L^2(\mathbb{G}))$ such that $p_m \searrow 0$ but $\lim_{m\to\infty}(y\omega)(p_m) = 0$ is not uniform on the unit ball $T(L^2(\mathbb{G}))_1$. Hence, there is $\varepsilon > 0$ such that for every $N \in \mathbb{N}$ there is $m \geq N$ and $\omega_n \in T(L^2(\mathbb{G}))_1$ with $|(y\omega_n)(p_m)| \geq \varepsilon$. Therefore,

$$\begin{aligned}\varepsilon^2 &\leq |(y\omega_n)(p_m)|^2 \;=\; |(\omega \otimes \omega_n)((1 \otimes p_m)W)|^2 \\ &\leq (\omega \otimes \omega_n)(W^*(1 \otimes p_m)W) \\ &= (\omega \triangleleft \omega_n)(p_m).\end{aligned}$$

Again, [**29**, Theorem III.5.4] implies that $\mathfrak{R}_\omega^{\triangleleft}$ is not weakly compact. □

COROLLARY 4.11. *Let $\mathbb{G}$ be a locally compact quantum group, and $0 \neq \omega \in T(L^2(\mathbb{G}))$ be positive. If $\mathfrak{R}_\omega^{\triangleleft}$ is weakly compact, then $\mathbb{G}$ is compact.*

Proof. By Proposition 4.10, $(\omega \otimes \mathrm{id})W \in K(L^2(\mathbb{G}))$. Also note that $(\omega \otimes \mathrm{id})W \in L^\infty(\hat{\mathbb{G}})$. Hence by Theorem 3.1 the dual quantum group $\hat{\mathbb{G}}$ is discrete; equivalently $\mathbb{G}$ is compact. □

COROLLARY 4.12. *Let $\mathbb{G}$ be a locally compact quantum group. The following are equivalent:*

(1) *$\mathbb{G}$ is a compact quantum group;*
(2) *for every $\omega \in T(L^2(\mathbb{G}))$, $\mathfrak{R}_\omega^{\triangleleft}$ is a weakly compact linear map;*
(3) *for every $\omega \in T(L^2(\mathbb{G}))$, $\mathfrak{R}_\omega^{\triangleleft}$ is a compact linear map.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from [**11**, Proposition 6.1] and [**22**, Proposition 1.14.3]. The implication (3) $\Rightarrow$ (2) is obvious.
(2) $\Rightarrow$ (3): It follows from Proposition 4.10 that $\mathfrak{R}_\omega^{\triangleleft}$ is compact for all $\omega \in T(L^2(\mathbb{G}))^+$. Now an arbitrary $\omega \in T(L^2(\mathbb{G}))$ is a finite linear combination of positive elements,

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therefore $\mathfrak{R}_\omega^{\triangleleft}$ is a finite linear combination of compact operators, whence is compact. □

PROPOSITION 4.13. *Let* $\mathbb{G}$ *be a locally compact quantum group. If* $\mathfrak{R}_\omega^{\triangleleft}$ *is weakly compact, then the range of* $(\mathfrak{R}_\omega^{\triangleleft})^*$ *is contained in* $C_0(\mathbb{G})$.

Proof. Note that $\mathfrak{R}_\omega^{\triangleleft}$ is weakly compact if and only if $(\mathfrak{R}_\omega^{\triangleleft})^*$ is weak*–weak continuous. Since $K(L^2(\mathbb{G}))$ is weak* dense in $B(L^2(\mathbb{G}))$, the range of $(\mathfrak{R}_\omega^{\triangleleft})^*$ is contained in the weak closure of the range of its restriction to $K(L^2(\mathbb{G}))$. But by [**11**, Formula 1.25], $(\mathfrak{R}_\omega^{\triangleleft})^*(K(L^2(\mathbb{G}))) \subseteq C_0(\mathbb{G})$. This completes the proof. □

This implies, for instance, that if image$(\mathfrak{R}_\omega^{\triangleleft})^*$ contains an invertible element of $L^\infty(\mathbb{G})$, then $\mathbb{G}$ is a compact quantum group. In particular, we have:

COROLLARY 4.14. *Let* $\mathbb{G}$ *be a locally compact quantum group. If the convolution operator* $\mathfrak{R}_\omega^{\triangleleft}$ *is weakly compact for some* $\omega \in T(L^2(\mathbb{G}))$ *whose restriction to* $L^\infty(\hat{\mathbb{G}})'$ *is not zero, then* $\mathbb{G}$ *is compact.*

