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ON THE DERIVED CUBOID

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We showed in [1] that the Eulerian family of cuboids with integral edges and face diagonals did not have integral inner diagonals. We now show that the derived family does not have integral inner diagonals except possibly when the generators are divisible by 705180. In this case there appears to be no inherent reason why the diagonals cannot be integral.

We seek solutions in nonzero integers to the following four equations in seven unknowns:

$$x^2+y^2 = t^2, \qquad x^2+z^2 = u^2, \qquad y^2+z^2 = v^2, \tag{1}$$

$$x^2+y^2+z^2 = w^2. \tag{2}$$

Lal and Blundon [2] gave 130 solutions of (1) none of which satisfied (2). Kraitchik [3] gave 241 solutions of (1) as well as formulae for a number of families of solutions. The simplest family is the Eulerian one given by

$$x = a(4b^2-c^2), \qquad y = b(4a^2-c^2), \qquad z = 4abc \tag{3}$$

where a, b, c form a primitive Pythagorean triple, that is,

$$a = m^2-n^2, \qquad b = 2mn, \qquad c = m^2+n^2 \tag{4}$$

$$m > n > 0, \qquad (m, n) = 1, \qquad 2 \mid mn \tag{5}$$

$$a^2+b^2 = c^2, \qquad (a, b) = (a, c) = (b, c) = 1, \qquad 4 \mid b, \qquad 2 \nmid ac \tag{6}$$

One readily sees that (3) is a primitive family, that is, $(x, y, z)=1$.

If x, y, z satisfy (1), then xy, xz, yz also satisfy (1). This triple when reduced is called the derived cuboid. When (3) is used and reduced by ab one gets

$$\begin{aligned} x = (4b^2-c^2)(4a^2-c^2), \qquad y = 4ac(4b^2-c^2),\\ z = 4bc(4a^2-c^2). \end{aligned} \tag{7}$$

We show $(x, y, z)=1$. Since x is odd we need only consider an odd prime p such that $p \mid z$. If $p \mid b$, $p \nmid y$. If $p \mid c$, $p \nmid x$. If $p \mid 4a^2-c^2$, then $p \nmid y$ since $4a^2-c^2+4b^2-c^2=2c^2$.

The simplest member of the derived cuboid family is given by $m=2$, $n=1$, $a=3$, $b=4$, $c=5$, $x=429$, $y=2340$, $z=880$. The expression for the inner diagonal

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squared is

$$x^2+y^2+z^2 = F_1F_2F_3F_4 \tag{8}$$

where

$$F_1 = (c-b+a)^2+(b-a)^2 = 5m^4-12m^3n+6m^2n^2+4mn^3+n^4$$

$$F_2 = (c+b-a)^2+(b-a)^2 = m^4-4m^3n+6m^2n^2+12mn^3+5n^4$$

$$F_3 = (c-b-a)^2+(b+a)^2 = m^4+4m^3n+6m^2n^2-12mn^3+5n^4$$

$$F_4 = (c+b+a)^2+(b+a)^2 = 5m^4+12m^3n+6m^2n^2-4mn^3+n^4$$

This general four factor structure contrasts with the unbalanced three factor structure for the Eulerian family given in [1].

If the F_i were pairwise relatively prime, then (2) could only be satisfied if all F_i were squares, which is not possible. Unfortunately, there is a subtle case which spoils the argument.

Since the F_i are odd, we need only consider an odd prime p. If $p \mid F_1$ and $p \mid F_2$, then $p \mid F_1+F_2$ and $p \mid F_1-F_2$, where

$$F_1+F_2 = 2[c^2+2(b-a)^2], \qquad F_1-F_2 = -4c(b-a).$$

This implies $p \mid c$ and $p \mid b-a$, which is impossible in view of (6) since $c^2-(b-a)^2=2ab$. Thus $(F_1, F_2)=1$. Similarly $(F_3, F_4)=1$.

