18

D-branes, gravity and gauge theory

As we learned in section 10.2, there are effectively two descriptions of the low energy dynamics of branes. One description uses the collective dynamics of the effetive world-volume field theory. In the case of N coincident D-branes, this is captured in string theory by the open string sectors which give a $U(N)$ gauge theory with sixteen supercharges. The other description treats the brane as a soliton-like source of the various low energy closed string fields in the superstring theory. As such it has a description in terms of a classical solution of the low energy field equations. In both cases, we must remember that there is a whole tower of stringy dynamics which sits on top of this low energy physics, and we must understand in which situations this tower can be made irrelevant, or at least kept under control by a sensible expansion. To control string loops, we must work in a regime where g_s is small, so that we can trust the classical action that we wrote down for the supergravity. Similarly, working in the $\alpha' \to 0$ limit ensures that we can safely ignore the possibility of the massive string states introducing corrections to our supergravity, and in the open string sector, that the truncation to gauge theory of the full Born–Infeld, etc., action, is sensible. In this chapter, we will follow this limit quite some way, and the two complementary descriptions will lead to a sharp statement of a duality between two traditionally disparate fields: large N gauge field theory and gravity. This is a natural and logical outcome of many of the gauge theory and geometry connections we have already been noticing throughout this book.

https://doi.org/10.1017/9781009401371.019 Published online by Cambridge University Press

18.1 The AdS/CFT correspondence

18.1.1 Branes and the decoupling limit

We have already learned from our moduli space probe computations that the specialisation of the results to gauge theory can be achieved by taking $\alpha' \to 0$ while keeping finite some characteristic gauge theory quantity of interest, such as a typical vacuum expectation value of a massless 'Higgs' field. In geometries already considered, this corresponded to, for example, keeping fixed a scaled radial coordinate $u = r/\alpha'$ as we send $\alpha' \to 0$, which also meant that $r \to 0$. In other words, we must approach the core or horizon of the solution closely. In these limits, we found that the remaining supergravity quantities which survived the limit in combinations have physical meaning in the gauge theory on the probe, such as the gauge coupling, etc. Let us see if we can take this further.

Let us consider the case of the extremal D3-brane, initially. At low energy, on the world-volume (ignoring the overall $U(1)$ corresponding to the centre of mass) there is a $U(N)$ gauge theory with $\mathcal{N} = 4$ supersymmetry in four dimensions, i.e. sixteen supercharges. The gauge coupling is $g_{\rm YM}^2 = 2\pi g_{\rm s}$. The gauge multiplet contains the vector, A_μ, six scalars ϕ_i, $i = 1, \ldots, 6$ (representing the transverse motions), and four two-component Weyl fermions, λ_a, $a = 1, \ldots, 4$, the fermionic superpartners of the eight physical bosonic degrees of freedom. There is a $SO(6) \simeq SU(4)$ R-symmetry under which the scalars transform as the **6** and the fermions transform as the **4**. The theory is conformally invariant, (i.e. it has vanishing β-function) with the conformal group being $SO(2, 4)$, which contains the Poincaré group, the dilatations, etc., as discussed in sections 3.1 and 10.1.9.

The low energy supergravity solution is:

$$\begin{aligned} ds^2 &= H_3^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + H_3^{1/2} dx^i dx^i, \\ e^{2\Phi} &= g_{\rm s}^2, \\ C_4 &= ({H_3}^{-1} - 1) g_{\rm s}^{-1} dx^0 \wedge \cdots \wedge dx^3, \end{aligned} \tag{18.1}$$

where $\mu = 0, \ldots, 3$, and $i = 4, \ldots, 9$, and the harmonic function H_3 is

$$H_3 = 1 + \frac{4\pi g_{\rm s} N {\alpha'}^2}{r^4}. \tag{18.2}$$

We are instructed to send $\alpha' \to 0$, and hold a quantity $u = r/\alpha'$ fixed. This limit focuses on the neighbourhood of the horizon of the brane, and

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a computation gives the following metric[270]:

$$\begin{aligned}\frac{ds^2}{\alpha'} &= \frac{u^2}{L^2}(-dt^2+dx_1^2+dx_2^2+dx_3^2)+L^2\frac{du^2}{u^2}+L^2d\Omega_5^2;\\ {}^*dC_{(4)} &= 16\pi\alpha'^2N\epsilon_{(5)}.\\ \text{where}\quad L^2 &= \sqrt{2g_{\rm YM}^2N}=\frac{\ell^2}{\alpha'}.\end{aligned}\tag{18.3}$$

We have written it such that we can see the lengths measured by the metric in units of the string length.

From our work in section 10.1.9, we recognise this solution as the metric of $\mathrm{AdS}_5\times S^5$, where the cosmological constant and the radius of the sphere is set by $\ell^2=\alpha'\sqrt{2g_{\rm YM}^2N}$, a combination of the supergravity/string theory parameters which gives the gauge coupling. Just as in the case of the Reissner–Nordström horizon, the near-horizon geometry of the D3-brane is a smooth 'throat' geometry, with size set by the charge of the solution. Since, as was discussed in chapter 10, $\mathrm{AdS}_5\times S^5$ is a maximally symmetric solution, just like Minkowski space, we see that the extremal D3-brane is an interpolating soliton solution, like extremal Reissner–Nordström[65] (see insert 1.4). The extremal M-brane solutions of section 12.6.1 also share this property[68].

Let us observe that the limit of small r is also a sensible restriction to low energy from the point of view of supergravity. Recall an effect which is familiar from considerations of ordinary gravity solutions such as black holes. There is a redshift effect, which means that the energy, as measured at asymptotic infinity, of a signal originating at radius r is decreased due to a multiplicative factor $\sqrt{g_{tt}(r)}=H_3^{-1/4}$, arising from having to climb out of the gravitational well produced by the solution. This redshift is infinite as $r\to 0$, and so the throat is decoupled from the asymptotic regime in the low energy limit.

Now we should ask about the regime of validity of this geometry. We have to examine the amount of curvature this solution has, and this is set by the size of a typical squared curvature invariant, R^2. We have sent α' to zero and also are keeping $g_{\rm s}$ small, to remain in the supergravity regime (string tree level). Looking at the essential controlling function (18.2), we see that we have one more parameter we can adjust, and this is N. If we make N large, we can keep the curvatures low. More properly, if we keep the effective coupling $\lambda=g_{\rm YM}^2N$ large enough, we can ensure that we stay at closed string tree level and decouple the higher string modes by sending $g_{\rm s},\alpha'\to 0$. Notice that this limit is the regime that open string and hence gauge theory perturbation theory breaks down. So we have a useful complementarity. The large N, strong 't Hooft coupling limit of

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the gauge theory has a description in terms of the supergravity solution above. This is the '*AdS/CFT correspondence*'[270, 271, 272]. The corrections to this in a $1/N$ expansion are the usual stringy loop corrections to the supergravity. In fact, the string coupling is to be identified with $1/N$, just as was anticipated long ago in general terms for gauge theories at large N (see insert 18.1). We have a concrete realisation of this conjectured string theory as type IIB on $\mathrm{AdS}_5 \times S^5$. Notice that the $SO(4,2)$ and $SO(6)$ isometries of each space become the conformal group and the R-symmetry of the gauge theory.

Before we go any further, let us therefore compute the five dimensional Newton constant, G_5 in terms of our compactification on an S^5 of radius ℓ. We get $1/G_5 = (\mathrm{Vol}(S^5)\ell^5)/G_{10}$. Looking at our expression for G_{10} in equation (7.44), and the one for ℓ in equation (18.3), it is prudent to substitute for ℓ^8, giving our first precise entry in the AdS/CFT dictionary:

$$G_5 = \frac{\pi \ell^3}{2N^2}, \tag{18.4}$$

since the volume of a unit S^5 is π^3. This will be useful to us many times later, since we will want to convert gravitational quantities to gauge theory ones. Notice that this formula also confirms for us in five dimensional terms that for fixed string length, the large N limit keeps us at tree level in the effective string theory, and hence just gravity. The effective closed string coupling is $g_{\mathrm{eff}} \sim 1/N$.

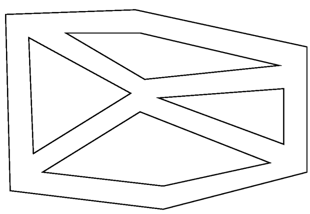
18.1.2 Sphere reduction and gauged supergravity

We have arrived at a remarkable connection between a particular large N gauge theory (pure $\mathcal{N} = 4$ supersymmetric $D = 4$ $SU(N)$) and a truncation of type IIB string theory on $\mathrm{AdS}_5 \times S^5$. For many purposes, it is useful to think of this as simply a five dimensional theory, obtained by the analogue of Kaluza–Klein reduction on S^5. The resulting five dimensional theory is in fact a five dimensional $\mathcal{N} = 8$ 'gauged supergravity', with 15 vector fields acting as gauge bosons of an $SO(6)$ gauge symmetry. There are in fact 42 scalars in the theory, which in general are charged under the $SO(6)$.

