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Positive Solution of a Subelliptic Nonlinear Equation on the Heisenberg Group

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Abstract. In this paper, we establish the existence of positive solution of a nonlinear subelliptic equation involving the critical Sobolev exponent on the Heisenberg group, which generalizes a result of Brezis and Nirenberg in the Euclidean case.

1 Introduction

Analysis on the Heisenberg group is important since it has many applications to the theory of several complex variables (see [FS]). Since Jerison and Lee solved the CR Yamabe problem [JL2], which also has applications to the theory of several complex variables, people have become interested in nonlinear equations on the Heisenberg group. Jerison and Lee also found extremals of Sobolev inequalities on the Heisenberg group [JL1]. Garofalo and Lanconelli [GL2] established existence, regularity and nonexistence results for a kind of semilinear equations on the Heisenberg group. In this paper, we establish the existence of positive solution of a nonlinear subelliptic equation involving critical Sobolev exponent on the Heisenberg group, which generalizes a result of Brezis and Nirenberg in the Euclidean case [BN].

Let $\mathbb{H}^n$ be the *Heisenberg group*, whose underlying manifold is $\mathbb{R}^{2n+1}$, endowed with the group law (see [GL1], [GL2] for the following fundamental facts about the Heisenberg group):

$$(1.1)\qquad (x,y,t)\circ(x',y',t') = \Big(x+x', y+y', t+t' - 2\sum_{j=1}^{n}(x_j y'_j - x'_j y_j)\Big),$$

where $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)\in\mathbb{R}^n$. Set $z_j = x_j+\sqrt{-1}y_j$, $j=1,\dots,n$. In these coordinates, the vector fields

$$(1.2)\qquad X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t},\quad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t},$$

$j=1,\dots,n$, generate the real Lie algebra of left-invariant vector fields on $\mathbb{H}^n$. It is easy to check that $[X_j, Y_k] = -4\delta_{jk}\frac{\partial}{\partial t}$, $j,k=1,\dots,n$. Let

$$(1.3)\qquad \begin{aligned} \|(x,y,t)\| &= \left|t^2+|z|^4\right|^{\frac{1}{2}},\\ d\big((x,y,t),(x',y',t')\big) &= \|(x,y,t)^{-1}\circ(x',y',t')\| \end{aligned}$$

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$(x, y, t)^{-1} = (-x, -y, -t)$. $d(\cdot, \cdot)$ define a distance on $\mathbb{H}^n$. The operator

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^{n} (X_j^2 + Y_j^2) \tag{1.4}$$

is called *subLaplacian*, since it is not elliptic and satisfies subelliptic estimate [FS].

$$\tau_{(x,y,t)} : (x', y', t') \longrightarrow (x, y, t) \circ (x', y', t'), \quad (x, y, t) \in \mathbb{H}^n \tag{1.5}$$

is called a *left translation.* $\Delta_{\mathbb{H}^n}$ is invariant with respect to left translations, since X_j, Y_j are left invariant vector fields for all j. Furthermore, there is a natural group of *dilations* on $\mathbb{H}^n$ given by

$$d_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t), \quad \lambda > 0 \tag{1.6}$$

with which $\Delta_{\mathbb{H}^n}$ commutes according to the formula

$$\Delta_{\mathbb{H}^n} \circ d_\lambda = \lambda^2 d_\lambda \circ \Delta_{\mathbb{H}^n}.$$

Let $W_0^{1,2}(D)$ denote the closure of $C_0^\infty(D)$ in the norm

$$\|u\|_{W_0^{1,2}} = \left(\int_D (|\nabla_{\mathbb{H}^n} u|^2 + |u|^2)\, dx\, dy\, dt \right)^{\frac{1}{2}}, \tag{1.7}$$

where $|\nabla_{\mathbb{H}^n} u|^2 = \sum_{j=1}^n (|X_j u|^2 + |Y_j u|^2)$, and $dx\, dy\, dt$ is the volume element of $\mathbb{R}^{2n+1}$. Let $Q = 2n + 2$ be the *homogeneous dimension* of $\mathbb{H}^n$. We will consider the following problem

$$\begin{cases} -\Delta_{\mathbb{H}^n} u = u^{\frac{Q+2}{Q-2}} + \lambda u & \text{on } D, \\ u > 0 & \text{on } D, \\ u = 0 & \text{on } \partial D. \end{cases} \tag{1.8}$$