Proof. Recall that $W \in L^\infty(\mathbb{G}) \otimes L^\infty(\hat{\mathbb{G}})$, and therefore

$$\mathfrak{R}_\omega^{\triangleleft}(\hat{x}') = (\mathrm{id} \otimes \omega) W^*(1 \otimes \hat{x}')W = \omega(\hat{x}')1$$

for all $\hat{x}' \in L^\infty(\hat{\mathbb{G}})'$. So, if $\omega(\hat{x}') \neq 0$ then $1 \in C_0(\mathbb{G})$, and hence $\mathbb{G}$ is compact. □

In the remaining of this section, we consider convolution operators $\mathfrak{R}_\omega^{\triangleright}$, $\omega \in T(L^2(\mathbb{G}))$.

First, we observe a similar result as Corollary 4.11 in this case.

COROLLARY 4.15. *Let* $\mathbb{G}$ *be a locally compact quantum group. If* $\mathfrak{R}_\omega^{\triangleright}$ *is weakly compact for a non-zero* $\omega \in T(L^2(\mathbb{G}))^+$, *then* $\mathbb{G}$ *is a finite quantum group.*

Proof. Recall that $V \in L^\infty(\hat{\mathbb{G}})' \otimes L^\infty(\mathbb{G})$, and therefore

$$\mathfrak{R}_\omega^{\triangleright}(\hat{x}) = (\mathrm{id} \otimes \omega) V(\hat{x} \otimes 1)V^* = \omega(1)\hat{x},$$

for all $\hat{x} \in L^\infty(\hat{\mathbb{G}})$. So, if $\omega(1) \neq 0$, then the identity map on $L^\infty(\hat{\mathbb{G}})$ is weakly compact. This implies that $L^\infty(\hat{\mathbb{G}})$ is a reflexive von Neumann algebra, hence finite dimensional. □

The same argument as the one given for Proposition 4.10 yields the following.

PROPOSITION 4.16. *Let* $\mathbb{G}$ *be a locally compact quantum group, and* $\omega \in T(L^2(\mathbb{G}))^+$. *If* $\mathfrak{R}_\omega^{\triangleright}$ *is weakly compact, then* $(\omega \otimes \mathrm{id})W \in K(L^2(\mathbb{G}))$.

The counter-part of Corollary 4.12 in this case reads as follows.

PROPOSITION 4.17. *Let* $\mathbb{G}$ *be a locally compact quantum group. The following are equivalent:*

(1) $\mathbb{G}$ *is a finite quantum group.*
(2) *For every* $\omega \in T(L^2(\mathbb{G}))$, $\mathfrak{R}_\omega^{\triangleright}$ *is a weakly compact linear map.*
(3) *For every* $\omega \in T(L^2(\mathbb{G}))$, $\mathfrak{R}_\omega^{\triangleright}$ *is a compact linear map.*

Proof. The only non-trivial implication is (2) ⇒ (1), which follows from [**11**, Proposition 6.4] and [**22**, Proposition 1.14.3]. □

PROPOSITION 4.18. *Let $\mathbb{G}$ be a regular locally compact quantum group and $\omega \in T(L^2(\mathbb{G}))$. If $\mathfrak{R}_\omega^\triangleright$ is weakly compact, then the range of $(\mathfrak{R}_\omega^\triangleright)^*$ is contained in $K(L^2(\mathbb{G}))$.*

Proof. First, since $\mathbb{G}$ is regular, by [**11**, Corollary 3.6], $(\mathfrak{R}_\omega^\triangleright)^*$ leaves $K(L^2(\mathbb{G}))$ invariant. Then, a similar argument to Proposition 4.13 completes the proof. □

The following is immediate.

COROLLARY 4.19. *A locally compact quantum group $\mathbb{G}$ is finite if and only if there is a non-zero $\omega \in T(L^2(\mathbb{G}))$ such that $\mathfrak{R}_\omega^\triangleright$ is weakly compact and* image$(\mathfrak{R}_\omega^\triangleright)^*$ *contains an invertible element.*

COROLLARY 4.20. *Let $\mathbb{G}$ be a commutative or co-commutative locally compact quantum group. If there is a non-zero $\omega \in T(L^2(\mathbb{G}))$ such that $R_\omega^\triangleright$ is weakly compact, $\mathbb{G}$ is a finite quantum group.*

Proof. First, since the restriction of $(\mathfrak{R}_\omega^\triangleright)^*$ to $L^\infty(\mathbb{G})$ is also weakly amenable, by classical results (Sakai's for the commutative and Lau's for co-commutative case), $\mathbb{G}$ is compact. On the other hand, commutative and co-commutative locally compact quantum groups are regular. Therefore by Proposition 4.18 $(\mathfrak{R}_\omega^\triangleright)^*(B(L^2(\mathbb{G}))) \subseteq K(L^2(\mathbb{G}))$. Since $(\mathfrak{R}_\omega^\triangleright)^*$ leaves $L^\infty(\mathbb{G})$ invariant, and does not vanish on it, $L^\infty(\mathbb{G}) \cap K(L^2(\mathbb{G})) \neq \{0\}$. Hence $\mathbb{G}$ is discrete by Theorem 3.1.