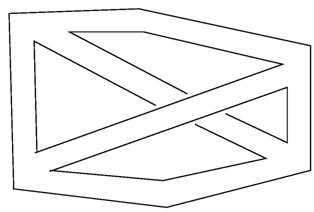
One way to think of how to arrive at the gauged supergravity theory and the resulting solution we are studying is as follows[302]. Start with the T^5 reduction of type IIB, which gives an $\mathcal{N} = 8$ theory in five dimensions with a *global* $E_{6(6)}$ symmetry. The 42 scalars ϕ_i in the resulting theory live on the coset $E_{6(6)}/USp(8)$. We discussed this theory in the context of U-duality in section 12.7, where we saw that wrapped branes filled out the various multiplets of the symmetry. Starting with this theory, it is possible

Insert 18.1. The large N limit and string theory

Quite general considerations[301] can lead to a search for a string theory description of gauge theory at large N. In the commonly used scaling, a power of g_{YM} is absorbed into the fields in order to write the Lagrangian as $\mathcal{L} \sim -\mathrm{Tr}F^2/(4g^2_{\mathrm{YM}})$, and this is the only appearance of g^2_{YM}. So there is an overall N/λ (where $\lambda = g^2_{\mathrm{YM}}N$ is the ''t Hooft coupling') in front of the entire Lagrangian. Hence, vertices in Feynman graphs come with a factor N, while propagators come with $1/N$. Feynman graphs are drawn with a double line, one line carrying an index in the fundamental, the other an antifundamental: the full propagator is in the adjoint. (This might remind the reader of open string diagrams of chapter 2.) A closed loop will contribute an N, since a free index can run over all its N values. The reader might like to consider some vacuum graphs (appropriate to any field theory with adjoint fields):



$$\left[\frac{1}{N}\right]^8 N^5 N^5 = N^2$$



$$\left[\frac{1}{N}\right]^6 N^4 N^2 = N^0$$

A graph with E edges (propagators), V vertices (interactions) and F faces (closed loops) can be drawn on a surface of Euler number $\chi = F - E + V = 2 - 2h$. The second equality is familiar from chapter 2, relating to the genus (number of handles h) of a closed Riemann surface. Overall, a graph comes with a factor N^χ and is some polynomial in λ, and so planar (sphere) diagrams dominate at large N, followed by toroidal, etc. As this is reminiscent of a closed string theory diagram of the same topology, this suggests the identification $g_{\mathrm{s}} \sim 1/N$. With reasoning along these lines, it was suspected that there might be stringy descriptions of large N gauge theories, where the string coupling is related to N as above. The difficulty was trying to find such a string theory. It surely could not be one of the strings used for 'theories of everything', since those would be too simple, it was thought. Now we see that we can use such strings, but propagating on interesting spacetimes, as we shall uncover.

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to make some of the global symmetry local, gauging by letting some of the vectors of $E_{6(6)}$ enter into the covariantised versions of the derivatives. In fact, to achieve gauge invariance, such a procedure ultimately leads one to go beyond minimal coupling and generate potentials for the scalars, and the largest subgroup that can be consistently gauged turns out to be $SO(6) \subset E_{6(6)}$. There is a non-trivial potential, $V(\phi_i)$, for the scalars, coming from the non-minimal coupling procedure, and the effective theory is of the form (looking just at the bosonic gravity and scalar sector):

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-G} \left\{ R - \mathcal{G}_{ij} \partial_\mu \phi^i \partial^\mu \phi^j - V(\phi_i) \right\}. \tag{18.5}$$

AdS_5 with the particular value of the cosmological constant $\Lambda = -6/\ell^2$ and with an $SO(6)$ gauge symmetry is a very special solution to this. It is a *fixed point* for the scalars, and so $\partial\phi_i/\partial x^\mu = 0$ and they all vanish $\phi_i = 0$, and so $SO(6)$ is preserved, since there are no non-zero fields charged under it in this case. The value of the potential is $V(\phi_i = 0) = -12/\ell^2$ and so we have:

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-G} \left\{ R - 2\Lambda \right\}, \tag{18.6}$$

for which the maximally symmetric solution is AdS_5 with $\Lambda = -6/\ell^2$.

This way of thinking of AdS is quite useful, since it leads immediately to an intuitive understanding of what is going on in the gauge theory in more complicated situations we will encounter in chapter 19.

The other way to think of our $SO(6)$ symmetric solution is in ten dimensional terms, which is how we began. However, it can be thought of as a Kaluza–Klein truncation of the ten dimensional theory by placing it on an S^5. The ansatz that is used is that the five-form $F_{(5)}$ is set by some constant times the volume five-form $\epsilon_{(5)}$ of the S^5:

$$F_{(5)} = 4\ell^4 \epsilon_{(5)} + 4\ell^{4*}\epsilon_{(5)}.$$

This is the *'Freund–Rubin ansatz'*[19], and with this choice, the ten dimensional equations of motion decompose into two sectors:

$$R_{\mu\nu} = -\frac{4}{\ell^2} g_{\mu\nu}, \qquad R_{mn} = +\frac{4}{\ell^2} g_{mn},$$

(with μ, ν having Lorentzian signature $(-++++)$ and m, n having Euclidean $(+++++)$) which is the precise generalisation of that which we saw happen for the Reissner–Nordström solution in section 10.1.11. The maximally symmetric solutions are of course AdS_5 and S^5.

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18.1.3 Extracts from the dictionary

Recall that we identified the scaled radial coordinate u as representing an energy scale in the gauge theory. It is natural to therefore to consider the limit $u \to 0$ as the infra-red (IR) and $u \to \infty$ as the ultra-violet (UV). We must not forget that our theory is defied as strongly coupled even in the UV, since it is conformal, and we must keep $\lambda = g_{\rm YM}^2 N$ large to remain within gravity.

The limit $u \to \infty$ defines a natural boundary of AdS. In the coordinates used in (18.3) any radial slice is in fact four dimensional Minkowski space, but $u = \infty$ is special for us, since on the one hand it takes a finite time for massless particles to propagate from $u = 0$, reflect at $u = \infty$, and return. On the other hand, the UV is the natural limit in which to discuss intuitive objects in gauge theory, like pointlike operator insertions.

Notice that large u seems like an IR limit for the AdS side of the duality, while it is UV on the side of the CFT. This is a feature of what is known as the 'UV/IR correspondence'. (See also the discussion before equation 6.1.)

When the common phrase 'the dual theory lives on the boundary', or some variation of it, is used, it should be understood that it is a short-hand for this UV identification. There are many properties (or quantities within) the dual which cannot unambiguously be placed at the boundary, and so we should be careful. It is better to think of the dual theory as being *everywhere**, and a slice at some value of u simply focuses on the effective theory obtained by working at a cutoff defined by the energy u, and the background geometry has metric $\gamma_{\mu\nu} = (\ell^2/u^2)h_{\mu\nu}$, where $h_{\mu\nu}$ is the metric induced on the boundary by restricting the five dimensional metric to constant u.

Recall that our coordinates inherited from the brane solution put us on AdS in local coordinates. We know from section 10.1.9 that we can consider this as a local patch of global AdS_5, and so it is natural to consider the same field theory dual to AdS in these coordinates. For example, for global AdS_5 we write:

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2}\right) dt^2 + \left(1 + \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2. \tag{18.7}$$

Going to the radial slice at infinity, we see that the dual theory is on a background $\mathbb{R} \times S^3$ with metric $ds^2 = -dt^2 + \ell^2 d\Omega_3^2$, which is the Einstein static universe. The local coordinates before had us studying a system for which the dual is on $\mathbb{R} \times \mathbb{R}^3$. There is the temptation to be confused at

* Frustratingly, perhaps, even better is not to think of the dual gauge theory as anywhere in the five or ten dimensional spacetime at all. It is simply the dual.

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this point, since we are supposed to think of the physics as independent of the coordinates, but somehow this seems to matter here. In fact, we must recall that in making the choices of coordinates here, we are also choosing a specific time slicing. This means that we are making choices which will affect our definition of the Hamiltonian of the theory. Further, since the radial coordinate seems to refer to the energy scales within the dual theory, being able to choose alternative radial foliations would seem to make good physical sense, since it refers to a choice of regulator at a given energy scale.

