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INTEGRAL KERNELS WITH REFLECTION GROUP INVARIANCE

Dedicated to P. G. Rooney on the occasion of his 65th birthday

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Root systems and Coxeter groups are important tools in multivariable analysis. This paper is concerned with differential-difference and integral operators, and orthogonality structures for polynomials associated to Coxeter groups. For each such group, the structures allow as many parameters as the number of conjugacy classes of reflections. The classical orthogonal polynomials of Gegenbauer and Jacobi type appear in this theory as two-dimensional cases. For each Coxeter group and admissible choice of parameters there is a structure analogous to spherical harmonics which relies on the connection between a Laplacian operator and orthogonality on the unit sphere with respect to a group-invariant measure. The theory has been developed in several papers of the author [4,5,6,7]. In this paper, the emphasis is on the study of an intertwining operator which allows the transfer of certain results about ordinary harmonic polynomials to those associated to Coxeter groups. In particular, a formula and a bound are obtained for the Poisson kernel.

The presentation begins with a quick review of the basic definitions and then some integral identities involving Laguerre polynomials and the Gaussian measure. Next there is a study of the analogy between partial derivatives and the differential-difference operators as applied to inner products on spaces of polynomials.

The intertwining operator is then defined and shown to be a bounded linear operator with respect to a useful norm on polynomials (absolutely convergent series of homogeneous parts). The author conjectures that the intertwining operator is a positive integral transform in general (in one dimension, it is a form of Weyl's fractional integral). The paper ends with a reasonably explicit integral for the Poisson kernel for the ball, a kernel which reproduces certain functions from their boundary values, and some examples coming from the group Z_2, including Gegenbauer and disk polynomials.

1. **Background.** Suppose that G is a finite reflection group (also called Coxeter group) on $\mathbb{R}^N$ with the set $\{\, v_i : i = 1,2,\dots,m\}$ of positive roots. Let σ_i denote the reflection along v_i, that is, $x\sigma_i : x - 2(\langle x, v_i\rangle \,/\, |v_i|^2)v_i$ for $x \in \mathbb{R}^N$ with the inner product

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$\langle x, y\rangle := \sum_{i=1}^{N} x_i y_i$ and the norm $|x| := \langle x, x\rangle^{1/2}$. Thus G is a finite orthogonal group generated by reflections; the set of reflections in G is $\{\sigma_i : 1 \le i \le m\}$; and there is a fixed vector x_0 so that $\langle v_i, x_0\rangle > 0$ for $1 \le i \le m$.

Choose positive parameters α_i, $1 \le i \le m$, such that $\alpha_i = \alpha_j$ whenever σ_i is conjugate to σ_j in G (assume also that $|v_i| = |v_j|$ in this case, so that $v_i w = \pm v_j$ for some $w \in G$).

Define $h(x) := \prod_{j=1}^{m} |\langle x, v_j\rangle|^{\alpha_j}$, a positively homogeneous G-invariant function of degree $\gamma := \sum_{j=1}^{m} \alpha_j$. The invariance refers to the action of G on functions on $\mathbb{R}^N$ defined by $R(w)f(x) = f(xw)$, $x \in \mathbb{R}^N$, $w \in G$. The hypotheses on α_i imply that $R(w)h = h$ for all $w \in G$.

The main concern of this work is the orthogonality structure for polynomials on the unit sphere $S := \{x \in \mathbb{R}^N : |x| = 1\}$ with respect to the measure $h^2 d\omega$, where $d\omega$ is the normalized rotation-invariant measure on S. The key device is the differential-difference operator

$$\nabla_h f(x) := \nabla f(x) + \sum_{j=1}^{m} \alpha_j \frac{f(x) - f(x\sigma_j)}{\langle x, v_j\rangle} v_j,$$

(∇ is the gradient), and its components T_i, where $T_i f(x) := \langle \nabla_h f(x), e_i\rangle$ (standard unit vectors $e_1, e_2, \dots, e_N$).

Let $\mathcal{P}_n$ denote the space of homogeneous polynomials of degree n in $x_1, x_2, \dots, x_N$. Say that a linear operator L is homogeneous of degree k if $L\mathcal{P}_n \subset \mathcal{P}_{n+k}$ for each $n = 0, 1, \dots$. It was shown in [5] that ∇_h is homogeneous of degree -1, and the set $\{T_i : i = 1, \dots, N\}$ generates a commutative algebra of operators containing the h-Laplacian $\Delta_h := \sum_{i=1}^{N} T_i^2$.

The main orthogonality theorem [4, p. 37] states that if $p \in \mathcal{P}_n$, then

$$\int_S pqh^2\, d\omega = 0 \text{ for all } q \in \sum_{j=0}^{n-1} \mathcal{P}_j$$

if and only if $\Delta_h p = 0$.

Accordingly, we define $\mathcal{H}_n^h := \mathcal{P}_n \cap (\ker \Delta_h)$, the space of *$h$-harmonic* polynomials of degree n. Also let $\mathcal{H}_n := \mathcal{P}_n \cap (\ker \Delta)$, the ordinary harmonic polynomials.

We will study the intertwining operator V which is homogeneous of degree 0 and is uniquely defined by $T_i V = V\partial_i$, $1 \le i \le N$, and $V1 = 1$ (where $\partial_i f(x) = \frac{\partial}{\partial x_i} f(x)$). Note that $V\mathcal{H}_n \subset \mathcal{H}_n^h$. We also study various inner products on polynomials and obtain bounds on the Poisson kernel for h-harmonic functions in the unit ball. The Gaussian distribution gets involved with Δ_h by use of polar coordinates. We collect some facts about such integrals.

Let μ be the normalized Gaussian measure on $\mathbb{R}^n$ defined by $d\mu(x) = (2\pi)^{-N/2} e^{-|x|^2/2}\, dx$. If f is a continuous function of polynomial growth on $\mathbb{R}^N$, then

$$\int_{\mathbb{R}^N} f\, d\mu = \left(2^{1-N/2} \Big/ \Gamma\left(\frac{N}{2}\right)\right) \int_0^\infty r^{N-1} e^{-r^2/2}\, dr \int_S f(rx)\, d\omega(x). \tag{1.1}$$

If f is positively homogeneous of degree $2k$, then

$$\int_{\mathbb{R}^N} fh^2\, d\mu = 2^{k+\gamma} \left(\Gamma\left(\frac{N}{2} + k + \gamma\right) \Big/ \Gamma\left(\frac{N}{2}\right)\right) \int_S fh^2\, d\omega \tag{1.2}$$

(recall $\gamma = \sum_{i=1}^{m} \alpha_i$).

This leads to an orthogonal decomposition of $L^2(h^2 d\mu)$. Let

$$\mathcal{H}_{n,k}^h = \{ L_k^{(n+\gamma-1+N/2)}(|x|^2/2)p(x) : p \in \mathcal{H}_n^h \}.$$

Formula (1.1) and the orthogonality theorem for Δ_h show $\mathcal{H}_{n,k}^h \perp \mathcal{H}_{m,\ell}^h$ for $n \neq m$.