Theorem *Let λ_1 be the first nonzero eigenvalue of $-\Delta_{\mathbb{H}^n}$ on $W_0^{1,2}(D)$ and $n \geq 2$. Then for every $\lambda \in (0, \lambda_1)$, there exists a solution of (1.8).*

See Section 2 for the facts about the eigenvalues of $-\Delta_{\mathbb{H}^n}$. Analogous to Theorem 2.1 in [BN], we can establish the existence of positive solution for $-\Delta_{\mathbb{H}^n} u = u^{\frac{Q-2}{Q+2}} + f(x, u)$ on D, $u = 0$ on ∂D, and $f(x, u)$ is a lower order perturbation of $u^{\frac{Q+2}{Q-2}}$. We omit details. Results about prescribed Webster scalar curvature on S^n will appear elsewhere.

2 Some Propositions of $\Delta_{\mathbb{H}^n}$

From now on, we take $p = \frac{Q+2}{Q-2}$, then $p + 1 = \frac{2Q}{Q-2}$. Let $\mathfrak{D}(\mathbb{H}^n) = \mathfrak{D}(\mathbb{R}^{2n+1})$, the Schwartz space on $\mathbb{R}^{2n+1}$.

Proposition 2.1 **([JL2, Corollary C] or [JL1, p. 326])** *The following Sobolev type inequality holds for all functions u for which both sides finite,*

$$S\left(\int_{\mathbb{H}^n} |u|^{p+1}\,dx\,dy\,dt\right)^{\frac{2}{p+1}} \le \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2\,dx\,dy\,dt. \tag{2.1}$$

The best constant for the Sobolev inequality (2.1) is $S = 2\pi n^2(4^n n!)^{-1+\frac{2}{p+1}}$. The equality is achieved by functions $e_{a,\lambda}$, where $a = (x_0, y_0, t_0) \in \mathbb{H}^n$, $\lambda > 0$, $e_{a,\lambda}\big((x, y, t)\big) = e_\lambda\big(a \circ (x, y, t)\big)$ and

$$e_\lambda(x, y, t) = \left(\frac{\lambda^2}{\lambda^4 t^2 + (\lambda^2|z|^2 + 1)^2}\right)^{\frac{n}{2}}. \tag{2.2}$$

e_λ in (2.2) is another form of extremal Φ^ε in [JL2, p. 326] for $\varepsilon = \frac{1}{\lambda}$. It follows that

$$-\Delta_{\mathbb{H}^n} e_\lambda = e_\lambda^p, \quad e_\lambda \in L^{p+1}, \tag{2.3}$$

and we can check that $\|e_\lambda\|_{p+1}$ is independent on λ.

Proposition 2.2 **([GL2, Theorem 3.2])** *For D bounded and $1 \le q < \frac{2Q}{Q-2}$, embedding $W_0^{1,2}(D) \longrightarrow L^q(D)$ is compact.*

For $u, v \in C_0^\infty(D)$, we have $(X_j u, v) = (\frac{\partial}{\partial x_j} u, v) + (\frac{\partial}{\partial t} u, 2y_j v) = (u, -X_j v)$ by Stokes' formula and similar formulae for Y_j, $j = 1, \dots, n$. Thus, $\int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n} u \nabla_{\mathbb{H}^n} v = -\int_{\mathbb{H}^n} u \Delta_{\mathbb{H}^n} v$, where $\nabla_{\mathbb{H}^n} u \nabla_{\mathbb{H}^n} v = \sum_{j=1}^n X_j u X_j v + Y_j u Y_j v$. The following *maximum principle* will be used later.