A large part of the rest of the dictionary of the AdS/CFT correspondence comes from equating the partition functions of the two theories:

$$Z_{\text{AdS}}(\phi_{0,i}) = Z_{\text{CFT}}(\phi_{0,i}). \tag{18.8}$$

Here the quantities $\phi_{0,i}$ have two interpretations: On the gravity side, these fields correspond to the boundary values (i.e. at $r = \infty$) for the bulk fields ϕ_i which propagate in AdS. This includes not just the 42 scalars, but *all fields*, including the graviton and the gauge fields. On the field theory side, the $\phi_{0,i}$ correspond to external sources or currents coupled to operators in the CFT. We can then obtain insertions of these operators by differentiation of the partition function with respect to the sources. This fits rather nicely, since all fields on the gravity side have specific $SO(6)$ gauge charges, which matches the corresponding R-charge of the inserted operator.

In fact, there is a specific[271, 272, 274] relation which can be derived between the dimension Δ of an operator which a scalar couples to and the mass m of the scalar in the bulk spacetime. As a solution to the wave equation in AdS_5, a scalar $\phi_i(r, \mathbf{x})$'s asymptotic behaviour is in fact:

$$\phi_i(r, \mathbf{x}) = e^{\frac{r}{\ell}(\Delta_i - 4)}\phi_m(\mathbf{x}) + e^{-\frac{r}{\ell}\Delta_i}\phi_v(\mathbf{x}), \tag{18.9}$$

where

$$\Delta_i = 2 + \sqrt{4 + m_i^2\ell^2}. \tag{18.10}$$

In fact, the first term is a non-normalisabile solution while the other term is normalisable. They both have a meaning in the theory. The first term is interpreted as switching on or 'inserting' the operator, while the latter term has the interpretation as controlling the vacuum expectation value of the operator. We shall see this interpretation in action with specific examples later on. In fact, there is a generalisation of this to the case of a p-form field in AdS_D. It couples to a $(D-p-1)$-form operator and the dimension is:

$$\Delta = \frac{D-1-2p}{2} + \left\{\frac{(D-1-2p)^2}{4} + m^2\ell^2 - p(p-D+1)\right\}^{1/2}. \tag{18.11}$$

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We list how the basic 42 scalars are interpreted in the gauge theory in table 18.1. In the table, the trace is over the adjoint of the gauge group,

Table 18.1. *An extract from the dictionary of the AdS/CFT correspondence*

Scalar	$m^2\ell^2$	Operator	$SO(6)$ charge	Δ
Φ	0	$\mathrm{Tr}\,[{}^*F\wedge F]$	$\mathbf{1}$	$\mathbf{4}$
$C_{(0)}$	0	$\mathrm{Tr}\,[F\wedge F]$	$\mathbf{1}$	$\mathbf{4}$
φ_1	-3	$\mathrm{Tr}[\lambda_{(a}\lambda_{b)}]$	$\mathbf{10}$	$\mathbf{3}$
$\bar{\varphi}_1$	-3	$\mathrm{Tr}[\bar{\lambda}_{(a}\bar{\lambda}_{b)}]$	$\overline{\mathbf{10}}$	$\mathbf{3}$
φ_2	-4	$\mathrm{Tr}[\phi_{(i}\phi_{j)}]-\frac{1}{6}\delta_{ij}\mathrm{Tr}\,[\phi_k\phi_k]$	$\mathbf{20}$	$\mathbf{2}$

under which every field transforms. In the first two lines we recognise our two friends from ten dimensions, the dilaton and the R–R scalar. As is to be expected, the dilaton couples to the basic Yang–Mills Lagrangian of dimension four, since its asymptotic value sets the string coupling (a fact we know from way back in chapter 2), and $g^2_{\rm YM}=2\pi g_{\rm s}$. Similarly, we know that the R–R scalar couples to the topological term of dimension four, controlling instanton, from our studies in chapter 9. These fields were not involved in the sphere reduction and so do not have and non-trivial $SO(6)$ charges.

The rest of the 42 scalars couple a different class of operators. The first set, with $m^2=-2/\ell^2$ couples to a symmetrised product of the scalars ϕ_i, with the trace removed.

N.B. The reader may be disturbed by the fact that the scalars can have a negative mass squared. It turns out that the presence of negative cosmological constant requires us to reexamine the issue of stable scalar fluctuations about the vacuum. The result is that there is a window of squared masses below zero up to the value $-4/\ell^2$ which is stable. This lower bound is the *'Breitenlohner–Freedman'* bound[303], and its negativity is a crucial feature which helps the dictionary to work.

Recalling that the scalars are in the $\mathbf{6}$ of $SO(6)$, a little group theory confirms that $\mathbf{6}\otimes\mathbf{6}=\mathbf{20}\oplus\mathbf{15}\oplus\mathbf{1}$, where the latter is the trace, and the previous two are the symmetric and antisymmetric combinations. In fact, there is a whole tower of Kaluza–Klein harmonics which can be made by such symmetrised products of the scalars. The reader might recognise these from group theory as the spherical harmonics of S^5, which

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are indeed made this way most commonly. These operators, which are the *'chiral primaries'* of the conformal field theory, couple to the basic tower of scalars which arise in the Kaluza–Klein spectrum with these same $SO(6)$ transformation properties.

The other set of scalars with $m^2 = -3/\ell^2$, couple to an antisymmetrised product of the fermions λ_a. These transform in the $\mathbf{4}$. Representation multiplication gives $\mathbf{4} \otimes \mathbf{4} = \mathbf{10} \oplus \mathbf{6}$, and since the λ_a are fermions, it is the $\mathbf{10}$ which is picked out. A similar structure exists for the $\bar{\lambda}_a$, which are in the $\bar{\mathbf{4}}$, and hence give the $\overline{\mathbf{10}}$.

Correlation functions of the various operators in the CFT can be determined through calculation involving the dynamics of the various scalars to which they couple, propagating in the AdS spacetime. This is a very powerful technique which we do not have time to explore here.

Just as we did in the case of black hole studies at the beginning of chapter 17, one can consider evaluating the AdS partition function in a saddle-point approximation:

$$e^{-I_{\mathrm{AdS}}(\phi_i)} = \left\langle e^{\int \phi_{0,i}\mathcal{O}^i} \right\rangle_{\mathrm{CFT}}, \tag{18.12}$$

where $I_{AdS}(\phi_i)$ is the classical gravitational action as a functional of the (super)gravity fields, and $\mathcal{O}^i$ are the dual CFT operators. Hence, in this approximation the AdS action becomes the generating function of the connected correlation functions in the CFT[271, 272]. This framework is also naturally extended to considering CFT states for which certain operators acquire expectation values by considering solutions of the gravitational equations which are only asymptotically AdS[273]. We shall do that below, but first we will explore a little of the technology of evaluating the action.

18.1.4 The action, counterterms, and the stress tensor

We need to make sense of the path integral of gravity on AdS, given on the left hand side of the correspondence dictionary in equation (18.12). This returns us to the issue of calculating the action of the theory, from which we can compute such quantities as the stress-energy-tensor, and if (as we did for asymptotically flat black holes in chapter 17) we were to Euclideanise, various thermodynamic quantities such as the energy, entropy, etc.

Recall from earlier discussions in chapter 17 that the action in D dimensions is defined as follows:

$$I_{\mathrm{bulk}} + I_{\mathrm{surf}} = -\frac{1}{16\pi G_5}\int_{\mathcal{M}} d^D x\sqrt{g}\,(R - 2\Lambda) - \frac{1}{8\pi G_5}\int_{\partial\mathcal{M}} d^{D-1}x\sqrt{h}K, \tag{18.13}$$

where, as we've seen in section 10.1.9, the cosmological constant is related to the length scale ℓ by

$$\Lambda = -\frac{(D-1)(D-2)}{2\ell^2}.$$

Recall that the second integral is the Gibbons–Hawking boundary term, which is required so that upon variation with metric fixed at the boundary, the action yields the Einstein equations. Here, K is the trace of the extrinsic curvature of the boundary $\partial\mathcal{M}$ as embedded in $\mathcal{M}$, which was discussed in insert 10.2.

Remember also that both of these expressions are divergent because the volumes of both $\mathcal{M}$ and $\partial\mathcal{M}$ are infinite (and the integrands are non-zero). The approach we used in section 17.2, (there, the first term vanished and the second term was divergent) to avoid this problem is to perform a 'background subtraction', producing a finite result by subtracting from equation (18.13) the contribution of a background reference spacetime, so that one can compare the properties of the solution of interest relative to those of the reference state. In our computation we ensured that the asymptotic boundary geometries of the two solutions can be matched in order to render the surface contribution finite.

In AdS, we can in fact follow a different approach, which has a natural meaning in the dual gauge theory[304]. We can supplement the action by a finite set of boundary integrals which depend only on the curvature scalar R (and its derivatives) of the induced boundary metric $h_{\mu\nu}$, which itself diverges as $r \to \infty$. These integrals look like a family of counterterms in the dual field theory, and with appropriate coefficients, they cancel the divergences (IR in AdS, UV in the gauge theory) as $r \to \infty$, giving a intrinsic definition of the action for asymptotically AdS spacetimes, with no reference to a background spacetime. Calling the set of boundary integrals† $I_{\rm ct}$, we define the complete action to be $I = I_{\rm bulk} + I_{\rm surf} + I_{\rm ct}$.

