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# $C_p$-CLASSES OF OPERATORS IN $C^*$-ALGEBRAS

S. GIOTOPOULOS

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**Abstract**

We construct a "suitable" representation of a $C^*$-algebra that carries single elements to rank one operators. We also prove an abstract spectral theorem for compact elements in the algebra. This leads naturally to an abstract definition of $C_p$-classes of compact elements in the algebra.

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## 1. Introduction

The classes $C_p$ $(0 < p \leqslant \infty)$ of linear operators on a Hilbert space $H$ were introduced by von Neumann and Schatten in [11] and have been studied in various articles (for example, [4, 7, 8 and 10]). Suppose $T$ is a compact operator on $H$ and $\mu_1, \mu_2, \ldots$ are the eigenvalues of $(T^*T)^{1/2}$ arranged in decreasing order and repeated according to their multiplicity. The numbers $\mu_n$ $(n = 1, 2, \ldots)$ are called the *characteristic numbers* of $T$ and are noted by $s_n(T)$ $(n = 1, 2, \ldots)$. We define

(i) $\|T\|_p = \{\Sigma_n [s_n(T)]^p\}^{1/p}$ $(0 < p < \infty)$,
(ii) $\|T\|_\infty = |s_1(T)| = \|T\|$,
(iii) $C_p = \{T \in C(H) \colon \|T\|_p < \infty\}$.

If $1 \leqslant p \leqslant \infty$, then (i) $C_p$ is a two-sided ideal of $\mathscr{L}(H)$, (ii) $\|\cdot\|_p$ is a norm on $C_p$ and with this norm $C_p$ is a Banach space (which is reflexive in case $p > 1$). The set $F(H)$ of all operators of finite rank on $H$ is an everywhere dense linear subspace of $C_p$.

This article is an attempt to introduce in an arbitrary $C^*$-algebra $A$ a class of elements analogous to the von Neumann-Schatten classes $C_p$ of compact operators on some Hilbert space. This is an application of some faithful representation of the algebra which carries single elements to rank one operators. This is Theorem 5 below.

An abstract spectral theorem for certain compact elements in $A$ is needed and it is proved in Theorem 7.

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## 2. Representations

In this section we are concerned with $C^*$-algebras which contain non-zero single elements. An element $s$ in a $C^*$-algebra $A$ is called single if whenever $asb = 0$, $a$, $b$ in $A$, then at least one of $as$, $sb$ is zero. $C^*$-algebras are obviously semi-simple Banach algebras and single elements in $C^*$-algebras "act compactly" [5, 6].

If an addition we assume that a $C^*$-algebra $A$ is separated by its single elements (i.e. the left annihilator $\operatorname{lan}\sigma$ of the set $\sigma$ of all non-zero single elements of $A$ is zero) then the representation constructed in [6, Theorem 6] is isometric and the Banach space $X$ is in fact a Hilbert space. In fact, in that case, $\operatorname{lan}\sigma = \operatorname{lan}(\operatorname{soc} A) = (0)$ and $X$ will be the direct sum of $\{Ae\}$, $(e \in \mathscr{E})$ [6], where $(e \in \mathscr{E})$ may be assumed self-adjoint ([5, Lemma 2.3] or [9, Lemma 4.9.2]). If $x, y \in Ae$ (for some $e \in \mathscr{E}$) then $y^*x \in eAe = \mathbb{C}e$, and thus we can define a scalar $\langle x, y\rangle$ by $y^*x = \langle x, y\rangle e$.

By a standard argument, $\langle x, y\rangle$ defines an inner product on the elements of $Ae$ making it a Hilbert space, with the inner product norm identified with the algebra norm. Hence, the space $X$ in [6, Theorem 6] can be regarded as a Hilbert space which we will denote in the following by $H$. The representation $a \to \pi(a)$ $(A \to \mathscr{L}(H))$ of [6, Theorem 6] is then an isometric $*$-representation of $A$ on the Hilbert space $H$ [3, 1.8.1]. Hence we have the following

THEOREM 1. *Let $A$ be a $C^*$-algebra which is separated by the set of its non-zero single elements. Then there exists an isometric representation $a \to \pi(a)$ of $A$ on a Hilbert space $H$ such that $\pi(s)$ has rank one, if and only if, $s$ is a non-zero single element of $A$.*

Moreover, we can easily see that the closed linear span, $[\pi(s)H : s \in \sigma]$, of all vectors $\pi(s)h$, $s \in \sigma$, $h \in H$, is all of $H$. The converse is also true, as the following lemma shows.