Thus $\mathbb{G}$ is both compact and discrete, hence finite. □

4.3. Almost periodic elements. In previous sections, we considered multiplication operators on $L^\infty(\mathbb{G})$ and $L^1(\mathbb{G})$. In this section, we present a new approach, by investigating (weak) compactness of the convolution maps of the form $L^1(\mathbb{G}) \ni \omega \mapsto \mu \star x \in L^\infty(\mathbb{G})$ for a fixed $x \in L^\infty(\mathbb{G})$.

This leads to the study of (weakly) almost periodic elements of $L^\infty(\mathbb{G})$: $x \in L^\infty(\mathbb{G})$ is called a (weakly) almost periodic if the above map is (weakly) compact.

Let us fist recall some definitions concerning this concept. We refer the reader to [**27**] and [**5**] for more details.

A (finite dimensional) *representation* of a locally compact group $\mathbb{G}$ on a finite dimensional Hilbert space H is an invertible element $T \in M(C_0(\mathbb{G})) \otimes B(H)$ with $(\Delta \otimes \mathrm{id})T = T_{12}T_{13}$. Identifying $M(C_0(\mathbb{G})) \otimes B(H) \cong \mathbb{M}_n(M(C_0(\mathbb{G})))$, we may consider T as a matrix $(T_{ij}) \in \mathbb{M}_n(M(C_0(\mathbb{G})))$. This matrix is invertible, and satisfies

$$\Delta(T_{ij}) = \sum_{k=1}^{n} T_{ik} \otimes T_{kj}.$$

We shall say that T is admissible if its transpose matrix (T_{ji}) is invertible.

A *matrix elements* of T is a linear span of the elements T_{ij}. The set of all matrix elements of admissible representations of $C_0(\mathbb{G})$ forms a unital $*$-subalgebra of $M(C_0(\mathbb{G}))$ [**27**, Proposition 2.12], which we denote by $\mathcal{AP}(C_0(\mathbb{G}))$. Also, we denote by $\mathbb{AP}(A)$ the closure of $\mathcal{AP}(A)$. Thus $\mathbb{AP}(A)$ is a unital C^*-algebra and Δ restricts to $\mathbb{AP}(A)$ to give a compact quantum group.

We say that $x \in L^\infty(\mathbb{G})$ is periodic if $\Delta(x)$ is a finite-rank tensor in $L^\infty(\mathbb{G})\bar{\otimes}L^\infty(\mathbb{G})$. Denote the collection of periodic elements of $L^\infty(\mathbb{G})$ by $\mathcal{P}^\infty(\mathbb{G})$, and denote its norm closure in $L^\infty(\mathbb{G})$ by $\mathbb{P}^\infty(\mathbb{G})$. It is known that $\mathbb{P}^\infty(\mathbb{G})$ is a C^*-subalgebra of $L^\infty(\mathbb{G})$ and an $L^1(\mathbb{G})$-submodule of $L^\infty(\mathbb{G})$, and

$$\mathcal{P}^\infty(\mathbb{G}) = \{x \in L^\infty(\mathbb{G}) : \mathcal{L}_x \text{ is finite rank}\} = \{x \in L^\infty(\mathbb{G}) : \mathcal{R}_x \text{ is finite rank}\},$$

where $\mathcal{L}_x$ and $\mathcal{R}_x$ are linear maps $L^1(\mathbb{G}) \ni f \mapsto x \star f \in L^\infty(\mathbb{G})$ and $L^1(\mathbb{G}) \ni f \mapsto f \star x \in L^\infty(\mathbb{G})$, respectively [**5**, Lemma 4.7].

In the following, S will denote the antipode of the quantum group $\mathbb{G}$.