One of the useful quantities which we will extract from the action is the stress tensor, which is obtained by the standard expression:

$$T^{\mu\nu} = \frac{2}{\sqrt{-\gamma}}\frac{\delta I}{\delta\gamma_{\mu\nu}} = \lim_{r\to\infty}\left(\frac{r^2}{\ell^2}\frac{2}{\sqrt{-h}}\frac{\delta I}{\delta h_{\mu\nu}}\right), \qquad (18.14)$$

where in the second expression we have used $h_{\mu\nu}$ which is the metric on the boundary induced by restricting the bulk metric by setting r to a

† That this construction is unique to asymptotically AdS spaces is apparent because the AdS curvature scale ℓ is essential in defining the counterterms. We are excluding non-polynomial terms, which could be introduced in the absence of a cosmological constant[305], giving a definition that is applicable to spacetimes with other asymptotic behaviour.

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constant. In the first expression, $\gamma_{\mu\nu}$ is the metric obtained by removing a conformal factor r^2/ℓ^2 to get the dual field theory's natural metric in the UV. From this stress-tensor we can extract quantities like the energy density, etc., in the usual way, for example $\rho = T_{\mu\nu}u^\mu u^\nu$, where u^μ are the components of a timelike unit vector.

It turns out that the counterterm action is[304, 306]:

$$I_{\text{ct}} = \frac{1}{8\pi G_5}\int_{\partial\mathcal{M}} d^{D-1}x\sqrt{h}\left[\frac{D-2}{\ell} + \frac{\ell}{2(D-3)}\mathcal{R} + \frac{\ell^3}{2(D-5)(D-3)^2}\left(\mathcal{R}_{ab}\mathcal{R}^{ab} - \frac{D-1}{4(D-2)}\mathcal{R}^2\right) + \cdots\right]. \tag{18.15}$$

Here, $\mathcal{R}$ and $\mathcal{R}_{ab}$ are the Ricci scalar and Ricci tensor for the boundary metric, respectively. The three counterterms are sufficient to cancel divergences for $D \leq 7$.

Let us consider an example. Take AdS_5 in global coordinates, as given in equation (18.7). As stated beneath that equation, the metric $\gamma_{\mu\nu}$ is that of the Einstein static universe of radius ℓ. Computing with the first two counterterms in equation (18.15), the stress tensor becomes:

$$\begin{aligned} T_{tt} &= \frac{1}{8\pi G_5}\left(\frac{3}{8\ell}\right) + O\left(\frac{1}{r}\right), \\ T_{ij} &= \frac{1}{8\pi G_5}\left(\frac{1}{8\ell}\right)G_{ij} + O\left(\frac{1}{r}\right), \end{aligned} \tag{18.16}$$

where G_{ij} refer to the metric components in the angular directions for an S^3 of unit radius. In the $r \to \infty$ limit we see that we get a finite result, which can be written in the suggestive form:

$$T_{\mu\nu} = \frac{1}{64\pi G_5\ell}(4u_\mu u_\nu + \gamma_{\mu\nu}) = \frac{N^2}{32\pi^2\ell^4}(4u_\mu u_\nu + \gamma_{\mu\nu}), \tag{18.17}$$

using the conversion formula (18.4). This is the standard form (see equation (10.23)) for a *conformally invariant* perfect fluid's stress tensor (since it is traceless) of density $\rho = 3p$ in four dimensions with a spacetime of metric $\gamma_{\mu\nu}$. The overall prefactor is $\rho/3$, as written. We have used the conversion formula (18.4) to change gravitational quantities to field theory ones. Note that the stress tensor is traceless, as expected for a conformally invariant theory. The field theory is in an S^3 box of radius ℓ, and so we can integrate the energy density to give the total energy (the dual to the spacetime's gravitational mass) which is:

$$E = \frac{3\pi\ell^2}{32G} = \frac{3N^2}{16\ell}. \tag{18.18}$$

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In fact, this result matches expectations from field theory[304]. Since it is defined in a box, there is a Casimir energy. For free fields, the Casimir energy on $S^3 \times R$, the Einstein static universe of radius ℓ, may be found in the literature[293] to be:

$$E_{\rm Cas} = \frac{1}{960\ell}(4n_0 + 17n_{1/2} + 88n_1) = \frac{3(N^2-1)}{16\ell}, \tag{18.19}$$

where n_0 denotes the number of real scalars, $n_{1/2}$ is the number of Weyl fermions, and n_1 the number of vectors. We have inserted the correct values for the dual $SU(N)$ gauge theory, $n_0 = 6(N^2-1)$, $n_{1/2} = 4(N^2-1)$ and $n_1 = N^2-1$, giving an expression which agrees with the result (18.18) that we got from the stress tensor in the large N limit. (See also insert 17.2 for comments on the AdS_3 case.)

18.2 The correspondence at finite temperature

We arrived at the correspondence between the supersymmetric gauge theory and pure AdS by taking the near horizon limit of the extremal D3-brane solution. It is natural to try to give an interpretation of the non-extremal solution. A key difference between the two is that the latter solution is at finite temperature. As we shall see, these properties relate to those of the field theory.

18.2.1 Limits of the non-extremal D3-brane

Taking the decoupling limit of the solution given in equation (10.34) for $p=3$, we see that $\alpha_3 \to 1$ and so $H_3 \to \ell^4/r^4$ again, and so we can write the solution as[307, 271]:

$$ds^2 = -\left(\frac{r^2}{\ell^2} - \frac{r_{\rm H}^4}{\ell^2 r^2}\right)dt^2 + \frac{r^2}{\ell^2}\sum_{i=1}^{3} dx_i^2 + \left(\frac{r^2}{\ell^2} - \frac{r_H^4}{\ell^2 r^2}\right)^{-1} dr^2 + \ell^2 d\Omega_5^2, \tag{18.20}$$

where $\ell^2 = \alpha_3^{\frac{1}{2}}\hat{r}_3^2 \to \hat{r}_3^2$. This is in fact the AdS_5-Schwarzschild black hole, in local coordinates (its horizon is $\mathbb{R}^3$ instead of S^3), times an S^5 of radius ℓ. It is sometimes called a 'flat' black hole. In fact, its mass and temperature are easily computed to be:

$$M = \frac{3\pi r_{\rm H}^4}{8G_5\ell^2}, \qquad T = \frac{r_{\rm H}}{\pi\ell^2}. \tag{18.21}$$

Interpreting the mass as an energy in the field theory[271], we see that in fact that there is a familiar energy-temperature relation following the

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Stephan–Boltzmann law:

$$E = \frac{3\pi^4 \ell^3 N^2}{4} T^4. \tag{18.22}$$

18.2.2 The AdS–Schwarzschild black hole in global coordinates

It is easy to write a global version of the AdS–Schwarzschild black hole solution:

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2} - \frac{r_0^4}{\ell^2 r^2}\right) dt^2 + \left(1 + \frac{r^2}{\ell^2} - \frac{r_0^4}{\ell^2 r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2, \tag{18.23}$$

and we have relabelled $r_{\rm H}$ as r_0 since this will in general *not* be an horizon radius. A computation of the stress tensor gives:

$$\begin{aligned} T_{tt} &= \frac{1}{8\pi G_5}\left(\frac{3}{8\ell} + \frac{3r_0^4}{2\ell^5}\right) + O\left(\frac{1}{r}\right), \\ T_{ij} &= \frac{1}{8\pi G_5}\left(\frac{1}{8\ell} + \frac{r_0^4}{2\ell^5}\right) G_{ij} + O\left(\frac{1}{r}\right), \\ \text{and so:}\quad T_{\mu\nu} &= \frac{1}{8\pi G_5}\left(\frac{1}{8\ell} + \frac{r_0^4}{2\ell^5}\right)(4u_\mu u_\nu + \gamma_{\mu\nu}). \end{aligned} \tag{18.24}$$

In the last line we have taken the $r \to \infty$ limit and put it into the perfect fluid form. The mass-energy can be written as

$$M = \frac{3\pi\ell^2}{32G_5} + \frac{3\pi r_0^4}{8G_5\ell^2}, \qquad \Longrightarrow \qquad E = \frac{3N^2}{16\ell} + \frac{3N^2 r_0^4}{4\ell^5}, \tag{18.25}$$

which after conversion using equation (18.4), gives the Casimir energy we derived before, since we are in the same box, together with an energy over extremality which matches the energy density derived for the flat black hole in the previous subsection.