The Laguerre polynomials are given by

$$L_n^{(A)}(t) = \frac{(A+1)_n}{n!} \sum_{j=0}^{n} \frac{(-n)_j}{(A+1)_j} \frac{t^j}{j!},$$

and satisfy

$$\Gamma(A+1)^{-1} \int_0^\infty L_n^{(A)}(t) L_m^{(A)}(t) t^A e^{-t}\, dt = \delta_{nm}(A+1)_n / n!$$

(see Szegö [15, p. 100ff]). Thus,

$$\begin{aligned} &\int_{\mathbb{R}^N} p_n(x) q_n(x) L_k^{(n+\gamma-1+N/2)}(|x|^2/2) L_\ell^{(n+\gamma-1+N/2)}(|x|^2/2) h(x)^2\, d\mu(x) \\ &\qquad = \delta_{k\ell} \frac{2^{n+\gamma}}{k!} \frac{\Gamma\left(\frac{N}{2} + \gamma + n + k\right)}{\Gamma\left(\frac{N}{2}\right)} \int_S p_n q_n h^2\, d\omega, \end{aligned} \tag{1.3}$$

for $p_n, q_n \in \mathcal{H}_n^h$. We introduce the normalization constants $c_N := \left(\int_{\mathbb{R}^N} h^2\, d\mu\right)^{-1}$ and $c_N' := \left(\int_S h^2\, d\omega\right)^{-1}$; by (1.2), $c_N' = 2^\gamma \left(\Gamma\left(\frac{N}{2} + \gamma\right) / \Gamma\left(\frac{N}{2}\right)\right) c_N$. Thanks to Selberg, Macdonald, Heckman, Opdam, and others, there is a closed form for c_N. For Weyl groups the formula is

$$c_N^{-1} = \prod_{j=1}^{m} \left(\frac{|v_j|^2}{2}\right)^{\alpha_j} \left(\Gamma(\langle v_h, v_j^* \rangle + \alpha_j + 1) / \Gamma(\langle v_h, v_j^* \rangle + 1)\right),$$

where $v_j^* = (2/|v_j|^2) v_j$ (the co-root) and $v_h = \frac{1}{2} \sum_{j=1}^{m} \alpha_j v_j$ (denoted by ρ in Opdam's paper [13]). Macdonald [12] conjectured this formula for root systems of Weyl groups and proved it for groups of the type A_N, B_N (or C_N), D_N by use of Selberg's integral [14]. Askey [1] also discussed integrals of this type. Opdam [13] proved the conjecture for all Weyl groups by use of shift operators. Although these have the effect of relating different parameter values, similar to the operator V discussed here, they are not homogeneous of degree 0 and are defined only on the G-invariant polynomials. For the Coxeter groups with just one conjugacy class of reflections ($H_3, H_4, I_2(k)$ dihedral, k odd),

$$c_N^{-1} = \prod_{j=1}^{m} \left(\frac{|v_j|^2}{2}\right)^{\alpha} \prod_{\ell=1}^{N} \frac{\Gamma(d_\ell \alpha + 1)}{\Gamma(\alpha + 1)},$$

where $\{d_1, d_2, \ldots, d_N\}$ are the degrees of the fundamental invariants ((2, 6, 10), (2, 12, 20, 30), (2, k) respectively). For the even dihedral group $I_2(2k)$ with

$$h(x) := |x^k - \bar{x}^k|^\alpha \cdot |x^k + \bar{x}^k|^\beta$$

(interpreting $x \in \mathbb{C} \cong \mathbb{R}^2$),

$$\int_{\mathbb{R}^2} h^2\, d\mu = 2^{2k(\alpha+\beta)} \frac{\Gamma\big(k(\alpha+\beta)+1\big)\Gamma(2\alpha+1)\Gamma(2\beta+1)}{\Gamma(\alpha+\beta+1)\Gamma(\alpha+1)\Gamma(\beta+1)}$$

(note that the choice of h implies that each $|v_j| = 2$). Heckman [9] recently showed that Opdam's shift operators could be defined for non-crystallographic reflection groups by use of the author's differential-difference operators. Subsequently, in an as yet unpublished work, Opdam proved the H_3 and H_4 formulas. (Of course, the dihedral group integrals are ordinary beta integrals.)

2. **The intertwining operator.** We review some concepts from the paper [7], which deals with inverting the operator $\sum_{i=1}^N x_i T_i$. For $w \in G$, $0 < t \leq 1$, let $p_w(t)$ be defined by

$$(1/|G|) \sum_{w \in G} p_w(t) w = \exp\Big((\log t) \sum_{j=1}^m \alpha_j (1-\sigma_j)\Big),$$

a central element of the group algebra $\mathbb{R}G$; thus $w \mapsto p_w(t)$ is a class function. Label the conjugacy classes of reflections by $1, 2, \ldots, \ell$, and let β_j be the value of α_i associated to class j, then $p_w(t)$ is a polynomial in $t^{\beta_1}, t^{\beta_2}, \ldots, t^{\beta_\ell}$ with integer coefficients. (For the irreducible Coxeter groups $\ell = 1$ except for $I_2(2m), B_N, F_4, G_2$ when $\ell = 2$.) Further, $p_w(t) \geq 0$ and $\sum_{w \in G} p_w(t) = |G|$ for $0 \leq t \leq 1$.

THEOREM 2.1 [7]. *Let $(f_1, \ldots, f_n)$ be an "h-exact 1-form" of polynomials (that is, $T_i f_j = T_j f_i$ for all i, j), then*

$$F(x) := \frac{1}{|G|} \sum_{w \in G} \int_0^1 p_w(t) \sum_{i=1}^N (xw)_i f_i(txw)\, dt$$

satisfies $T_i F(x) = f_i(x)$; further, if g is another polynomial with $T_i g(x) = f_i(x)$ for all i, then $F(x) = g(x) - g(0)$.

This formula is used inductively to define V (from [7]).

DEFINITION 2.2. Let V be the linear operator on polynomials defined by $V1 = 1$, and if $f \in \mathcal{P}_{n+1}$, then

$$Vf(x) := \frac{1}{|G|} \sum_{w \in G} \sum_{i=1}^N (xw)_i \big(V(\partial_i f)(xw)\big) \int_0^1 p_w(t) t^n\, dt$$

for $n = 0, 1, 2, \ldots$.