Theorem 4.21. *Let $\mathbb{G}$ be a locally compact quantum group. Then, the following are equivalent:*

(1) *$\mathbb{G}$ is a compact quantum group.*
(2) $\mathbb{AP}(C_0(\mathbb{G})) = C_0(\mathbb{G})$.
(3) $\mathcal{AP}(C_0(\mathbb{G})) \cap C_0(\mathbb{G}) \neq \{0\}$.
(4) $\mathbb{P}^\infty(\mathbb{G}) = C_0(\mathbb{G})$.
(5) $\{\mathcal{P}^\infty(\mathbb{G}) \cap D(S)\} \cap C_0(\mathbb{G}) \neq \{0\}$.

Proof. (1) $\Rightarrow$ (2) is well known, see for example [**27**, Remark 2.3].
(2) $\Rightarrow$ (3) is obvious.
(3) $\Rightarrow$ (1): suppose $x = \sum_{i,j=1}^n c_{i,j} u_{i,j}$ is a non-zero element in the intersection $\mathcal{AP}(C_0(G)) \cap C_0(\mathbb{G})$, where $c_{i,j} \in \mathbb{C}$, and $u_{i,j}$'s are matrix elements of admissible representations π_{ij} of $\mathbb{G}$. Since admissible representations are closed under taking direct sums [**27**, Proposition 2.4], considering $\oplus\pi_{ij}$, we may assume x is a linear combination of matrix elements of one admissible representation, say $U \in M_n(C_0(\mathbb{G}))$.

Let B_U be the unital C^*-subalgebra of $M(C_0(\mathbb{G}))$ generated by all matrix elements of U. Then, the proof of Proposition 2.7 of [**27**] shows that $(B_U, \Delta_{|B_U})$ is a compact matrix quantum group, and therefore we can apply the results of [**31**]. In particular, formula (4.32) of [**31**] implies that there exist linear functionals $f_{r,s} \in L^1(\mathbb{G})$ such that

$$\sum_{s=1}^n S(u_{k,s})(x \star f_{r,s}) = c_{r,k} 1,$$

for all $r, k = 1, \ldots, n$. This implies that $1 \in C_0(\mathbb{G})$ and $\mathbb{G}$ is compact.
(4) $\Rightarrow$ (1) is obvious, because $1 \in \mathbb{P}^\infty(G)$.
(1)&(2) $\Rightarrow$ (4): since $\mathbb{G}$ is compact, $\mathcal{P}^\infty(\mathbb{G}) = \mathcal{AP}(C_0(\mathbb{G}))$ by [**5**, Theorem 6.4]. Hence, (4) follows from (2).
(1)&(2) $\Rightarrow$ (5): combining Theorem 4.14 and Theorem 6.4 of [**5**], we obtain $\mathcal{P}^\infty(\mathbb{G}) = \mathcal{AP}(C_0(\mathbb{G}))$. Hence, (5) follows from (2).
(5) $\Rightarrow$ (1): suppose that $0 \neq x \in \{\mathcal{P}^\infty(\mathbb{G}) \cap D(S)\} \cap C_0(\mathbb{G})$ and $\Delta(x) = \sum_{i=1}^n x_i \otimes y_i$. Then, the proof of Theorem 4.14 of [**5**] shows that there exist linear functionals $\omega_i, \tau_j \in L^1(\mathbb{G}), i, j = 1, \ldots, n$, such that $(\omega_j \star x \star \tau_i)_{i,j=1}^n = (U_{i,j})_{i,j=1}^n \in \mathbb{M}_n(M(C_0(\mathbb{G})))$ is an invertible finite-dimensional representation of $C_0(\mathbb{G})$. Since $x \in C_0(\mathbb{G})$, $U_{i,j} \in C_0(\mathbb{G})$ and hence $(U_{i,j})_{i,j=1}^n \in \mathbb{M}_n(C_0(\mathbb{G}))$. This implies that $1 \in C_0(\mathbb{G})$ and so $\mathbb{G}$ is a compact quantum group. □

Using this Theorem, we give a different proof of Theorem 4.3.