The horizon of the solution is located at the largest root, r_+, of the equation $G^{rr} = 0$:

$$V \equiv \left(1 + \frac{r^2}{\ell^2} - \frac{r_0^4}{\ell^2 r^2}\right) = 0 \qquad \Longrightarrow \qquad r_+ = \frac{1}{2}\left(\sqrt{\ell^4 + 4r_0^2} - \ell^2\right). \tag{18.26}$$

Notice that $r_+ \neq r_0$ for the global case, in general. The temperature of

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the solution can be computed to be:

$$T = \frac{V'}{4\pi}\bigg|_{r=r_+} = \frac{1}{2\pi\ell^2}\left(r_+ + \frac{r_0^4}{r_+^3}\right). \tag{18.27}$$

This expression is very interesting, since for a given temperature T, there are in fact *two* values of r_+ which solve the above relation, as is clear from figure 18.1. Notice that there is a minimum temperature below which there are no black hole solutions. We see also that there are two classes of black hole solutions. There is one branch which, for large r_+, the temperature goes linearly with the radius. The other branch goes at small r_+ as the inverse cube of r_+. These 'small' black holes have the familiar behaviour of five dimensional black holes in asymptotically flat spacetime, since their temperature decreases with increasing size. The term 'small' is appropriate, since they are smaller than the characteristic size set by the AdS scale ℓ, and so they have the characteristics of the asymptotically Minkowskian holes. Similarly, the 'large' black holes are obtained when ℓ is small compared to the horizon size. These cases are most apparent when taking the large or small ℓ limit of the equation (18.26). The large black hole limit gives the case $r_+ = r_0$ (which we previously called r_H) and the linear temperature behaviour seen in the case of the planar black holes obtained in local coordinates in equation (18.21).

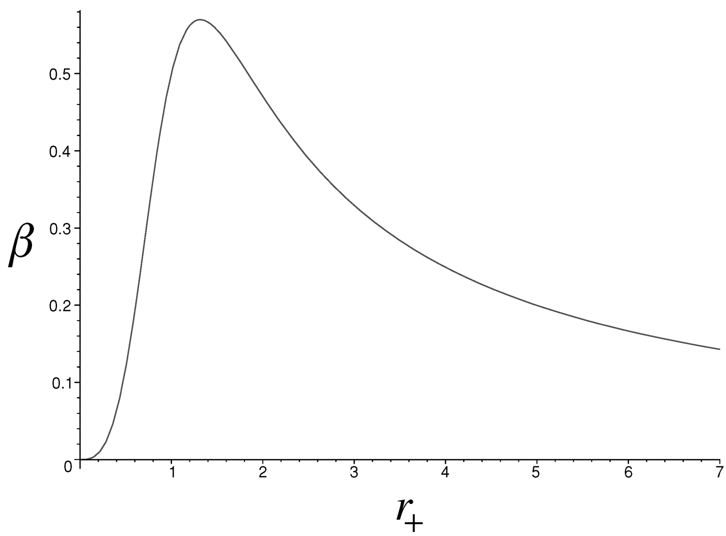

Fig. 18.1. The inverse temperature *vs.* horizon radii, r_+, for AdS black holes. There are two classes of holes, small and large, and a minimum temperature.

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18.3 The correspondence with a chemical potential

It is extremely natural, given what we saw in the previous sections, to ask about the role of charged black holes in this AdS scenario. There are $SO(6)$ gauge fields in the supergravity and so a black hole can be a charged source of them. We will focus on the Abelian case, and the $U(1)^3$ Cartan subalgebra is the maximal case. It is easy to see what in the dual gauge theory such a black hole would correspond to. An electric field will be supported by a potential of the form $A_t \sim r^{-2}$. Since this is a rank one massless field in AdS with this asymptotic, it must correspond, by the dictionary of equation (18.11), to a dimension four operator or current in the gauge theory. This is just what we would expect for an R-current, to which the gauge fields correspond. In other words, putting in a charged source is equivalent to switching on an external current source or chemical potential in the theory. It is instructive to construct the precise geometry, as our first example of non-trivial ten dimensional geometries which are dual to gauge theory.

18.3.1 Spinning D3-branes and charged AdS black holes

Given that the $SO(6)$ R-charge comes from Kaluza–Klein reduction from ten dimensions on an S^5, it is natural to guess that the appropriate geometry to seek is one which has some momentum in the compact directions which will be equivalent to the conserved R-charges in the theory. The internal velocities – which couple to momenta in a canonical formalism – will be the chemical potentials in the gauge theory, and hence conjugate to conserved R-charges[308, 309].

So we seek a 'spinning' D3-brane solution[308, 311]. There are six dimensions transverse to a D3-brane, and so we have three independent planes in which a rotation axis can be placed, to define three different angular momenta.

Let us first review some geometrical parameterisations which will be useful[312]. Using the angles θ, ψ on an S^2, we may introduce three direction cosines μ_i, with $\sum_i \mu_i^2 = 1, 0 < \theta \leq 2\pi, 0 < \psi \leq \pi$:

$$\mu_1 = \sin\theta, \qquad \mu_2 = \cos\theta\,\sin\psi, \qquad \mu_3 = \cos\theta\,\cos\psi. \tag{18.28}$$

In terms of these, and three more angles ϕ_i, $i = 1, 2, 3$, the metric on a round S^5 of unit radius can be written as follows ($0 \leq \phi_i < 2\pi$):

$$d\Omega_5^2 = \sum_i^3 (d\mu_i^2 + \mu_i^2\, d\phi_i^2). \tag{18.29}$$

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Now we can write the metric for the rotating solution, and it is[311]:

$$
\begin{aligned}
ds_{10}^2 &= H_3^{-1/2}\left(-\left[1-\frac{r_{\rm H}^4}{r^4\Delta}\right]dt^2+dx_1^2+dx_2^2+dx_3^2\right)\\
&\quad + H_3^{1/2}\left[\frac{\Delta dr^2}{\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3-r_{\rm H}^4/r^4}-\frac{2r_{\rm H}^4\cosh\beta_3}{r^4\,H_3\Delta}\,dt\left(\sum_{i=1}^3\ell_i\,\mu_i^2\,d\phi_i\right)\right.\\
&\quad \left.+r^2\sum_{i=1}^3\mathcal{H}_i\left(d\mu_i^2+\mu_i^2\,d\phi_i^2\right)+\frac{r_{\rm H}^4}{r^4}\frac{1}{H_3\Delta}\left(\sum_{i=1}^3\ell_i\,\mu_i^2\,d\phi_i\right)^2\right],\\
C_{(4)} &= \frac{g_{\rm s}^{-1}(H_3^{-1}-1)}{\sinh\beta_3}\Big(\cosh\beta_3\,dt-\sum_{i=1}^3\ell_i\,\mu_i^2\,d\phi_i\Big)\wedge dx_1\wedge dx_2\wedge dx_3,\\
e^{\Phi} &= g_{\rm s},
\end{aligned}
\tag{18.30}
$$

where the functions Δ, H_3, and $\mathcal{H}_i$ are given by

$$
\begin{aligned}
\Delta &= \mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\sum_{i=1}^3\frac{\mu_i^2}{\mathcal{H}_i}, \qquad H_3 = 1+\frac{r_{\rm H}^4}{r^4}\frac{\sinh^2\beta_3}{\Delta} = 1+\frac{\alpha_3}{\Delta}\frac{r_3^4}{r^4},\\
\mathcal{H}_i &= 1+\frac{\ell_i^2}{r^2}, \qquad i=1,2,3.
\end{aligned}
\tag{18.31}
$$

It will be useful at this point to refer to the expressions given for the boost form for the non-extremal solution given in section 10.2.2. The structure of the solution can be seen to be closely related to our non-extremal solution presented there, the key difference being that there is a deformation of the round S^5 produced by spoiling the three directions ϕ_i with deformations controlled by the three parameters ℓ_i. The $SO(6)$ isometry of rotation invariance is broken to $U(1)^3$ generated by the obvious Killing vectors $\partial/\partial\phi_i$. There are a number of interesting limits of this solution. One of them is discussed in insert 18.2, where we find an interesting form to which we will return later.