THEOREM 2.3. *V is one-to-one on each $\mathcal{P}_n$, $n \in Z_+$; $T_i V = V\partial_i$ for $1 \leq i \leq N$; V is uniquely determined by the conditions $V\mathcal{P}_n \subset \mathcal{P}_n$, $V1 = 1$, $T_i V = V\partial_i$ for all i; and $R(w_0)V = VR(w_0)$ for $w_0 \in G$.*

PROOF. Most of these statements were proved in [7]. Note, for example, that induction shows $(V\partial_i f)_{i=1}^N$ is an h-exact 1-form, and so Theorem 2.1 implies $T_i Vf(x) = V\partial_i f(x)$

$(f \in \mathcal{P}_{n+1})$. To show that V commutes with the action of G assume that $R(w_0)Vg = V\big(R(w_0)g\big)$ for $g \in \mathcal{P}_n$, $w_0 \in G$. For $f \in \mathcal{P}_{n+1}$, $\nabla\left(R(w_0)f\right)(x) = \nabla f(xw_0)w_0^{-1}$ (chain-rule, and w_0 is orthogonal). Thus

$$\partial_i[R(w_0)f](x) = \sum_{j=1}^{N} \partial_j f(xw_0)w_{0,ij} = \sum_{j=1}^{N} w_{0,ij}R(w_0)\partial_j f(x),$$

and

$$\sum_{i=1}^{N}(xw)_i \sum_{j=1}^{N} w_{0,ij}V\big(R(w_0)\partial_j f\big)(xw) = \sum_{j=1}^{N}(xww_0)_j\big(R(w_0)V\partial_j f\big)(xw)$$

by inductive hypothesis,

$$= \sum_{j=1}^{N}(xww_0)_j\big(V(\partial_j f)\big)(xww_0).$$

Now in the expression for $V\big(R(w_0)f\big)(x)$, these calculations lead to

$$V\big(R(w_0f)(x) = \frac{1}{|G|}\sum_{w\in G}\sum_{j=1}^{N}(xww_0)_j V\partial_j f(xww_0)\int_0^1 p_w(t)t^n\,dt.$$

Replace the summation index by $w = w_0w'w_0^{-1}$ and use the fact that $p_w(t) = p_{w'}(t)$ to see that $V\big(R(w_0)f\big)(x) = Vf(xw_0)$. ∎

There is a useful norm on $\sum_{n=0}^{\infty}\mathcal{P}_n$ for which V is a bounded linear operator. For any polynomial p, let $\|p\|_\infty := \sup_{|x|\le 1}|p(x)|$. For formal sums $f(x) = \sum_{n=0}^{\infty} f_n(x)$ with $f_n \in \mathcal{P}_n$, let $\|f\|_A := \sum_{n=0}^{\infty}\|f_n\|_\infty$, and let $A := \{f : \|f\|_A < \infty\}$, a subalgebra of the space of functions continuous on the closed unit ball and infinitely differentiable in the interior. There is another approach to norms for homogeneous polynomials by way of iterated directional derivatives. We define the ∂ and T versions together.

DEFINITION 2.4. Let $\|1\|_\partial = \|1\|_T = 1$, and for $f \in \mathcal{P}_{n+1}$, let

$$\|f\|_T = \frac{1}{n+1}\sup\{\,\|\langle y, \nabla_h f\rangle\|_T : |y| = 1\},$$

and

$$\|f\|_\partial = \frac{1}{n+1}\sup\{\,\|\langle y, \nabla f\rangle\|_\partial : |y| = 1\},$$

for $n = 0, 1, 2, \ldots$ (note that $x \mapsto \langle y, \nabla_h f\rangle$ is in $\mathcal{P}_n$ for each fixed y). Equivalently, for $f \in \mathcal{P}_n$,

$$\|f\|_T = \frac{1}{n!}\sup\left\{\left|\prod_{i=1}^{n}\langle y_i, \nabla_h\rangle f(x)\right| : y_1, y_2, \ldots, y_n \in S\right\};$$

a similar expression holds for $\|f\|_\partial$.

PROPOSITION 2.5. *For $f \in \mathcal{P}_n$, $\|Vf\|_T = \|f\|_\partial$.*

PROOF. For any $y_1, y_2, \ldots, y_n \in S$,

$$\prod_{i=1}^{n} \langle y_i, \nabla_h \rangle\, Vf(x) = V\Big(\prod_{i=1}^{n} \langle y_i, \nabla \rangle f(x)\Big).$$

In the latter expression, the argument of V is a constant and $V1 = 1$. Thus the suprema over $y_1, \ldots, y_n$ of the two expressions are the same. ∎

Van der Corput and Schaake [2] strengthened the Bernstein inequality and proved that $\|f\|_\partial = \|f\|_\infty$ for $f \in \mathcal{P}_n$ (in particular, $|\nabla f(x)|^2$ is maximized on S at the same x maximizing $|f(x)|$ and $|\nabla f(x)|^2 = n^2|f(x)|^2$ there).

PROPOSITION 2.6. *For $f \in \mathcal{P}_n$, $\|f\|_\infty \leq \|f\|_T$.*

PROOF. Inductively, assume $|g(x)| \leq |x|^n \|g\|_T$ for $g \in \mathcal{P}_n$ (obvious for $n = 0$), and let $f \in \mathcal{P}_{n+1}$. By Theorem 2.1,

$$f(x) = \frac{1}{|G|} \sum_{w \in G} \int_0^1 p_w(t) \sum_{i=1}^{N} (xw)_i (T_i f)(txw)\, dt.$$

For fixed $z \in S$, $\|\langle z, \nabla_h f\rangle\|_T \leq \sup_{y \in S} \|\langle y, \nabla_h f\rangle\|_T = (n+1)\|f\|_T$. By the inductive hypothesis, $|\langle z, \nabla_h f(x)\rangle| \leq \|\langle z, \nabla_h f\rangle\|_T \leq (n+1)\|f\|_T$ (each $x, z \in S$). Using the homogeneity we have

$$\Big|\sum_{i=1}^{N} (xw)_i T_i f(txw)\Big| \leq t^n |xw|^{n+1}(n+1)\|f\|_T,$$

and so

$$|f(x)| \leq |x|^{n+1}(n+1)\|f\|_T (1/|G|) \sum_{w \in G} \int_0^1 p_w(t) t^n\, dt = |x|^{n+1}\|f\|_T$$

because $p_w(t) \geq 0$ and $\sum_{w \in G} p_w(t) = |G|$ for $0 \leq t \leq 1$. ∎

The following is a corollary to this proposition and the van der Corput-Schaake inequality.

THEOREM 2.7. *V extends to a bounded operator on A, where $Vf = \sum_{n=0}^{\infty} Vf_n$, for $f = \sum_{n=0}^{\infty} f_n$ in A, $\|Vf\|_A \leq \|f\|_A$, and $|Vf(x)| \leq \sum_{n=0}^{\infty} |x|^n \|f_n\|_\infty \leq \|f\|_A$ $(|x| \leq 1)$.*

The author conjectures that, in fact, $|Vf(x)| \leq \sup\{|f(y)| : |y| \leq |x|\}$ and the functional $f \mapsto Vf(x)$ is positive for each x. For $G = Z_2$, $h(x) = x_1^\alpha$ we already know V as an explicit fractional integral of Weyl type (see Theorem 5.1).