Corollary 4.22. *Let $\mathbb{G}$ be a locally compact quantum group. If $\mathfrak{R}_\mu$ is of finite rank for a non-zero $\mu \in L^1(\mathbb{G})$, then $\mathbb{G}$ is compact.*

Proof. Since $\mathfrak{R}_\mu$ is of finite rank, for every $x \in L^\infty(\mathbb{G})$ the map $L^1(\mathbb{G}) \ni \nu \mapsto \nu \star (\mu \star x) \in L^\infty(\mathbb{G})$ is of finite rank, and so $\mu \star x \in \mathcal{P}^\infty(\mathbb{G})$ by [**5**, Lemma 4.7]. Thus,

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the range of $\mathfrak{R}_\mu^*$ is contained in $\mathcal{P}^\infty(\mathbb{G}) \cap C_0(\mathbb{G})$. On the other hand, since $\{\lambda_*(\hat{\omega}) : \hat{\omega} \in L^1(\hat{\mathbb{G}})\} \subseteq D(S)$ (cf. [**16**]), by choosing $x = \lambda_*(\hat{\omega})$ for a non-zero $\hat{\omega} \in L^1(\hat{\mathbb{G}})$, we get

$$\mu \star \lambda_*(\hat{\omega}) = \lambda_*(\lambda(\mu)\hat{\omega}) \in \mathcal{P}^\infty(\mathbb{G}) \cap D(S) \cap C_0(\mathbb{G}).$$

Hence, by Theorem 4.21 $\mathbb{G}$ is compact. □

In particular, for $f \in L^1(\mathbb{G})^{lcf}$, $L^\infty(\mathbb{G}) \star f \subseteq \mathcal{P}^\infty(\mathbb{G})$, but it is not clear that for $f \in L^1(\mathbb{G})^{lcc}$, $L^\infty(\mathbb{G}) \star f \subseteq \mathbb{P}^\infty(\mathbb{G})$. It is obvious that $\mathcal{L}_{x\star f} : L^1(\mathbb{G}) \to L^\infty(\mathbb{G})$ is compact, that is $L^\infty(\mathbb{G}) \star f$ is contained in the set of almost periodic elements in sense of [**24**], but we don't know if this is in the set of completely almost periodic elements in sense of [**24**]. It is known only in the commutative case that these two spaces are equal (cf. [**24**]). However, one may hope that for $f \in L^1(\mathbb{G})^{lcc}$, $L^\infty(\mathbb{G}) \star f \subseteq \mathbb{P}^\infty(\mathbb{G})$ and that $\mathbb{P}^\infty(\mathbb{G}) = \mathbb{AP}(C_0(\mathbb{G}))$ (cf. [**5**]).

5. Convolution operators on $M(\mathbb{G})$ and $L^1(\mathbb{G})^{**}$.

In this section, we consider the convolution product, and (weak) compactness of the corresponding operators on the measure algebra $M(\mathbb{G})$ and the second dual $L^1(\mathbb{G})^{**}$.

5.1. The measure algebra $M(\mathbb{G})$.

We first investigate the relations between $L^1(\mathbb{G})^{rcc}$ and $M(\mathbb{G})^{rcc}$. Similar statements and proofs are valid for compact, weakly compact and weakly completely continuous elements.

Note that since $L^1(\mathbb{G})$ is an ideal in $M(\mathbb{G})$, we have $M(\mathbb{G})^{rcc} \cap L^1(\mathbb{G}) \subseteq L^1(\mathbb{G})^{rcc}$.

PROPOSITION 5.1. *Suppose $\mathbb{G}$ is a co-amenable locally compact quantum group, and $\mu \in L^1(\mathbb{G})$ is such that $\mathfrak{R}_\mu : L^1(\mathbb{G}) \to L^1(\mathbb{G})$ is a compact operator. Then $\mathfrak{R}_\mu : M(\mathbb{G}) \to M(\mathbb{G})$ is as well a compact operator.*

Proof. Suppose (ω_α) is an approximate identity for $L^1(\mathbb{G})$. Since $L^1(\mathbb{G})$ is an ideal in $M(\mathbb{G})$, for every α, $\mathfrak{R}_{\mu\star\omega_\alpha}$ is a compact operator on $M(\mathbb{G})$. Moreover, $\mathfrak{R}_{\mu\star\omega_\alpha} \to \mathfrak{R}_\mu$ in norm, which implies $\mathfrak{R}_\mu$ is a compact operator on $M(\mathbb{G})$. □

PROPOSITION 5.2. *Let $\mu \in M(\mathbb{G})$ and $\mu(1) \neq 0$. If $\mathfrak{R}_\mu$ is weakly compact, then $\mathbb{G}$ is a compact quantum group.*

Proof. Let $\omega \in L^1(\mathbb{G})_1^+$ be a normal state. Then, $\mu \star \omega \in L^1(\mathbb{G})$ with $\mu \star \omega(1) = \mu(1)\omega(1) = \omega(1) \neq 0$, and the map $\mathfrak{R}_{\mu\star\omega}$ is weakly compact. Hence $\mathbb{G}$ is compact by Corollary 4.7. □