For the case of F_1 and F_3,

$$F_1+F_3 = 2c(3c-2b), \qquad F_1-F_3 = 4a(c-2b)$$

The only case not immediately disposed of is $p \mid 3c-2b$ and $p \mid a$. If $p \mid a$, then $p \mid m-n$ or $p \mid m+n$. If $p \mid m-n$ and $p \mid 3(m-n)^2+2mn$ then $p \mid m$ and $p \mid n$ which is impossible. If $p \mid m+n$ and $p \mid 3(m+n)^2-10mn$, then $p=5$ and $5 \mid m+n$. One way to see that 5 is actually a common factor of F_1 and F_3 when $5 \mid m+n$ is from the identities

$$F_1(m, n) = \tfrac{1}{4}F_2(m+n, m-n), \qquad F_3(m, n) = \tfrac{1}{4}F_3(m+n, m-n).$$

Notice that 25 cannot be a common factor. Thus $(F_1, F_3)=1$ unless $5 \mid m+n$ in which case $(F_1, F_3)=5$. In like manner we get $(F_2, F_4)=1$ unless $5 \mid m-n$, when $(F_2, F_4)=5$. Similar arguments give $(F_2, F_3)=1$ unless $5 \mid m$, when $(F_2, F_3)=5$ and $(F_1, F_4)=1$ unless $5 \mid n$, when $(F_1, F_4)=5$.

If $5 \nmid mn(m^2-n^2)$ then the F_i are pairwise relatively prime and must all be squares to satisfy (2). But F_1 and F_2 cannot be squares simultaneously, because one is congruent to 1 and the other to 5 modulo 8 and 5 is not a quadratic residue, so that (2) is impossible. If $5 \mid m+n$ and (2) is satisfied, then $5F_1$, F_2, $5F_3$, and F_4 are all squares. But F_2 and F_4 cannot both be squares by the above modulo 8 argument. Similarly if $5 \mid m-n$, F_1 and F_3 cannot be squares simultaneously. However this argument breaks down when $5 \mid mn$.

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When $5 \mid m$, we need F_1, $5F_2$, $5F_3$, and F_4 all to be squares for the same m and n. That they can be squares individually is shown by $m=100$, $n=31$ for F_1 and $m=80$, $n=29$ for $5F_2$. The four quantitites were examined by congruence considerations to see if they could be squares simultaneously. If p is an odd prime modulus, and $m\equiv 0$ (modulo p), then these are congruent to n^4, $25n^4$, $25n^4$, and n^4, respectively, so that all could be squares in this case. For $m\equiv 1, \ldots, p-1$ and $n\equiv 0, \ldots, p-1$ they were examined on a computer to see if all could be squares for any pair of values, being congruent to 0 or a quadratic residue. There were no permissible pairs for $p=3$, 7, 23, and 73. All primes under 1000 were examined. Thus m must be divisible by these numbers in addition to 5. We show further that $4 \mid m$. It can be seen that

$$F_1 = (2xy)^2+(-2x^2+4xy-y^2)^2 \tag{9}$$

for $x=m$, $y=m-n$. Furthermore, $(2xy, -2x^2+4xy-y^2)=1$ in view of (5). Then if F_1 is a square, (9) represents a primitive Pythagorean triangle and we can set

$$xy = st, \qquad -2x^2+4xy-y^2 = s^2-t^2$$

where $(s, t)=1$ and $2 \mid st$. But y is odd, whence x is even. Then modulo 8, $-2x^2+4xy-y^2\equiv -1$ and $4 \mid s$ which implies $4 \mid m$. Thus $705180 \mid m$. Similarly when $5 \mid n$, $5F_1$, F_2, F_3 and $5F_4$ must be squares and $705180 \mid n$.

Solutions for which the quartics are squares are very sparse (Mordell [4]). For example the values of m and n for which F_1 is a square and $2 \mid mn$ begin 4, 1; 100, 31; 600, 14239; -338136, 8698591; 19799271700, 3629305951, none of which satisfy the divisibility requirements.

References

1. W. G. Spohn, *On the integral cuboid*, Amer. Math. Monthly, **79** (1972) 57–59.

2. M. Lal and W. J. Blundon, *Solutions of the Diophantine equations* $x^2+y^2=l^2$, $y^2+z^2=m^2$, $z^2+x^2=n^2$, Math. Comp., **20** (1966) 144–147.

3. M. Kraitchik, *Théorie des Nombres*, Tome III, Analyse Diophantine et Applications aux Cuboïdes Rationnels, Gauthier-Villars, Paris, 1947.

4. L. J. Mordell, *Diophantine Equations*, Academic Press, London and New York, 1969.

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