3. **Inner products on polynomials.** Consider the pairing on $\mathcal{P}_n$ given by $[p,q]_\partial := p(\partial_x)q(x)$, $p,q \in \mathcal{P}_n$; where $p(\partial_x)$ means that x_i is replaced by $\frac{\partial}{\partial x_i}$ in $p(x)$. Note that $[p,q]_\partial = \sum_m m_1!\cdots m_N! p_m q_m$, for $m \in Z_+^N$ with $m_1 + \cdots + m_N = n$ and $p(x) = \sum_m p_m x_1^{m_1}\cdots x_N^{m_N}$ and similarly for q. The reproducing kernel for this pairing is $\langle x,y\rangle^n/n!$; that is, $(\langle x,\partial_y\rangle^n/n!)q(y) = q(x)$ for $q \in \mathcal{P}_n$, $x \in \mathbb{R}^N$. With the goal of constructing the Poisson kernel for h-harmonic functions, we consider the action of V on this pairing.

DEFINITION 3.1. For $x,y \in \mathbb{R}^N$, let $K(x,y) := V_x e^{\langle x,y\rangle}$. Here, V_x indicates the variable for the transformation; also $\|e^{\langle x,y\rangle}\|_A = e^{|y|}$. Further, let $K_n(x,y) := V_x(\langle x,y\rangle^n/n!)$, $n \in Z_+$.

PROPOSITION 3.2. *For $n \in Z_+$, $x,y \in \mathbb{R}^N$,*
(i) $|K_n(x,y)| \le \max_{w\in W}|\langle xw,y\rangle|^n/n!$;
(ii) $K_n(xw,yw) = K_n(x,y)$, $w \in G$;
(iii) $K_n(x,y) = K_n(y,x)$;
(iv) $(\nabla_h)_x K_n(x,y) = K_{n-1}(x,y)y$;
(v) $K_{n+1}(x,y) = \frac{1}{|G|}\sum_{w\in G}\langle xw,y\rangle K_n(xw,y)\int_0^1 p_w(t)t^n\,dt$ *for* $n \ge 0$ *and* $K_0(x,y) = 1$.

PROOF. Part (v) is used to prove the others. Indeed,

$$K_{n+1}(x,y) = V_x\big(\langle x,y\rangle^{n+1}/(n+1)!\big) = \frac{1}{|G|}\sum_{w\in G}\int_0^1 p_w(t)\sum_{i=1}^N (xw)_i\Big(V\Big(\frac{\partial}{\partial x_i}f\Big)(xwt)\Big)\,dt,$$

where $f(x) = \langle x,y\rangle^{n+1}/(n+1)!$, but $\frac{\partial}{\partial x_i}f(x) = y_i\langle x,y\rangle^n/n!$ and thus $V\frac{\partial}{\partial x_i}f = K_n(x,y)y_i$. Let $d(x,y) := \max_{w\in G}|\langle xw,y\rangle|$ (thus $\min_{w\in G}|xw \pm y|^2 = |x|^2 + |y|^2 - 2d(x,y)$). Use Formula (v) and assume $|K_n(x,y)| \le d(x,y)^n/n!$ for some $n \ge 0$, then

$$|K_n(xw,y)| \le d(x,y)^n/n!, \text{ all } w \in G,$$

and

$$|K_{n+1}(x,y)| \le \frac{1}{|G|}\sum_{w\in G}\big(d(x,y)^{n+1}/n!\big)\int_0^1 p_w(t)t^{n+1}\,dt = d(x,y)^{n+1}/(n+1)!$$

(since $p_w(t) \ge 0$ and $\sum_w p_w(t) = |G|$). This shows (i).

Part (ii) follows from the commutation relation $R(w)V = VR(w)$, $w \in G$ (Theorem 2.3). Indeed,

$$K_n(xw,yw) = R(w)K(x,yw) = V_xR(w)(\langle x,yw\rangle^n/n!) = V_x(\langle xw,yw\rangle^n/n!) = V_x(\langle x,y\rangle^n/n!) = K_n(x,y).$$

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We prove part (iii) inductively by use of (v). Assume $K_n(x,y) = K_n(y,x)$ (clear for $n = 0$), then

$$\begin{aligned} K_{n+1}(y,x) &= \frac{1}{|G|}\sum_{w\in G}\langle yw,x\rangle K_n(yw,x)\int_0^1 p_w(t)t^n\,dt \\ &= \frac{1}{|G|}\sum_{w\in G}\langle xw^{-1},y\rangle K_n(xw^{-1},y)\int_0^1 p_w(t)t^n\,dt \\ &= K_{n+1}(x,y), \end{aligned}$$

after the change of summation variable $w' = w^{-1}$ and by the fact $p_w(t) = p_{w^{-1}}(t)$ (established in [7]).

Finally,

$$\begin{aligned} (\nabla_h)_x K_n(x,y) &= (\nabla_h)_x V_x(\langle x,y\rangle^n/n!) = V_x\nabla_x(\langle x,y\rangle^n/n!) \\ &= V_x\big(\langle x,y\rangle^{n-1}/(n-1)!\big)y = K_{n-1}(x,y)y. \end{aligned}$$

■

COROLLARY 3.3. *If $p \in \mathcal{P}_n$, then $K_n(x,T^y)p(y) = p(x)$, for all $x \in \mathbb{R}^N$; where $K_n(x,T^y)$ is the operator formed by replacing y_i by T_i with respect to the variable y, in $K_n(x,y)$.*

PROOF. If $q \in \mathcal{P}_n$, then $q(x) = (\langle x,\partial_y\rangle^n/n!)q(y)$ and $V_xq(x) = K_n(x,\partial_y)q(y)$. Apply V_y to both sides (the left side is constant in y) to obtain $V_xq(x) = K_n(x,T^y)V_yq(y)$; formally $V_y\partial_y = T^yV_y$. The required identity holds for all Vq with $q \in \mathcal{P}_n$, and V is one-to-one on $\mathcal{P}_n$. ■

DEFINITION 3.4. The bilinear form $[p,q]_h := p(T^x)q(x)$, for $p,q \in \mathcal{P}_n$, $n = 0,1,\ldots$.

THEOREM 3.5. *For $p,q \in \mathcal{P}_n$,*

$$[p,q]_h = K_n(T^x,T^y)p(x)q(y) = [q,p]_h.$$

PROOF. By Corollary 3.3, $p(x) = K_n(x,T^y)p(y)$. The operators T^x and T^y commute and thus

$$[p,q]_h = K_n(T^x,T^y)p(y)q(x) = K_n(T^y,T^x)p(y)q(x)$$

by 3.2(iii). The latter expression equals $[q,p]_h$. ■

In fact, $[p,q]_h$ is positive-definite. We establish this by expanding p,q in series of products of $|x|^2$ and h-harmonic polynomials and then relating the form $[p,q]_h$ to the $L^2(S;h^2d\omega)$ inner product.