Most pertinent to this section is the decoupling limit of the solution, where we scale the rotation parameters in order to keep them finite in the limit. We write $r=\alpha' u$, $r_{\rm H}=\alpha' u_{\rm H}$, and since $r_3=\alpha'^2\hat{r}_3$, in the limit $\alpha'\longrightarrow 0$, we see:

$$
\begin{aligned}
\sinh\beta_3 \quad\text{and}\quad \cosh\beta_3 &\longrightarrow \frac{1}{\alpha'}\frac{\hat{r}_3^2}{u_{\rm H}^2},\\
H_3 &\longrightarrow \frac{1}{\alpha'^2}\frac{\hat{r}_3^4}{u^4}\frac{1}{\Delta},\\
\ell_i &\longrightarrow \alpha' q_i,\\
\mathcal{H}_i &\longrightarrow 1+\frac{q_i^2}{u^2},\\
\Delta(\ell_i,r) &\longrightarrow \Delta(q_i,u).
\end{aligned}
\tag{18.32}
$$

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Insert 18.2. D3-brane distributions

Particularly interesting is the straightforward extremal limit $\beta_3 \to \infty$, $r_{\rm H} \to 0$, holding $r_{\rm H}^4 e^{2\beta_3}/4 = r_3^4$ fixed, giving

$$\begin{aligned} ds_{10}^2 &= H_3^{-1/2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) \\ &\qquad + H_3^{1/2}\left[\frac{\Delta dr^2}{\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3} + r^2\sum_{i=1}^{3}\mathcal{H}_i\left(d\mu_i^2 + \mu_i^2 d\phi_i^2\right)\right], \\ H_3 &= 1 + \frac{1}{\Delta}\frac{\ell^4}{r^4}, \\ C_{(4)} &= g_{\rm s}^{-1}(H_3^{-1} - 1)\, dt\wedge dx_1\wedge dx_2\wedge dx_3, \qquad e^{\Phi} = g_{\rm s}. \end{aligned} \tag{18.33}$$

The terms corresponding to rotation have disappeared, leaving a solution which is supersymmetric[311, 313]. It, of course, still has N D3-branes composing it, but it is not spherically symmetric. The change of variables[314]:

$$\begin{aligned} y_1 &= \sqrt{(r^2+\ell_1^2)}\,\mu_1\cos\phi_1 = \sqrt{(r^2+\ell_1^2)}\,\sin\theta\cos\phi_1, \\ y_2 &= \sqrt{(r^2+\ell_1^2)}\,\mu_1\sin\phi_1 = \sqrt{(r^2+\ell_1^2)}\,\sin\theta\sin\phi_1, \\ y_3 &= \sqrt{(r^2+\ell_2^2)}\,\mu_2\cos\phi_2 = \sqrt{(r^2+\ell_2^2)}\,\cos\theta\sin\psi\cos\phi_2, \\ y_4 &= \sqrt{(r^2+\ell_2^2)}\,\mu_2\sin\phi_2 = \sqrt{(r^2+\ell_2^2)}\,\cos\theta\sin\psi\sin\phi_2, \\ y_5 &= \sqrt{(r^2+\ell_3^2)}\,\mu_3\cos\phi_3 = \sqrt{(r^2+\ell_3^2)}\,\cos\theta\cos\psi\cos\phi_3, \\ y_6 &= \sqrt{(r^2+\ell_3^2)}\,\mu_3\sin\phi_3 = \sqrt{(r^2+\ell_3^2)}\,\cos\theta\cos\psi\sin\phi_3, \end{aligned} \tag{18.34}$$

places the solution back into the familiar form:

$$ds^2 = H_3^{-1/2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + H_3^{1/2}\,(d\vec{y}\cdot d\vec{y}).$$

Now, H_3 is not of our simple forms discussed in chapter 10. It is still harmonic, as it ought to be, and we may write it in the integral form:

$$H_3(\vec{y}) = L^4\int d^6w\,\frac{\rho_3(\vec{w})}{|\vec{y}-\vec{w}|^4}, \quad \text{with} \quad \int d^6w\,\rho_3(\vec{w}) = 1, \tag{18.35}$$

where the density function ρ_3 – which may be derived implicitly from the change of variables (18.34) – encodes a general *distribution* of branes[313], which we shall study in more detail later in section 19.2.4.

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The last term in the metric in equations (18.30) vanishes in this limit, and after some algebra, the metric can be written as:

$$\frac{ds^2}{\alpha'} = \sqrt{\Delta}\left(-(\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3)^{-1} f\,dt^2 + f^{-1}du^2 + \frac{u^2}{\ell^2}[dx_1^2 + dx_2^2 + dx_3^2]\right) + \frac{1}{\sqrt{\Delta}}\sum_{i=1}^{3}\mathcal{H}_i\left(\ell^2 d\mu_i^2 + \mu_i^2\left(\ell d\phi_i + (\mathcal{H}_i^{-1} - 1)dt\right)^2\right), \tag{18.36}$$

where

$$f = \frac{u^2}{\ell^2}\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3 - \frac{1}{\ell^2}\frac{u_{\rm H}^4}{u^2}. \tag{18.37}$$

Now we can consider dimensional reduction to five dimensions of this solution. Pulling a factor $(\mathcal{H}_1\mathcal{H}_2\mathcal{H}_2)^{-1/3}$ into $\sqrt{\Delta}$ puts it into the standard Kaluza–Klein form for reduction to five dimensions, and we get:

$$\begin{aligned}\frac{ds^2}{\alpha'} &= -(\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3)^{-2/3} f\,dt^2 + (\mathcal{H}_1\,\mathcal{H}_2\,\mathcal{H}_3)^{1/3}(f^{-1}\,du^2 + \frac{u^2}{\ell^2}\,d\vec{x}\cdot d\vec{x}),\\ X_i &= \mathcal{H}_i^{-1}\,(\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3)^{1/3}, \qquad A_t^i = 1 - \mathcal{H}_i^{-1}.\end{aligned} \tag{18.38}$$

We have two scalar fields from the reduction, since $X_1X_2X_3 = 1$. There are three $U(1)$ gauge fields since there are are three independent isometry directions, ϕ_i.

The meaning of this solution might be more apparent if one sets all the q_i, and hence the $\mathcal{H}_i$, to be equal. Then comparing to equation (17.19), we recognise this as a family of charged five dimensional black holes, written in isotropic coordinates. One difference is that, just as earlier in section 18.2 these are actually 'flat' black holes, in the sense that the horizons are of $\mathbb{R}^3$ topology. There are spherical and hyperbolic versions which can be readily written down. Similarly, there are such generalisations in the case of the full ten dimensional rotating solution. In the case where all of the charges are different, we see that it is a quite general family, with three charges under the $U(1)^3$, and two scalar fields.

They are solutions of a $U(1)^3$ truncation of the $\mathcal{N} = 8$ $SO(6)$ gauged supergravity, with action:

$$I = -\frac{1}{16\pi G_5}\int d^5x\sqrt{-G}\Big(R - \frac{1}{2}(\partial\varphi_1)^2 - \frac{1}{2}(\partial\varphi_2)^2 - \frac{1}{4}\sum_i X_i^{-2}\,(F^i)^2 + \frac{4}{\ell^2}\sum_i X_i^{-1} + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma\lambda}\,F_{\mu\nu}^1\,F_{\rho\sigma}^2\,A_\lambda^3\Big). \tag{18.39}$$

In the above, the gauge fields and their field strengths are labelled 1, 2 or 3 for each of the three $U(1)$ sectors. The final term is a Chern–Simons

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type term, which only will be non-zero if there are both magnetic and electric charges present, which will not be the case here.

The two scalars φ_1 and φ_2 are contained in the three X_i, via a generalisation of the exponential ansatz that we used in simpler Kaluza–Klein cases. We write them as components of a two-vector, $\vec{\varphi} = (\varphi_1, \varphi_2)$, and:

$$X_i = e^{-\frac{1}{2}\vec{a}_i\cdot\vec{\varphi}}, \tag{18.40}$$

where the $\vec{a}_i$ sum to zero in order to ensure $X_1X_2X_3 = 1$, and we make the conventional choice[311]:

$$\vec{a}_1 = \left(\tfrac{2}{\sqrt{6}}, \sqrt{2}\right), \qquad \vec{a}_2 = \left(\tfrac{2}{\sqrt{6}}, -\sqrt{2}\right), \qquad \vec{a}_3 = \left(-\tfrac{4}{\sqrt{6}}, 0\right), \tag{18.41}$$

where $\vec{a}_i$ satisfy the dot products $\vec{a}_i \cdot \vec{a}_j = 4\delta_{ij} - \frac{4}{3}$.

18.3.2 The AdS–Reissner–Nordström black hole

A special case of this is to set all of the angular momenta to be equal, $q_i = q$ which makes all the $X_i = 1$, setting all of the scalars to zero. Then with $F^i_{(2)} = F_{(2)}/\sqrt{3}$, the action becomes[310, 308]:

$$I = -\frac{1}{16\pi G_5}\int d^5x\sqrt{-G}\left(R - \frac{1}{4}F^2 + \frac{12}{\ell^2}\right), \tag{18.42}$$

where the cosmological constant is $\Lambda = -6/\ell^2$ (we omit the Chern–Simons term, since we only have electric charges present) and the solution is:

$$\begin{aligned} ds_5^2 &= -\mathcal{H}^{-2} f\,dt^2 + \mathcal{H}(f^{-1}\,dr^2 + \frac{u^2}{\ell^2}\,d\vec{x}\cdot d\vec{x}),\\ A_t &= 1 - \mathcal{H}^{-1},\\ \mathcal{H} &= 1 + \frac{q^2}{r^2}, \quad f = \frac{u^2}{\ell^2}\mathcal{H}^3 - \frac{u_{\rm H}^4}{\ell^2 u^2}. \end{aligned} \tag{18.43}$$

As stated before, this is the form of our old friend from chapter 17, the Reissner–Nordström black hole in five dimensions (17.19), but now in anti-de Sitter spacetime and with an horizon with topology $\mathbb{R}^3$. We can make the global cousin of this which would have a spherical horizon by replacing $\ell^{-2}d\vec{x}\cdot d\vec{x}$) by $d\Omega_3^2$ and adding a 1 to the function f. We shall study this solution shortly[308].