PROPOSITION 5.3. *Let $\mathbb{G}$ be a co-amenable locally compact quantum group, and let $\mu \in M(\mathbb{G})$. If $\mathfrak{R}_\mu$ is weakly compact, then $\mu \in L^1(\mathbb{G})$.*

Proof. We first show that the range of $\mathfrak{R}_\mu$ is contained in $L^1(\mathbb{G})$. For this, note that since $\mathfrak{R}_\mu$ is an adjoint weakly compact operator, it is weak*–weak continuous on $M(\mathbb{G})$. But since $L^1(\mathbb{G})$ is weak* dense in $M(\mathbb{G})$, the range of $\mathfrak{R}_\mu$ is contained in the weak closure of $\mathfrak{R}_\mu(L^1(\mathbb{G}))$, which is contained in $L^1(\mathbb{G})$.

Now, since $\mathbb{G}$ is co-amenable, by [**2**, Theorem 3.1], $M(\mathbb{G})$ has a unit, say ϵ. Then,

$$\mu = \mathfrak{R}_\mu(\epsilon) \in L^1(\mathbb{G}).$$

□

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If $\mathbb{G}$ is a co-amenable compact quantum group, then $RM^{wc}\{L^1(\mathbb{G})\} = L^1(\mathbb{G})$, and conversely, if $RM^{wc}\{L^1(\mathbb{G})\} = L^1(\mathbb{G})$, then $\mathbb{G}$ is a compact quantum group [**12**, Theorem 4.2]. We prove a similar result for the measure algebra $M(\mathbb{G})$.

COROLLARY 5.4. *Let $\mathbb{G}$ be a co-amenable locally compact quantum group. Then, $RM^{wc}\{L^1(\mathbb{G})\} = M(\mathbb{G})$ if and only if $\mathbb{G}$ is a finite quantum group.*

Proof. We only need to prove the forward implication. On the one hand, since $L^1(\mathbb{G}) \subseteq M(\mathbb{G}) = RM^{wc}(L^1(\mathbb{G}))$, it follows from [**23**, Theorem 3.8] that $\mathbb{G}$ is compact.
On the other hand, since $\mathbb{G}$ is co-amenable, by Proposition 5.3, $RM^{wc}\{L^1(\mathbb{G})\} \subseteq L^1(\mathbb{G})$. Thus $L^1(\mathbb{G}) = M(\mathbb{G})$, and therefore by [**12**, Theorem 3.7] $\mathbb{G}$ is discrete.
Hence $\mathbb{G}$ is finite. □

5.2. The group of qauntum point masses. There is a canonical way to assign to every locally compact quantum group $\mathbb{G}$ a locally compact group $\tilde{\mathbb{G}}$, such that if $\mathbb{G} = L^\infty(G)$ for a locally compact group G, then $\tilde{\mathbb{G}} = G$ (cf. [**15**]). In the case of a co-amenable $\mathbb{G}$, the group $\tilde{\mathbb{G}}$ is identified with the set of all non-zero characters on $C_0(\mathbb{G})$, i.e. continuous *-homomorphism from $C_0(\mathbb{G})$ onto $\mathbb{C}$, with the multiplication induced from the convolution product of $M(\mathbb{G})$.
The following is another generalization of Sakai's theorem.

THEOREM 5.5. *Let $\mathbb{G}$ be a co-amenable locally compact quantum group. If there is a non-zero $\mu \in M(\mathbb{G})$ such that the convolution operator $\mathfrak{R}_\mu : M(\mathbb{G}) \to M(\mathbb{G})$ is compact, then $\tilde{\mathbb{G}}$ is a compact group.*

Proof. Suppose $\tilde{\mathbb{G}}$ is not compact. Considering $\tilde{\mathbb{G}}$ as the group of all multiplicative linear functionals on $C_0(\mathbb{G})$, we may choose a net $\chi_\alpha \in \tilde{\mathbb{G}}$ such that $\chi_\alpha \to 0$ in the $\sigma(M(\mathbb{G}), C_0(\mathbb{G}))$-topology.
Since the set $\{\mu \star \chi : \chi \in \tilde{\mathbb{G}}\}$ is relatively compact in $L^1(\mathbb{G})$, there are $\chi_1, \chi_2, \ldots, \chi_k$ such that

$$\inf_{i=1,2,\ldots,k} \| \mu \star \chi - \mu \star \chi_i \| \leq 1,$$

for all $\chi \in \tilde{\mathbb{G}}$. Passing to a subnet if necessary, we may assume that for some j, $1 \leq j \leq k$

$$\| \mu \star \chi_\alpha - \mu \star \chi_j \| \leq 1,$$

for all α. This implies that $\mu \star \chi_j$ is zero on $C_0(\mathbb{G})$, hence $\mu = 0$, a contradiction. □