LEMMA 2. *Let $A$ be a $C^*$-algebra and $\pi$ a faithful representation of $A$ on a Hilbert space $H$ such that $H = [\pi(s)H : s \in \sigma]$. Then $A$ is separated by its single elements.*

PROOF. Suppose that for some $a$ in $A$ we have $as = 0$ for all $s \in \sigma$. Since $\pi$ is faithful, we have $\pi(as) = 0$ and hence $\langle \pi(as)x, y \rangle = 0$, or equivalently $\langle \pi(s)x, \pi(a^*)y \rangle = 0$, $(s \in \sigma,\ x,\ y \in H)$. By assumption, the latter implies that $\pi(a^*)y^{\perp} H$ for all $y \in H$ and therefore $\pi(a^*) = 0$. Hence $a^* = 0$ and so $a = 0$, and the lemma follows.

REMARK. The assumption that $\pi$ is faithful cannot be discarded, as the following example shows. Let $\mathscr{A}$ be a $C^*$-algebra with no non-zero single elements and suppose $\pi$ is the representation of $A \oplus \mathscr{L}(H)$ on the Hilbert space $H$ defined by $\pi(S \oplus T) = T$. The set of single elements of $\mathscr{A} \oplus \mathscr{L}(H)$ consists of all operators of the form $0 \oplus R$, with $R$ either zero or rank one operator, in $\mathscr{L}(H)$. Clearly, $\pi$ is not faithful and

$$H = \left[\pi(0 \oplus R)H : R \in \mathscr{L}(H), \text{rank } R \leqslant 1\right].$$

However, a non-zero operator $S \oplus 0$ annihilates every operator $0 \oplus R$ and thus $\mathscr{A} \oplus \mathscr{L}(H)$ is not separated by its single elements.

Let $\pi$ and $\rho$ be two representations of $A$, perhaps acting on different spaces $H$ and $K$. We say that $\pi$ and $\rho$ are *equivalent* $(\pi \sim \rho)$ if there is a unitary operator $U: H \to K$ such that $U\pi(a)U^* = \rho(a)$, for all $a$ in $A$. Equivalent representations are indistinguishable in the sense that any geometric property of one must be shared by the other, and it is correct to think of the unitary operator $U$ as representing nothing more than a change of "coordinates".

LEMMA 3. *Let $A$ be a $C^*$-algebra, and $\pi$ and $\rho$ two isometric representations of $A$ acting on the Hilbert spaces $H$ and $K$ respectively, which map single elements to operators of rank one and such that $[\pi(s)H : s \in \sigma] = H$ and $[\rho(s)K : s \in \sigma] = K$. Then $\pi$ and $\rho$ are equivalent.*

PROOF. By Zorn's lemma, there exists a family $\mathscr{E} = \{e_i\}$ of minimal idempotents in $A$ which is maximal subject to the condition,

$$(Ae_iA) \cap (Ae_jA) = (0), \qquad i \neq j,\ e_i,\ e_j \in \mathscr{E}.$$

We may also assume that $\{e_i\}$ are self-adjoint and hence each of the minimal left ideals $Ae_i$ of $A$ is a Hilbert space with the $C^*$-algebra norm. Consider now, a family of vectors of unit norm $\{x_i\} \subset H$, such that $\pi(e_i) = x_i \oplus x_i$, and denote

$H_i = \{\pi(a)x_i : a \in A\}$ for all $i$. Since

$$\begin{aligned}\|\pi(a)x_i\|^2 &= \|\pi(a)x_i\| \cdot \|\pi(a)x_i\| \\ &= \|\pi(a)x_i \otimes \pi(a)x_i\| = \|\pi(a)(x_i \otimes x_i)\pi(a^*)\| \\ &= \|\pi(a) \cdot \pi(e_i) \cdot \pi(a^*)\| = \|\pi(ae_ia^*)\| \\ &= \|ae_ia^*\| = \|(ae_i)(ae_i)^*\| = \|ae_i\|^2\end{aligned}$$

we have that $\{H_i\}$ are isometrically isomorphic to the minimal left ideals $\{Ae_i\}$. Minimal left ideals are always closed so $\{H_i\}$ are closed. More precisely $H_i$ is a closed $\pi(A)$-invariant subspace of $H$, where the $x_i$ are cyclic vectors for $H_i$, for all $i$. Now, for any elements $a$, $b$ in $A$ we have that