18.3.3 Thermodynamic phase structure

By changing to a new radial coordinate r, in the same manner in which we did for equation (17.19), we write the black holes we have obtained in

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static coordinates in the form in which we have previously done our thermodynamic studies. For comparison to the earlier case of AdS–Schwarzschild in section 18.2, we shall, as promised, work with the spherical cousins, obtained as stated below equation (18.43):

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2 d\Omega_3, \tag{18.44}$$

where

$$V(r) = 1 - \frac{m}{r^2} + \frac{q^2}{r^4} + \frac{r^2}{\ell^2}. \tag{18.45}$$

Here, m is related to the mass M of the solution as

$$M = \frac{3\pi}{8G} m. \tag{18.46}$$

The $U(1)$ charge Q is related to q by

$$Q = \frac{\sqrt{3}\pi}{8\pi G} q. \tag{18.47}$$

Let r_+ denote the largest real positive root of $V(r)$. This defines the horizon:

$$r_+^6 + \ell^2 r_+^4 - \ell^2 m r_+^2 + q^2\ell^2 = 0.$$

The derivative of V is

$$V' = \frac{1}{r_+^5\ell^2}\left[2r_+^6 + 2mr_+^2\ell^2 - 4q^2\ell^2\right] = \frac{2}{r_+^5\ell^2}\left[2r_+^6 + r_+^4\ell^2 - q^2\ell^2\right],$$

and so for a non-singular horizon we must have $2r_+^6 + r_+^4\ell^2 \geq q^2\ell^2$. Now, as we've seen many times before, V' controls the temperature of the black hole via

$$\beta = \frac{4\pi}{V'} = \frac{2\pi r_+^5\ell^2}{2r_+^6 + r_+^4\ell^2 - q^2\ell^2}. \tag{18.48}$$

When the inequality above is saturated the horizon is degenerate, β diverges, and we get the zero temperature extremal black hole‡.

As before, we will choose a gauge in which A is regular at the horizon:

$$A = \sqrt{\frac{3}{4}}\left(-\frac{q}{r^2} + \Phi\right)dt, \quad \text{where} \quad \Phi = \frac{q}{r_+^2}. \tag{18.50}$$

‡ Note that this extremal case is *not* supersymmetric, as in asymptotically flat cases. The supersymmetric case is $m = 2q$, and then

$$V(r) = \left(1 - \frac{q}{r^2}\right)^2 + \frac{r^2}{l^2}, \tag{18.49}$$

which is clearly positive everywhere. This means that the curvature singularity at $r = 0$ is *naked* for this value[310].

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It is useful to rewrite the temperature in terms of the potential:

$$\beta = \frac{4\pi l^2 r_+}{2\ell^2(1 - \Phi^2/\Phi_c^2) + 4r_+^2}, \tag{18.51}$$

which will be useful later. Here, $\Phi_c = \sqrt{3/4}$. It is useful to observe the behaviour of the temperature as a function of black hole size r_+, as we did previously for the AdS–Schwarzschild case.

The reader may notice that there are qualitatively two distinct types of behaviour, determined by whether Φ is less than or greater than the critical value Φ_c. In particular, for $\Phi \geq \Phi_c$, β diverges (T vanishes) at $r_+^2 = \ell^2(\Phi^2/\Phi_c^2 - 1)/2$. This regime of large potential has a unique black hole radius associated with each temperature. Meanwhile for $\Phi < \Phi_c$, β goes smoothly towards zero as $r_+ \to 0$. This latter behaviour is just like that we observed in the case of AdS–Schwarzschild in figure 18.1. This small potential regime has two branches of allowed black hole solutions, a branch with larger radii and one with smaller. Correspondingly, the smaller branch of holes is unstable, having negative specific heat. Both cases are plotted in figure 18.2.

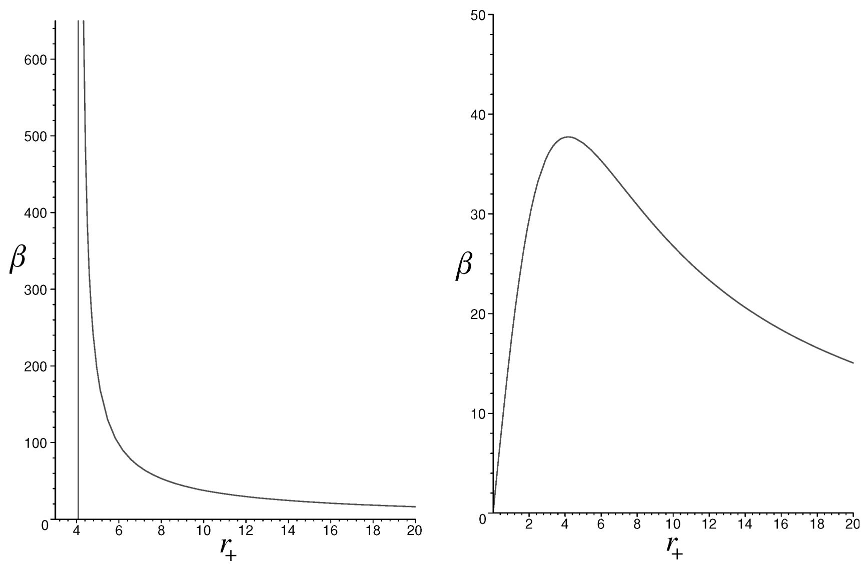

Fig. 18.2. The inverse temperature *vs.* horizon radii, r_+, at fixed potential for $\Phi \geq \Phi_c$, $\Phi < \Phi_c$. The divergence in the first graph (shown with a vertical line) is at zero temperature, where the black hole is extremal. This divergence goes away for $\Phi < \Phi_c$, in general, and the curve is similar to that of the situation with zero potential.

https://doi.org/10.1017/9781009401371.019 Published online by Cambridge University Press

We will study the Euclidean section ($t \to i\tau$) of the solution, at fixed temperature set by the period, β, of the imaginary time. We will work with fixed temperature and potential, defining thus the grand canonical thermodynamic ensemble using the Euclidean version of the action given in equation (18.42).

In fact, as both spaces we use are asymptotically AdS, it turns out that we need not consider the Gibbons–Hawking boundary term, since its contributions vanishes. The boundary terms from the gauge field will vanish if we keep the potential A_t fixed at infinity. Imposing the equations of motion we can obtain:

$$I^{\mathrm{E}} = \frac{1}{16\pi G}\int_M d^5x\sqrt{g}\left[\frac{F^2}{6}+\frac{8}{\ell^2}\right], \tag{18.52}$$

and we get, after substitution and integrating:

$$\begin{aligned} I &= \frac{2\pi^2}{16\pi G\ell^2}\beta\left(\ell^2 r_+^2 - r_+^4 - \frac{q^2\ell^2}{r_+^2}\right) \\ &= \frac{2\pi^2}{16\pi G\ell^2}\beta\left(\ell^2 r_+^2\frac{4}{3}(\Phi_{\mathrm{c}}^2-\Phi^2) - r_+^4\right). \end{aligned} \tag{18.53}$$

This is the grand canonical ensemble, at fixed temperature and fixed potential. The grand canonical (Gibbs) potential is $W = I^{\mathrm{E}}/\beta = E - TS - \Phi Q$. Using the expression in equation (18.53), we may compute the state variables of the system as follows:

$$\begin{aligned} E &= \left(\frac{\partial I^{\mathrm{E}}}{\partial\beta}\right)_\Phi - \frac{\Phi}{\beta}\left(\frac{\partial I^{\mathrm{E}}}{\partial\Phi}\right)_\beta = \frac{3\pi}{8\pi G_5}m = M, \\ S &= \beta\left(\frac{\partial I^{\mathrm{E}}}{\partial\beta}\right)_\Phi - I^{\mathrm{E}} = \frac{2\pi^2 r_+^3}{4G_5} = \frac{A_{\mathrm{H}}}{4G_5}, \qquad \text{and} \\ Q &= -\frac{1}{\beta}\left(\frac{\partial I^{\mathrm{E}}}{\partial\Phi}\right)_\beta = \frac{\sqrt{3}\pi}{8G_5}q. \end{aligned} \tag{18.54}$$

Together, they satisfy: $dE = TdS + \Phi dQ$.