5.3. The second dual $L^1(\mathbb{G})^{**}$. In this last section, we prove some results concerning (weakly) compact multipliers of the second dual $L^1(\mathbb{G})^{**}$ endowed with the Arens products.
Recall that for $m, n \in L^1(\mathbb{G})^{**}$ and $x \in L^\infty(\mathbb{G})$, the *left Arens product* $m\Box n \in L^1(\mathbb{G})^{**}$ is defined via $\langle m\Box n, x\rangle = \langle m, n\Box x\rangle$, where $n\Box x = (\iota \otimes n)\Delta(x) \in L^\infty(\mathbb{G})$.
For every $m \in L^1(\mathbb{G})^{**}$ we denote by $\mathfrak{R}_m^\Box$ and $\mathfrak{L}_m^\Box$ the corresponding right, respectively, left multiplication operator by m.
For any $m \in L^1(\mathbb{G})^{**}$, $\mathfrak{R}_m^\Box$ is weak*–weak* continuous. The *left topological center* of $L^1(\mathbb{G})^{**}$ is defined as

$$\mathfrak{Z}_t(L^1(\mathbb{G})^{**}, \Box) = \{m \in L^1(\mathbb{G})^{**} : \mathfrak{L}_m^\Box \text{ is weak* continuous on } L^1(\mathbb{G})^{**}\}.$$

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The right Arens product $\diamond$, and the corresponding right topological center $\mathfrak{Z}_t(L^1(\mathbb{G})^{**}, \diamond)$ are defined similarly. It is known that

$$L^1(\mathbb{G}) \subseteq \mathfrak{Z}_t(L^1(\mathbb{G})^{**}, \square) = \{m \in L^1(\mathbb{G})^{**} : \mathfrak{L}_m^{\square} = \mathfrak{L}_m^{\diamond}\} \subseteq L^1(\mathbb{G})^{**}.$$

The algebra $L^1(\mathbb{G})$ is said to be strongly Arens irregular (SAI) if

$$\mathfrak{Z}_t(L^1(\mathbb{G})^{**}, \square) = L^1(\mathbb{G}) = \mathfrak{Z}_t(L^1(\mathbb{G})^{**}, \diamond).$$

It is well-known that for every locally compact group G, $L^1(G)$ is SAI (cf. [**18**]).

The following observation shows that unlike the classical case, in the general setting of locally compact quantum groups, $RM\{(L^1(\mathbb{G})^{**}, \square)\}$ and $LM\{(L^1(\mathbb{G})^{**}, \square)\}$ may contain elements other than multiplication operators.

PROPOSITION 5.6. *Let $\mathbb{G}$ be a locally compact quantum group. Then the following hold:*

(1) *$\mathbb{G}$ is co-amenable if and only if $RM\{(L^1(\mathbb{G})^{**}, \square)\} = L^1(\mathbb{G})^{**}$.*
(2) *$\mathbb{G}$ is co-amenable and $LUC(\mathbb{G}) = L^\infty(\mathbb{G})$ if and only if*

$$LM\{(L^1(\mathbb{G})^{**}, \square)\} = L^1(\mathbb{G})^{**}.$$

Proof. (1) $\mathbb{G}$ is co-amenable if and only if $(L^1(\mathbb{G})^{**}, \square)$ is right unital. So, if $RM\{(L^1(\mathbb{G})^{**}, \square)\} = L^1(\mathbb{G})^{**}$, then $L^1(\mathbb{G})^{**}$ is unital, and therefore $\mathbb{G}$ is co-amenable. Conversely, suppose $\mathbb{G}$ is co-amenable, and suppose that $e \in L^1(\mathbb{G})^{**}$ is the right unit. Then, $T(n) = T(n\square e) = n\square T(e)$. Thus, $T = \mathfrak{R}_m^{\square}$ for $m = T(e)$.
(2) According to [**12**, Theorem 2.4.(ii)] $LUC(\mathbb{G}) = L^\infty(\mathbb{G})$ if and only if $(L^1(\mathbb{G})^{**}, \square)$ is left unital. Suppose that $e \in L^1(\mathbb{G})^{**}$ is left unit. Then, $T(n) = T(e\square n) = T(e)\square n$. Thus, $T = L_m^{\square}$ for $m = T(e)$. □