THEOREM 3.6. *Let $p,q \in \mathcal{P}_n$ and express*

$$\begin{aligned} p(x) &= \sum_{j\le n/2}|x|^{2j}p_{n-2j}(x), \\ q(x) &= \sum_{j\le n/2}|x|^{2j}q_{n-2j}(x), \end{aligned}$$

with $p_{n-2j}, q_{n-2j} \in \mathcal{H}^h_{n-2j}$, *then*

$$[p,q]_h = \sum_{j\le n/2} 4^j j!(n-2j+\gamma+N/2)_j [p_{n-2j}, q_{n-2j}]_h.$$

PROOF. The series expansions were shown to uniquely exist in [4, p. 37]. Recall $\Delta_h = \sum_{i=1}^N T_i^2$, thus

$$[p,q]_h = \sum_{j\le n/2}\sum_{\ell\le n/2} \Delta_h^\ell p_{n-2\ell}(T^x)\big(|x|^{2j} q_{n-2j}(x)\big).$$

By the identity,

$$\Delta_h|x|^{2j} f_m(x) = 4j(m+j+\gamma-1+N/2)|x|^{2j-2} f_m(x) + |x|^{2j}\Delta_h f_m(x), \tag{3.6}$$

for $f_m \in \mathcal{P}_m$, $m = 0,1,2,\ldots$ (see [4, p. 38]), we see that

$$\Delta_h^\ell\big(|x|^{2j} q_{n-2j}(x)\big) = 4^\ell(-j)_\ell(-n+j-\gamma+1-N/2)_\ell |x|^{2j-2\ell} q_{n-2j}(x),$$

which is zero if $\ell > j$.

If $\ell < j$, then $[|x|^{2j} q_{n-2j}(x), |x|^{2\ell} p_{n-2\ell}(x)]_h = 0$ by the same argument and the pairing is symmetric (Theorem 3.5). The only remaining terms are those with $j = \ell$, namely, $4^j j!(n-2j+\gamma+N/2)_j p_{n-2j}(T^x) q_{n-2j}(x)$. ■

LEMMA 3.7. *The adjoint of T_i acting on $L^2(\mathbb{R}^N, h^2\,d\mu)$ is given by $T_i^* g(x) = x_i g(x) - T_i g(x)$ for polynomials g.*

PROOF. Integration by parts shows

$$\begin{aligned}\int_{\mathbb{R}^N}\Big(\frac{\partial}{\partial x_i} f(x)\Big) g(x) h(x)^2\,d\mu(x) &\\ = -\int_{\mathbb{R}^N} f(x)\Big(\frac{\partial}{\partial x_i} g(x)\Big) h(x)^2\,d\mu(x)&\\ + \int_{\mathbb{R}^N} f(x) g(x)\Big\{-2h(x)\frac{\partial h(x)}{\partial x_i} + h(x)^2 x_i\Big\}\,d\mu(x)&\end{aligned}$$

(f, g polynomials). For a fixed root v_j,

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{f(x)-f(x\sigma_j)}{\langle x, v_j\rangle} g(x) h(x)^2\,d\mu(x) &\\ = \int_{\mathbb{R}^N}\frac{f(x)g(x)}{\langle x, v_j\rangle} h(x)^2\,d\mu(x) - \int_{\mathbb{R}^N}\frac{f(x\sigma_j)g(x)}{\langle x, v_j\rangle} h(x)^2\,d\mu(x)&\\ = \int_{\mathbb{R}^N}\frac{f(x)g(x)}{\langle x, v_j\rangle} h(x)^2\,d\mu(x) + \int_{\mathbb{R}^N}\frac{f(x)g(x\sigma_j)}{\langle x, v_j\rangle} h(x)^2\,d\mu(x);&\end{aligned}$$

in the second integral replace x by $x\sigma_j$ which changes $\langle x, v_j\rangle$ to $\langle x\sigma_j, v_j\rangle = -\langle x, v_j\rangle$ and leaves $h(x)^2 d\mu(x)$ invariant (assume $\alpha_j \ge 1$ for integrability). Note also that

$$h(x)\frac{\partial}{\partial x_i} h(x) = \sum_{j=1}^m \alpha_j \frac{(v_j)_i}{\langle x, v_j\rangle} h(x)^2.$$

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Combining these ingredients, we obtain

$$\int_{\mathbb{R}^N} T_i f(x) g(x) h(x)^2 \, d\mu(x) = \int_{\mathbb{R}^N} \Big(f(x) \Big(x_i g(x) - \frac{\partial}{\partial x_i} g(x) \Big) + \sum_{j=1}^{m} \alpha_j (v_j)_i f(x) \big(-2g(x) + g(x) + g(x\sigma_j) \big) / \langle x, v_i \rangle \Big) h(x)^2 \, d\mu(x).$$

∎

THEOREM 3.8. *If $p, q \in \mathcal{H}_n^h$, then*

$$[p,q]_h = c_N \int_{\mathbb{R}^N} pqh^2 \, d\mu = 2^n \Big(\frac{N}{2} + \gamma \Big)_n c'_N \int_S pqh^2 \, d\omega.$$

PROOF. Since $p(T^x)q(x)$ is a constant,

$$\begin{aligned}
[p,q]_h &= c_N \int_{\mathbb{R}^N} \big(p(T^x) q(x) \big) h(x)^2 \, d\mu(x) \\
&= c_N \int_{\mathbb{R}^N} q(x) \big(p(T^x)^* 1 \big) h(x)^2 \, d\mu(x) \\
&= c_N \int_{\mathbb{R}^N} q(x) \big(p(x) + p_0(x) \big) h(x)^2 \, d\mu(x)
\end{aligned}$$

with a polynomial p_0 of degree less than n. Repeated use of the relation $T_i^* g(x) = x_i g(x) - T_i g(x)$, and the fact that $\deg(T_i g) < \deg(g)$ shows that the terms of highest degree in $p(T^x)^* 1$ are exactly $p(x)$. But $\int_{\mathbb{R}^N} qp_0 h^2 \, d\mu = 0$ since $q \in \mathcal{H}_n^h$ (recall Formula (1.1)). This shows $[p,q]_h = c_N \int_{\mathbb{R}^N} pqh^2 \, d\mu$. The rest of the theorem follows from identity (1.2). ∎

Thus $[p,q]_h$ is positive-definite. There is a natural isomorphism of polynomials that maps $\mathcal{P}_n$ and $\mathcal{P}_m$ into orthogonal subspaces of $L^2(\mathbb{R}^N, h^2 \, d\mu)$ (for $n \neq m$); indeed the image of $\mathcal{P}_n$ is $\sum_{j \leq n/2} \oplus \mathcal{H}_{n-2j,j}^h$ (see Section 1). The idea of forming $e^{-\Delta_h/2}$ comes from Macdonald's use of $e^{\Delta/2}$ in [12] in connection with $[p,q]_\partial$. Observe that $e^{-\Delta_h/2}$ maps $\mathcal{P}_n$ to $\sum_{j \leq n/2} \mathcal{P}_{n-2j}$.