$$\pi(ae_i)x_i = \pi(a) \cdot \pi(e_i)x_i = \pi(a)(x_i \otimes x_i)x_i = \pi(a)x_i \in H_i$$

and

$$e_jb^*ae_i \in Ae_jA \cap Ae_iA = (0) \qquad (i \neq j).$$

Hence, if $i \neq j$, and $\langle\ ,\ \rangle$ is the inner product on $H$, then

$$\begin{aligned}\langle \pi(a)x_i, \pi(b)x_j\rangle &= \langle \pi(ae_i)x_i, \pi(be_j)x_j\rangle \\ &= \langle \pi(e_jb^*ae_i)x_i, x_j\rangle \\ &= \langle 0, x_j\rangle = 0,\end{aligned}$$

showing that $H_i$ and $H_j$ are orthogonal. Now, let $\pi_i$ be the restriction of $\pi$ on $H_i$. Then $\pi_i$ is irreducible since every non-zero vector of $H_i$ is cyclic. In fact

$$\begin{aligned}\{\pi_i(A)\pi_i(a)x_i\} &= \{\pi_i(A)\pi_i(ae_i)x_i\} = \{\pi_i(Aae_i)x_i\} \\ &= \{\pi_i(Ae_i)x_i\} = \{\pi_i(A)\pi_i(e_i)x_i\} \\ &= \{\pi_i(A)x_i\} \qquad (a \in A).\end{aligned}$$

To prove that $\pi(a) = \oplus \pi_i(a)$ $(a \in A)$, it is sufficient to show that $H = \oplus H_i$. But, this is true since $h \in H$ implies $\pi(e_i)h \in H_i$ and therefore $H = [\pi(s)H : s \in \sigma] \subseteq \oplus H_i \subseteq H$.

This completes the proof that there is a family $\{\pi_i\}$ of irreducible subrepresentations of $\pi$ such that $\pi(a)$ is the direct sum of $\pi_i(a)$ $(a \in A)$.

Also, if we denote $K_i = \{\rho(a)y_i : a \in A\}$ and $\rho_i$ the restriction of $\rho$ on $K_i$, then (as above) $K = \oplus K_i$ and $\rho(a) = \oplus \rho_i(a)$ $(a \in A)$. $H_i$ and $K_i$ are isometrically isomorphic since

$$\|\pi(a)x_i\| = \|ae_i\| = \|\rho(a)y_i\| \qquad (a \in A),$$

and therefore we may define a unitary operator $U_i$: $H_i \to K_i$ by $U_i\pi(a)x_i = \rho(a)y_i$, so that $U_i\pi_i(a)U^* = \rho_i(a)$ $(a \in A)$ for all $i$. The operator $U = \oplus U_i$ is clearly a unitary operator from $H$ onto $K$ and

$$U\pi(a)U^* = \bigoplus U_i\pi_i(a)U_i^* = \bigoplus \rho_i(a) = \rho(a) \qquad (a \in A).$$

REMARKS. (a) The assumptions that $[\pi(s)H : s \in \sigma] = H$ and $[\rho(s)K : s \in \sigma] = K$ cannot be discarded, as the following example shows. Let $H$ be an infinite dimensional Hilbert space and $A$ the $C^*$-subalgebra of $\mathscr{L}(H \oplus H)$ defined as follows:

$$A = \left\{ \begin{pmatrix} \lambda I & 0 \\ 0 & \lambda I + K \end{pmatrix}, \begin{array}{l} \text{where } \lambda \in \mathbf{C}, I \text{ is the identity} \\ \text{and } K \text{ is a compact operator on } H \end{array} \right\}.$$

Let $\pi$ be the identity representation of $A$ on $H \oplus H$, and $\rho$ the representation of $A$ on $H$ given

$$\rho\left(\begin{pmatrix} \lambda I & 0 \\ 0 & \lambda I + K \end{pmatrix}\right) = \lambda I + K.$$

Clearly, the single elements of $A$ are all elements of the form $\left(\begin{smallmatrix} 0 & 0 \\ 0 & R \end{smallmatrix}\right)$ with $R$ either zero or rank one operator on $H$. We can easily see that $\pi$ and $\rho$ carry single elements to rank one operators. But there is no unitary operator $U : H \oplus H \to H$.