PROPOSITION 5.7. *Let $\mathbb{G}$ be a locally compact quantum group and let T be a weak*–weak* continuous left multiplier of $(L^1(\mathbb{G})^{**}, \square)$. Then $T(\mathfrak{Z}_t(L^1(\mathbb{G})^{**}, \square)) \subseteq \mathfrak{Z}_t(L^1(\mathbb{G})^{**}, \square)$. In particular, if $L^1(\mathbb{G})$ is SAI, then $T(L^1(\mathbb{G})) \subseteq L^1(\mathbb{G})$.*

Proof. Let $m \in \mathfrak{Z}_t(L^1(\mathbb{G})^{**}, \square)$. Then, $\mathfrak{L}_m^{\square}$ is weak*–weak* continuous on $L^1(\mathbb{G})^{**}$. So $\mathfrak{L}_{T(m)}^{\square} = T \circ \mathfrak{L}_m^{\square}$ is weak*–weak* continuous and hence $T(m) \in \mathfrak{Z}_t(L^1(\mathbb{G})^{**}, \square)$. □

In particular, the above proposition implies that for a locally compact group G, if a left multiplier of $(L^1(\mathbb{G})^{**}, \square)$ is an adjoint map, then it is a second adjoint map, hence an element of $M(G)$.

This fails if move one level up to multipliers of the third dual $L^1(\mathbb{G})^{***}$ [**21**].

COROLLARY 5.8. *Let $\mathbb{G}$ be a locally compact quantum group and let T be a weak*–weak* continuous weakly compact left multiplier of $(L^1(\mathbb{G})^{**}, \square)$. Then, $T(L^1(\mathbb{G})^{**}, \square)) \subseteq \mathfrak{Z}_t(L^1(\mathbb{G})^{**}, \square)$. In particular, if $L^1(\mathbb{G})$ is SAI, then $T(L^1(\mathbb{G})^{**}) \subseteq L^1(\mathbb{G})$.*

Proof. Since T is weak*–weak* continuous and weakly compact, it is weak*–weak continuous, and since $\mathfrak{Z}_t(L^1(\mathbb{G})^{**}, \square) \subseteq L^1(\mathbb{G})^{**}$ is weak* dense, the range of T is a subset of $\overline{T(\mathfrak{Z}_t(L^1(\mathbb{G})^{**}, \square))}^{w}$. Hence, by Proposition 5.7, $T(L^1(\mathbb{G})^{**}) \subseteq \mathfrak{Z}_t(L^1(\mathbb{G})^{**}, \square)$. □

Let $\mathbb{G}$ be a co-amenable compact quantum group. Suppose that $e \in L^1(\mathbb{G})^{**}$ is a right unit. Then $\mathfrak{R}_e^{\square} = \mathrm{id} : L^1(\mathbb{G})^{**} \to L^1(\mathbb{G})^{**}$. Thus, in contrast to the case of

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$L^1(\mathbb{G})$, compactness of $\mathbb{G}$ does not imply weak compactness of every right multiplier of $L^1(\mathbb{G})^{**}$.

But we have the following generalization of Corollary 4.7, which also generalizes [**23**, Theorem 3.8].

THEOREM 5.9. *A locally compact quantum group $\mathbb{G}$ is compact if and only if there is a weakly compact right multiplier (equivalently, left multiplier) T of $L^1(\mathbb{G})^{**}$, and $m \in L^1(\mathbb{G})^{**}$ such that $T(m) \in L^1(\mathbb{G})$ and $\langle T(m), 1\rangle \neq 0$.*

Proof. For every $\omega \in L^1(\mathbb{G})$ we have

$$\mathfrak{R}_{T(m)}(\omega) = \omega \star T(m) = T(\omega \Box m) = T \circ \mathfrak{R}_m(\omega).$$

Since T is weakly compact, so is its restriction to $L^1(\mathbb{G})$. Therefore, $\mathfrak{R}_{T(m)} = T \circ \mathfrak{R}_m$ is weakly compact on $L^1(\mathbb{G})$, and since $\langle T(m), 1\rangle \neq 0$, $\mathbb{G}$ is compact by Corollary 4.7. □

REMARK 2. Let $\mathbb{G}$ be an amenable locally compact quantum group with the two-sided invariant mean $m \in L^1(\mathbb{G})^{**}$. Then, the map $n \mapsto n \Box m = \langle 1, n\rangle m$ on $L^1(\mathbb{G})^{**}$ has rank one, and hence is compact.

Thus, Theorem 5.9 does not hold if we remove the assumption of $T(m) \in L^1(\mathbb{G})$.

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