PROPOSITION 3.9. *Let $f \in \mathcal{H}_m^h$, $m, j \in Z_+$, then*

$$e^{-\Delta_h/2} |x|^{2j} f(x) = (-1)^j j! 2^j L_j^{(m+\gamma+N/2-1)} (|x|^2/2) f(x),$$

an element of $\mathcal{H}_{m,j}^h$.

PROOF. By Formula (3.6),

$$e^{-\Delta_h/2} |x|^{2j} f(x) = \sum_{\ell=0}^{j} \frac{(-1)^\ell 2^\ell}{\ell!} (-j)_\ell (-m - j - \gamma + 1 - N/2)_\ell |x|^{2j-2\ell} f(x),$$

(and use the reversed form for the Laguerre polynomial,

$$L_j^{(A)}(t) = \frac{(-1)^j}{j!} \sum_{\ell=0}^{j} \frac{(-j)_\ell (-j-A)_\ell}{\ell!} t^{j-\ell} (-1)^\ell.$$

∎

The pairing $[\cdot,\cdot]_h$ has an obvious extension to all polynomials with the convention $[p,q]_h = 0$ if $p \in \mathcal{P}_n$, $q \in \mathcal{P}_m$, $n \neq m$. Macdonald proved the following for the pairing $[\cdot,\cdot]_\partial$ (the relatively easy proof does not seem to be adaptable to Δ_h).

THEOREM 3.10. *For p, q polynomials,*

$$[p,q]_h = c_N \int_{\mathbb{R}^N} (e^{-\Delta_h/2}p)(e^{-\Delta_h/2}q)h^2\, d\mu.$$

PROOF. By Theorem 3.6, it suffices to establish this for p, q of the form

$$p(x) = |x|^{2j}p_m(x),\ q(x) = |x|^{2j}q_m(x) \text{ with } p_m, q_m \in \mathcal{H}_m^h.$$

By (3.6),

$$\begin{aligned}[p,q]_h &= 4^j j!\Big(m+\gamma+\frac{N}{2}\Big)_j [p_m, q_m]_h \\ &= 4^j j!\Big(m+\gamma+\frac{N}{2}\Big)_j 2^m\Big(\frac{N}{2}+\gamma\Big)_m c_N' \int_S p_m q_m h^2\, d\omega \\ &= 2^{m+2j} j!\Big(\frac{N}{2}+\gamma\Big)_{m+j} c_N' \int_S p_m q_m h^2\, d\omega.\end{aligned}$$

The righthand side of the formula (by 3.9) equals

$$\begin{aligned}&c_N \int_{\mathbb{R}^N} (j!2^j)^2 L_j^{(n+\gamma-1+N/2)}(|x|^2/2)^2 p_m(x)q_m(x)h(x)^2\, d\mu(x) \\ &\qquad = 2^{-\gamma}\frac{\Gamma\big(\frac{N}{2}\big)}{\Gamma\big(\frac{N}{2}+\gamma\big)} c_N'(j!)^2 2^{m+2j+\gamma}\frac{\Gamma\big(\frac{N}{2}+\gamma+m\big)}{j!\Gamma\big(\frac{N}{2}\big)}\int_S p_m q_m h^2 d\omega\end{aligned}$$

using Formula (1.3). ∎

Note that

$$\begin{aligned}&\Delta_h L_k^{(m+\gamma+N/2-1)}(|x|^2/2)p_m(x) \\ &\qquad = -2(m+\gamma+N/2+k-1)L_{k-1}^{(m+\gamma+N/2-1)}(|x|^2/2)p_m(x)\end{aligned}$$

for $p_m \in \mathcal{H}_m^h$ (a simple calculation using identity (3.6)). Thus $\Delta_h \mathcal{H}_{m,k}^h = \mathcal{H}_{m,k-1}^h (k \geq 1)$, and we can characterize $\mathcal{H}_{m,k}^h$ as $\Big(\ker(\Delta_h^{k+1}) \cap \sum_{j=0}^{m+2k} \mathcal{P}_j\Big) \cap \Big(\ker(\Delta_h^k) \cap \sum_{j=0}^{m+2k-1} \mathcal{P}_j\Big)^{\perp}$ (as a subspace of $L^2(\mathbb{R}^N, h^2\, d\mu)$).

PROPOSITION 3.11. *For $k, m \in Z_+$ with $2k \leq m$,*

$$V\Big(\sum_{j=0}^{k} \mathcal{H}_{m-2j,j}\Big) \subset \sum_{j=0}^{k} \mathcal{H}_{m-2j,j}^h.$$

PROOF. Note $\mathcal{H}_{m-2j,j} = \big\{L_j^{(n+N/2-1)}(|x|^2/2)p(x) : p \in \mathcal{H}_{m-2j}\big\}$. The two spaces are the images of $e^{-\Delta/2}$, $e^{-\Delta_h/2}$ applied to $\mathcal{P}_m \cap \ker(\Delta^{k+1})$, $\mathcal{P}_m \cap \ker(\Delta_h^{k+1})$ respectively. But $Ve^{-\Delta/2} = e^{-\Delta_h/2}V$, $V(\mathcal{P}_m \cap \ker \Delta^{k+1}) = \mathcal{P}_m \cap \ker(\Delta_h^{k+1})$, and V is one-to-one. ∎

There is an identity of Hecke for harmonic polynomials which can be adapted to h-harmonic polynomials.

PROPOSITION 3.12. *If $p \in \mathcal{H}_n^h$, then*

$$p(y) = e^{-|y|^2/2} c_N \int_{\mathbb{R}^N} p(x) K(x,y) h(x)^2 \, d\mu(x)$$

for $y \in \mathbb{R}^N$.

PROOF. Choose $m \geq n$, and let $f_m(x) = \sum_{j=0}^m K_j(x,y)$. By Corollary 3.3 and Theorem 3.10,

$$p(y) = [f_m, p]_h = c_N \int_{\mathbb{R}^N} p(x) e^{-\Delta_h/2} f_m(x) h(x)^2 \, d\mu(x).$$

Now $e^{-\Delta_h/2} f_m(x) = \sum_{j \leq m/2} \big((-|y|^2/2)^j / j!\big) \sum_{i=0}^{m-2j} K_i(x,y)$. Let $m \longrightarrow \infty$ and use the dominated convergence theorem to get the stated formula. ∎

4. **The Poisson kernel.** This is the reproducing kernel $P(x,y)$ which is defined by the property:

$$f(y) = c_N' \int_S f(x) P(x,y) h(x)^2 \, d\omega(x)$$

for each $f \in \mathcal{H}_n^h$, $n \in Z_+$, $|y| < 1$.