(b) Erdos in [5, Theorem 3.7] proved that for an arbitrary $C^*$-algebra $A$ "there exists an isometric representation $\pi$ of $A$ on a Hilbert space $H$ such that the image of each non-zero element has rank one". Along the lines of the proof of this theorem it is shown that $\pi$ is the direct sum of irreducible representations, and therefore one can reobtain Lemma 3 as a consequence of the referred to Theorem 3.7 of [5]. In our case Erdos's Theorem can be deduced from Lemma 3.

Ylinen, drawing upon the representation referred to in that theorem, proved the following theorem [12].

THEOREM 4. *Let $A$ be a $C^*$-algebra. Then there exists an isometric representation $\pi$ of $A$ on a Hilbert space $H$ such that $t$ is compactly acting element of $A$, if and only if, $\pi(t)$ is a compact operator on $H$. Furthermore, $t$ is an element of the* soc$(A)$ *(i.e., the operator $a \to tat$ on $A$ has finite rank), if and only if, $\pi(t)$ is a finite rank operator on $H$.*

Suppose now, that $\pi$ is any faithful representation of a $C^*$-algebra $A$ on a Hilbert space $H$. Choose any closed two-sided ideal $J$ of $A$ and define $H_J = [\pi(J)H]$. Clearly, since $J$ is an ideal of $A$, $H = H_J \oplus H_J^{\perp}$ gives a decomposition of $H$ into reducing subspaces for $\pi(A)$. Define representations $\pi_J$ and $\pi_J^{\perp}$ of $A$ on $H_J$ and $H_J^{\perp}$ to be the restrictions of $\pi$ on $H_J$ and $H_J^{\perp}$ respectively. Then in a natural sense we have that $\pi(a) = \pi_J(a) \oplus \pi_J^{\perp}(a)$ $(a \in A)$ and $a \in J$ implies that

$\pi_J(a) = 0$. Suppose now that $J = \text{cl}(\text{soc}(A))$ and $\pi$ and $\rho$ are two isometric representations of $A$, acting on the Hilbert spaces $H$ and $K$ respectively (as in Proposition 4). Define $\pi'$ and $\rho'$ to be the representations on $J$ given by $\pi'(a) = \pi_J(a)$ and $\rho'(a) = \rho_J(a)$ ($a \in J$), respectively. Since $J$ is separated by its single elements, Lemma 3 implies that $\pi'$ and $\rho'$ are equivalent and hence by [1, Theorem 1.3.4] their extensions $\pi_J$ and $\rho_J$ are also equivalent.

We summarize what we have just proved and discussed, in the following theorem.

THEOREM 5. *Let $A$ be a $C^*$-algebra and $J = \text{cl}(\text{soc}(A))$. Then there exists an isometric representation $\pi$ of $A$ on some Hilbert space $H$ such that the image of each non-zero single element has rank one. Moreover, if $\pi$ and $\rho$ are such representations of $A$, then*

(i) *$\pi_J$ and $\rho_J$ are equivalent, and*

(ii) *$a \in J$ if and only if $\pi(a) = \pi_J(a) \oplus 0$ is a compact operator.*

(iii) *In particular, $a \in \text{soc}(A)$ if and only if $\pi(a)$ is a finite rank operator.*

## 3. Applications

Theorem 5 gives us a method of investigating the properties of compactly acting elements in $C^*$-algebras. For example, it can be used to introduce in an arbitrary $C^*$-algebra $A$, a class of elements analogous to von Neumann-Schatten classes $C_p$ of compact operators on a Hilbert space $H$, by a reduction to the concrete case.

First, we need the following well-known proposition, which we state without proof.

PROPOSITION 6. *If $K$ is any compact operator on some Hilbert space $H$, and $\{T_n\}$ a sequence in $\mathscr{L}(H)$ converging to $T$, say, in the strong operator topology, then the sequence $\{T_n K\}$ converges to $TK$ in the norm topology.*

The following result is related to the spectral theorem for compact normal operators on Hilbert spaces.

THEOREM 7. *Let $A$ be a $C^*$-algebra and let $a$ be a normal element in $J =$ $\text{cl}(\text{soc}(A))$. Then $a$ may be represented as a sum*

$$(*) \qquad a = \sum r_n e_n$$

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*in which*

(i) $\{r_n\}$ *is a finite or a countable family of non-zero complex numbers consisting of the non-zero elements of the spectrum of* $a$ *(repeated according to their multiplicity).*

(ii) $\{e_n\}$ *is a countable family of mutually orthogonal self-adjoint single idempotents. We have,* $ae_n = e_n a = e_n a e_n = r_n e_n$, *for each* $n$; $a$ *is self-adjoint if and only if each* $r_n$ *is real, and* $a$ *is positive if and only if each* $r_n > 0$.