In order to study the phase structure we must study the free energy $W = I^{\mathrm{E}}/\beta$ as a function of the temperature. It is shown in figure 18.3. The interpretation of this is as follows. At any non-zero temperature, for large potential ($\Phi > \Phi_{\mathrm{c}}$) the charged black hole is thermodynamically preferred, as its free energy (relative to the background of AdS with a fixed potential) is strictly negative for all temperatures.

This behaviour differs sharply from the small potential ($\Phi < \Phi_{\mathrm{c}}$) situation, which is qualitatively the same as the uncharged case. In that situation, the free energy is positive for some range $0 < T < T_{\mathrm{c}}$, and it is only

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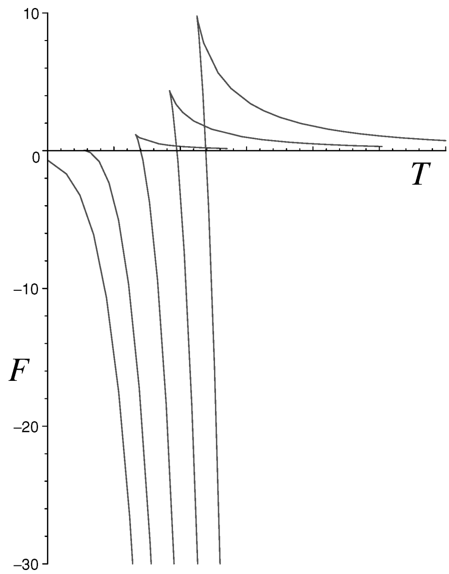

Fig. 18.3. A graph of the free energy vs. temperature for fixed potential ensemble. There is a crossover from the cusp behaviour in the case $\Phi < \Phi_c$ to the single branch ($\Phi > \Phi_c$) behaviour. The two branches consisting of smaller (unstable) and large (stable) black holes are visible. The entire unstable branch has positive free energy while the stable branch's free energy goes negative.

above T_c that the thermodynamics is dominated by AdS–Schwarzschild black holes (the larger, stable branch), as their free energy is negative.

So for high enough temperature in all cases the physics is dominated by non-extremal black holes[§]. This phase represents a sort of 'unconfined' phase of the dual gauge theory, while AdS without a black hole is a 'confined' phase[271]. There is a lot of evidence for this which we cannot uncover here due to lack of space. However, a clear sign of this an examination of the behaviour of the physical quantities we have computed, such as the energy and entropy. One can take the quantities in equations (18.54), converting them to the gauge theory quantities using equation (18.4), and find that there is an overall factor N^2. In an unconfined gauge theory, all of the N^2 adjoint degrees of freedom contribute on the same footing, and we see this here are an overall factor of N^2 in extensive quantities.

[§] The $\Phi = 0$ special case of this transition, from AdS to AdS–Schwarzschild black holes, was studied first by Hawking and Page[291]. The more general phase diagram was worked out later in the AdS/CFT context[308].

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At low temperatures, and for $\Phi > \Phi_c$, we have something very new. Notice that as we go to $T = 0$, the free energy curve approaches a maximum value which is less than zero. This implies that even at zero temperature the thermodynamic ensemble is dominated by a black hole. From the temperature curve (18.2) it is clear that it is the extremal black hole. For $\Phi = \Phi_c$, at $T = 0$ we recover AdS space, returning to the 'confined' phase. So this suggests that even at zero temperature the system prefers to be in a state with non-zero entropy (given by the area of the extremal black hole)¶.

The resulting thermodynamic phase structure for the fixed potential ensemble is summarised in figure 18.4. It represents in the dual gauge theory the phase diagram for the introduction of a chemical potential into the gauge theory, and there is a phase boundary across which there is a first order phase transition to the deconfined phase. It is intriguing that this may be a (highly simplified) prototype computation for the phase structure of more realistic gauge theories in analogous situations. One can imagine the chemical potential here being analogous to baryon number in QCD. This would then be an analogue of the finite temperature and density phase diagram, a subject of some current experimental interest, at the time of writing. Perhaps one future use of this gauge/gravity duality might be to model the generic phase structure of more realistic gauge theories using black hole and other objects within the gauge dual. On the one hand, it seems unrealistic to expect a direct connection, but on the other, there may be universality classes of behaviour which are quite robust to modification of the details, and so may be captured by studies of the sort presented here.

18.4 The holographic principle

As we have seen there is a close relationship between the physical properties of five dimensional AdS backgrounds and those of a four dimensional conformally invariant gauge theory. It is a remarkable duality, and is in fact the sharpest known example of what is called *holographic* behaviour[286, 287]: the physics involving gravity in a given number of dimensions is conjectured to be completely captured by a non-gravitational description in fewer dimensions.

¶ Notice that this $T = 0$ situation can be seen to display the 'confined' behaviour characteristic of the ordinary zero-temperature phase, despite the presence of the black hole. This follows from the fact that the horizon at extremality is infinitely far away down a throat. There is the possibility that the extremal black hole might decay away by emission of charged quanta, which is possible since it it not supersymmetric, and so this $T = 0$ part of the story should be studied further.

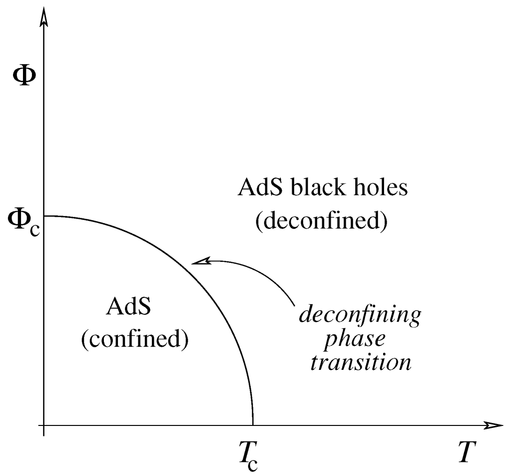

Fig. 18.4. The phase diagram for charged spherical black holes in global AdS. There is a transition from pure AdS to the black hole at finite temperature and potential. In the dual gauge theory, the black holes represent a deconfined phase of the theory. There is a boundary across which there is a first order phase transition between the two phases. The $\Phi = 0$ axis is the Schwarzschild case, with the Hawking–Page transition[291]. The $T = 0$ axis is the extremal charged case, also confining in the gauge theory.

The idea of why such a conjectured phenomenon should be a reality is motivated by the behaviour of black holes. They seem to represent all their degrees of freedom on their horizon, from the point of view of an observer who remains outside, and the universal result that their entropy is one quarter of the area of the horizon is a precise statement that the number of degrees of freedom within the volume that is occupied by the black hole is in fact only of order one per unit area of the horizon, as measured in Planck units.

The idea then is that in any quantum theory of gravity, the number of degrees of freedom in any volume are again just of order one per unit area of the surface surrounding the volume. This is enforced by the expectation that an attempt to examine the structure of the theory right down to the shortest distances, in order to learn about the microscopic degrees of freedom, will eventually probe energy densities which will dynamically favour the formation of a black hole, for which we believe the result is true. The largest obtainable entropy for a given volume is that held by a black hole which fills that volume. This puts an upper limit on the number of degrees of freedom as that given by the total surrounding area.

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The AdS/CFT correspondence can be examined in the light of just this type of argument and seen to realise precisely this type of arrangement[315]. In this case, it takes the physics of gravity in five dimensional anti-de Sitter spacetime and makes a hologram of it in terms of a gauge theory. This is also true for anti-de Sitter spacetimes of other dimension too: the hologram is again a non-gravitational conformal field theory in one dimension fewer. Some of the best known examples are as follows. There is AdS_3, which is dual to the 1+1 dimensional gauge theory arising from D1- and D5-branes intersecting. This was responsible for controlling a number of universal properties of five dimensional black holes which we uncovered in chapter 17. The cases of AdS_4 and AdS_7 are also natural in this context. They arise as near-horizon limits (times S^7 and S^4 respectively) of the M2- and M5-brane geometries discussed in chapter 12 (the reader can check this directly). In fact, as hinted at previously (see section 12.6.2), there are important conformal field theories, with sixteen supercharges (in 2+1 and a 5+1 dimensions), on the world-volumes of these branes, whose direct Lagrangian definitions are not known. However, the theories certainly exist as limits of more familiar theories, and the AdS/CFT relation can be taken as a definition of the properties of these theories via the holographic duality.

The holographic expectation has been elevated to the status of a principle, although at present there is a scarcity of well-understood examples outside the AdS/CFT examples and their close cousins. A very active area of research is the endeavour to find further examples, since this is clearly an important clue regarding the nature of fundamental physics about which we should learn more.