Let $P_n(x,y)$ denote the component of degree n; that is, $P_n(x,y) = \sum_{j=1}^{d_n} q_{n,j}(x) q_{n,j}(y)$, where $\{ q_{n,j} : j = 1, 2, \ldots, d_n \}$ is a real orthonormal basis for $\mathcal{H}_n^h$ (in $L^2(S, c_N' h^2 d\omega)$), and $d_n = \dim \mathcal{H}_n^h = \dim \mathcal{P}_n - \dim \mathcal{P}_{n-2}$. Thus, for each fixed y, $x \longmapsto P_n(x,y)$ is in $\mathcal{H}_n$. Further $P(x,y) = \sum_{n=0}^\infty P_n(x,y)$ as a formal series.

THEOREM 4.1. *For $n \in Z_+$,*

$$P_n(x,y) = \sum_{j \leq n/2} \frac{\left(\frac{N}{2} + \gamma\right)_n 2^{n-2j}}{(2 - n - \gamma - N/2)_j j!} |x|^{2j} |y|^{2j} K_{n-2j}(x,y),$$

for $x, y \in \mathbb{R}^N$.

PROOF. The kernel $P_n(x,y)$ is uniquely defined by the reproducing property for $\mathcal{H}_n^h$. Let $f \in \mathcal{H}_n^h$, then $f(y) = K_n(T^x, y) f(x)$ (Corollary 3.3). Fix y, and let $p(x) = K_n(x,y)$ so that $f(y) = [p, f]_h$. Expand $p(x) = \sum_{j \leq n/2} |x|^{2j} p_{n-2j}(x)$ with $p_{n-2j} \in \mathcal{H}_{n-2j}^h$, then $f(y) = [p, f]_h = [p_n, f]_h = 2^n \left(\frac{N}{2} + \gamma\right)_n c_N' \int_S p_n f h^2 d\omega$ (by Theorem 3.8). Thus, $P_n(x,y) = 2^n \left(\frac{N}{2} + \gamma\right)_n p_n(x)$ with p_n being the (orthogonal) projection of p on $\mathcal{H}_n^h$. It was shown in [4, p. 38] that

$$p_n(x) = \sum_{j \leq n/2} \left(4^j j! \left(-\frac{N}{2} - \gamma - n + 2 \right)_j \right)^{-1} |x|^{2j} \Delta_h^j p(x).$$

But $\Delta_h K_n(x,y) = |y|^2 K_{n-2}(x,y)$ (for Δ_h acting on x), so that $\Delta_h^j p(x) = |y|^{2j} K_{n-2j}(x,y)$. ∎

By formally adding $\sum_{n=0}^\infty P_n(x,y)$ and reversing the summation, we are led to the following theorem.

THEOREM 4.2. *Fix $y \in \mathbb{R}^n$ with $|y| < 1$, then*

$$P(x,y) = V_x\Big((1-|y|^2)(1-2\langle x,y\rangle + |y|^2)^{-N/2-\gamma}\Big),$$

and

$$0 \le P(x,y) \le (1-|y|^2)/\big(\min_{w\in G}|wx \pm y|\big)^{N+2\gamma}$$

for $|y| < 1 = |x|$.

PROOF. Denote by f_y the function which is the argument of V_x in the statement. We claim $f_y \in A$, with $|f_y|_A = (1-|y|^2)(1-|y|)^{-N-2\gamma}$. Indeed,

$$\begin{aligned} f_y(x) &= (1-|y|^2)(1+|y|^2)^{-N/2-\gamma}\Big(1-\frac{2\langle x,y\rangle}{1+|y|^2}\Big)^{-N/2-\gamma} \\ &= (1-|y|^2)(1+|y|^2)^{-N/2-\gamma}\sum_{n=0}^{\infty}\frac{\big(\frac{N}{2}+\gamma\big)_n}{n!}\frac{2^n}{(1+|y|^2)^n}\langle x,y\rangle^n. \end{aligned}$$

But $\|\langle x,y\rangle^n\|_\infty = |y|^n$ and $0 \le 2|y| < 1+|y|^2$ for $|y| < 1$, so

$$\begin{aligned} \|f_y\|_A &= (1-|y|^2)(1+|y|^2)^{-N/2-\gamma}\Big(1-\Big(\frac{2|y|}{1+|y|^2}\Big)\Big)^{-N/2-\gamma} \\ &= (1-|y|^2)(1-|y|)^{-N-2\gamma}. \end{aligned}$$

Thus, $Vf_y(x)$ is defined and continuous for $|x| \le 1$. Further,

$$\begin{aligned} Vf_y(x) &= (1-|y|^2)\sum_{m=0}^{\infty}\Big(\frac{N}{2}+\gamma\Big)_m 2^m K_m(x,y)(1+|y|^2)^{-m-N/2-\gamma} \\ &= (1-|y|^2)\sum_{m=0}^{\infty}\Big(\frac{N}{2}+\gamma\Big)_m 2^m K_m(x,y)\sum_{j=0}^{\infty}(-1)^j\frac{\big(m+\frac{N}{2}+\gamma\big)_j}{j!}|y|^{2j}. \end{aligned}$$

Using the bound $|K_m(x,y)| \le d(x,y)^m$ (from Proposition 3.2(i)) in the first equation shows

$$|Vf_y(x)| \le (1-|y|^2)(1-2d(x,y)+|y|^2)^{-N/2-\gamma},$$

the stated bound. In the double sum the part homogeneous of degree n in y is

$$\sum_{j\le n/2}\frac{2^{n-2j}\big(\frac{N}{2}+\gamma\big)_n}{\big(-\frac{N}{2}-\gamma-n+2\big)_j}|y|^{2j}K_{n-2j}(x,y),$$

which equals $P_n(x,y)$ for $|x| = 1$. The fact that $P(x,y) \ge 0$ follows from the maximum principle for Δ_h (in [4, p. 41]). ∎

Note that the bound on $P(x,y)$ shows that for fixed x, y on S, $P(x,ry) \longrightarrow 0$ as $r \longrightarrow 1_-$ except possibly for $y \in \{\pm xw : w \in G\}$, the G-orbit of x and its antipode.

5. **Examples.** At the time of writing, a closed form for V is known only for $h(x) = x_1^\alpha$ (the group Z_2). Even so, this allows a simple determination of the Poisson kernel for the disc polynomials (Ikeda [10], see also [3] for formulas). They are orthogonal on the disk $\{x \in \mathbb{R}^2 : |x| \le 1\}$ with the measure $(1 - x_1^2 - x_2^2)^\lambda dx_1 dx_2$ and are realized as the restrictions to S of h-harmonic polynomials on $\mathbb{R}^3$, for $h(x) = x_3^\alpha$ with $\alpha = \lambda + 1/2$, which are even in x_3.