Such a representation (∗) of $a$, having properties (i) and (ii) of Theorem 7, is said to be a *spectral representation* of $a$.

PROOF. From Theorem 5, we have that $a$ is in cl(soc($A$)) if and only if $\pi(a)$ is a compact operator and $a$ is normal, which is if and only if $\pi(a)$ is normal. By the spectral theorem for compact operators we have that $\pi(a) = \Sigma\lambda_i P_i$ where $\{\lambda_i\}$ is the sequence of distinct non-zero eigenvalues of $\pi(a)$, and $P_i$ is the finite rank projection upon the eigenspace corresponding to the eigenvalue $\lambda_i$. Every projection $P_i$ is then the strong limit of polynomials in $\pi(a)$, i.e. there exists a sequence of polynomials $q_n(\cdot)$ such that $q_n(\pi(a)) \to P_i$ (strongly). From Proposition 6 we have $q_n(\pi(a))\pi(A) \to P_i\pi(a)$ (in norm). Since $P_i\pi(a) = \lambda_i P_i$ we have $(1/\lambda_i)q_n(\pi(a))\pi(a) \to P_i$ (in norm) and so, every projection $P_i$ belongs to the closed algebra generated by $\pi(a)$. Hence $P_i \in \pi(A)$, and therefore for every $i$ there exists a self-adjoint idempotent $f_i$ in $A$ such that $P_i = \pi(f_i)$. From Theorem 5, since the $\{P_i\}$ are finite dimensional, we have that the $\{f_i\}$ are in soc($A$) and therefore each $f_i$ is a finite sum of orthogonal self-adjoint single idempotents $f_i = \Sigma_j e_{ij}$ say. Using Theorem 5 we obtain $a = \Sigma_i \lambda_i f_i = \Sigma_i \lambda_i(\Sigma_j e_{ij}) = \Sigma_i\Sigma_j \lambda_i e_{ij}$ and hence by a suitable modification of the notation, $a = \Sigma r_n e_n$. By multiplying the above equality by $e_n$ we have $ae_n = e_n a = e_n a e_n = r_n e_n$. The latter part of the Theorem is now clear.

REMARK. From Theorem 5 it follows that $\{r_n\}$ in the spectral representation (∗) of the element $a$, in Theorem 7, does not depend on the choice of the representation $\pi$ of $A$.

If $a$ is a non-zero element in cl(soc($A$)) then $a^*a$ is positive and clearly to cl(soc($A$)) also. Let $a^*a = \Sigma r_n e_n$ be a spectral representation of $a^*a$. The numbers $\sqrt{r_n}$ (denoted by $s_n(a)$) are called the *characteristic numbers* of $a$.

DEFINITION 8. Let $A$ be a $C^*$-algebra. Define

(i) $A_p = \{a \in \text{cl(soc}(A)): [\Sigma_n s_n^p(a)]^{1/p} < +\infty\}$ $(0 < p < \infty)$ and $A_\infty =$ cl(soc($A$)),

(ii) for $a \in A_p$, $\|a\|_p = [\Sigma_n s_n^p(a)]^{1/p}$ $(0 < p < \infty)$ and for $a \in A_\infty$, $\|a\|_\infty = \|a\| = \max\{s_n(a): n = 1, 2, \dots\}$.

From the above remark, $\|\cdot\|_p$ is well defined. An immediate consequence of the above definition is the following.

COROLLARY 9. (i) $a \in A_p$, *if and only if*, $\pi(a) \in C_p$.
(ii) $\|a\|_p = \|\pi(a)\|_p$.

Hence, by a reduction to the concrete case we can easily see that $\|a\|_p$ ($p \geqslant 1$) is a norm on $A_p$ making it a Banach space, and soc($A$) is a $\|\cdot\|_p$-dense subspace of $A_p$. Well-known results for the $C_p$ class can be extended to results for the $A_p$ class by means of Corollary 9.

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Section of Mathematical Analysis and its Application
Department of Mathematics
University of Athens
Panepistemiopolis
157 81 Athens
Greece