THEOREM 5.1. *For $N = 1$, $h(x) = |x|^\alpha$, $\alpha > 0$,*

$$Vf(x) = b_\alpha \int_{-1}^1 f(xt)(1-t)^{\alpha-1}(1+t)^\alpha \, dt,$$

where

$$b_\alpha = 2^{-2\alpha}\Gamma(2\alpha+1)/\big(\Gamma(\alpha)\Gamma(\alpha+1)\big).$$

PROOF. Direct verification by way of beta integrals shows that

$$Vx^{2n} = \big((1/2)_n/(\alpha+1/2)_n\big)x^{2n}$$

and

$$Vx^{2n+1} = \big((1/2)_{n+1}/(\alpha+1/2)_{n+1}\big)x^{2n+1}, \quad n \in Z_+.$$

Further $Tx^{2n} = 2nx^{2n}$ and $Tx^{2n+1} = (2n+1+2\alpha)x^{2n}$ (note

$$Tf(x) = f'(x) + \alpha\big(f(x) - f(-x)\big)/x,$$

and induction shows $VT = T\big(\frac{d}{dx}\big)$. Alternatively,

$$\begin{aligned} TVf(x) &= b_\alpha \int_{-1}^1 \Big(tf'(xt) + \alpha\big(f(xt) - f(-xt)\big)/x\Big)(1-t)^{\alpha-1}(1+t)^\alpha \, dt \\ &= b_\alpha \int_{-1}^1 \Big(tf'(xt)(1-t)^{\alpha-1}(1+t)^\alpha + 2\big(f(xt)/x\big)\alpha t(1-t^2)^{\alpha-1}\Big)\, dt \\ &= b_\alpha \int_{-1}^1 f'(xt)\big(t + (1-t)\big)(1-t)^{\alpha-1}(1+t)^\alpha \, dt \\ &= Vf'(x); \end{aligned}$$

integration by parts is used in the second term. ∎

To illustrate Proposition 3.11 for $N = 1$, $\gamma = \alpha$, we note that $VL_n^{(-1/2)}(x^2/2) = \frac{(1/2)_n}{(\alpha+1/2)_n}L_n^{(\alpha-1/2)}(x^2/2)$ (in $\mathcal{H}_{0,n}^h$), and $V\big(xL_n^{(1/2)}(x^2/2)\big) = \frac{(1/2)_{n+1}}{(\alpha+1/2)_{n+1}}xL_n^{(\alpha+1/2)}(x^2/2)$ (in $\mathcal{H}_{1,n}^h$). Further, for $N = 2$, and $h(x) = x_2^\alpha$, the h-harmonic polynomials are Gegenbauer polynomials so that V acts as a transform from trigonometric polynomials to the former, a classical formula of Dirichlet type. Indeed.

$$\begin{aligned} &b_\alpha \int_{-1}^1 (\cos\theta + it\sin\theta)^n(1-t)^{\alpha-1}(1+t)^\alpha \, dt \\ &\quad = \frac{n!}{(2\alpha)_n}C_n^\alpha(\cos\theta) + i\frac{n!}{(2\alpha+1)_n}\sin\theta\, C_{n-1}^{\alpha+1}(\cos\theta), \end{aligned}$$

$n = 1, 2, 3, \ldots$ (a result of Erdélyi [8] for the part even in θ). The Poisson kernel for this family was discussed in [6]. A transform, and the Poisson kernel, for Jacobi polynomials can also be obtained by using $h(x) = |x_1|^\beta |x_2|^\alpha$ on $\mathbb{R}^2$ and expressing V as a double integral.

Turning to $N = 3$, $h(x) = |x_3|^\alpha$, we get the Poisson kernel explicitly as

$$\begin{aligned} &P(x, y) \\ &\quad = b_\alpha \int_{-1}^{1} (1 - |y|^2)\big(1 - 2(x_1y_1 + x_2y_2 + x_3y_3t) + |y|^2\big)^{-\alpha-3/2}(1-t)^{\alpha-1}(1+t)^\alpha \, dt \\ &\quad = \frac{1 - |y|^2}{|x - y|^{2\alpha+3}} \, {}_2F_1\left(\begin{matrix}\alpha + 3/2, \alpha \\ 2\alpha + 1\end{matrix}; -\frac{4x_3y_3}{|x - y|^2}\right) \end{aligned}$$

for $|y| < |x| = 1, x, y \in \mathbb{R}^3$. As in [6], consider the hypergeometric function analytic on $\mathbb{C} \setminus [1, \infty]$ (cut along $\{z \in \mathbb{R} : z \geq 1\}$. Note $-4x_3y_3 = |x - y|^2 - |x\sigma - y|^2$, where $x\sigma = (x_1, x_2, -x_3)$. The restriction of this kernel to polynomials even in x_3 was already determined in integral form by Kanjin [11].

References

1. R. Askey, *Some basic hypergeometric extensions of integrals of Selberg and Andrews*, SIAM J. Math. Anal. **11**(1980), 938–951.
2. J. van der Corput and G. Schaake, *Ungleichungen für Polynome und trigonometrische Polynome*, Compositio Math. **2**(1935), 321–361.
3. C. Dunkl, *An addition theorem for Heisenberg harmonics*. Conference on Harmonic Analysis in honor of A. Zygmund, Wadsworth International, 1983, 690–707.
4. ———, *Reflection groups and orthogonal polynomials on the sphere*, Math. Z. **197**(1988), 33–60.
5. ———, *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc. **311** (1989), 167–183.
6. ———, *Poisson and Cauchy kernels for orthogonal polynomials with dihedral symmetry*, J. Math. Anal. Appl. **143**(1989), 459–470.
7. ———, *Operators commuting with Coxeter group actions on polynomials*. Invariant Theory and Tableaux, D. Stanton (ed.), Springer-Verlag, 1990, 107–117.
8. A. Erdèlyi, *Axially symmetric potentials and fractional integration*, SIAM J. **13**(1965), 216–228.
9. G. J. Heckman, *A remark on the Dunkl differential-difference operators*. Proceedings of Bowdoin conference on Harmonic Analysis of Reductive Groups, August 1989, to appear.
10. M. Ikeda, *On spherical functions for the unitary group, I, II, III*, Mem. Fac. Engrg, Hiroshima Univ. **3**(1967), 17–75.
11. Y. Kanjin, *A convolution measure algebra on the unit disc*, Tôhoku Math. J. **28**(1976), 105–115.
12. I. G. Macdonald, *Some conjectures for root systems*, SIAM J. Math. Anal. **13**(1982), 988–1007.
13. E. M. Opdam, *Some applications of hypergeometric shift operators*, Inventiones Math. **98**(1989), 1–18.
14. A. Selberg, *Bemerkninger om et multiplet integral*, Norsk. Mat. Tidsskrift **26**(1944), 71–78.
15. G. Szegö, *Orthogonal polynomials*. 3rd ed., AMS Colloquium Publications, Providence, 1967.

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