Proposition 2.3 **(The Maximum Principle)** *Let $u \in W_0^{1,2}(D)$ satisfy $\Delta_{\mathbb{H}^n} u \ge 0$, i.e., $\int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n} u \nabla_{\mathbb{H}^n} v \ge 0$ for all $v \in W_0^{1,2}(D)$ with $v \ge 0$. Then either $u \equiv 0$ or u cannot attain its maximum in D.*

This proposition follows from the following *weak Harnack inequality* by standard argument [GT, p. 188].

Proposition 2.4 *If $u \in W_0^{1,2}(D)$, $\Delta_{\mathbb{H}^n} u \ge 0$ in D, and $u \ge 0$ in a ball $B_{4R} = B_{4R}(y) \subset D$, there exist $p_0 > 0$, $C > 0$ such that*

$$\left(\frac{1}{|B_{2R}|}\int_{B_{2R}} |u|^{p_0}\right)^{\frac{1}{p_0}} \le C\,\mathrm{essinf}_{B_R}\, u. \tag{2.4}$$

Only in this proposition and its proof, we use the following distance on $\mathbb{H}^n$ as in [CDG]:

$$\hat{d}(x, y) = \inf\{T > 0 \mid \text{there exists } \gamma\colon [0, T] \longrightarrow \mathbb{H}^n, \gamma(0) = x, \gamma(T) = y\},$$

where γ must be a piecewise C^1 curve and for each $\xi \in \mathbb{R}^{2n+1}$, $t \in [0, T]$,

$$\bigl(\gamma'(t)\cdot\xi\bigr)^2 \le \sum_{j=1}^{n}\Bigl(X_j\bigl(\gamma(t)\bigr)\cdot\xi\Bigr)^2 + \Bigl(Y_j\bigl(\gamma(t)\bigr)\cdot\xi\Bigr)^2.$$

$B_R(y) = \{y' \in \mathbb{H}^n \mid \hat{d}(y, y') \le R\}$. This distance is equivalent to the distance defined in (1.3), but we will not use this fact. Harnack inequality is proved for more general nonlinear subelliptic equation in [CDG]. They consider vector fields in $\mathbb{R}^n$, $X_1, \dots, X_m$ ($m < n$), satisfying Hömander condition, *i.e.*, the rank of Lie algebra generated by $X_1, \dots, X_m$ at each point is n. The Heisenberg group is a special and simple case of this. Although Proposition 2.4 is about subsolution, its proof is similar to [CDG, pp. 1788–1791]. In the case of Heisenberg group, computations become more simple.

Proof We denote the weak derivatives corresponding to X_j, Y_j also by X_j, Y_j, $j = 1, \dots, n$, and for $\nabla_{\mathbb{H}^n}$. Choose $\beta < -1$, $q = \frac{1+\beta}{2} < 0$. For $\eta \in C_0^\infty(B_{2R})$ with $0 \le \eta \le 1$, let $\overline{u} = u + k$ and $\phi = \eta^2\overline{u}^\beta$ for constant $k > 0$. Since $\Delta_{\mathbb{H}^n} u \ge 0$,

$$0 \le \int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n}\overline{u}\nabla_{\mathbb{H}^n}\phi\, dx\, dy\, dt = \int_{\mathbb{H}^n} 2\eta\overline{u}^\beta\nabla_{\mathbb{H}^n}\overline{u}\nabla_{\mathbb{H}^n}\eta + \beta\eta^2\overline{u}^{\beta-1}\nabla_{\mathbb{H}^n}\overline{u}\nabla_{\mathbb{H}^n}\overline{u} \tag{2.5}$$

by calculus of weak differentiation [GT, Section 7.4]. Now let $v = \overline{u}^q$, then

$$|\beta|\int_{\mathbb{H}^n} |\eta\nabla_{\mathbb{H}^n} v|^2 \le 2|q|\int_{\mathbb{H}^n} |v\nabla_{\mathbb{H}^n}\eta\cdot\eta\nabla_{\mathbb{H}^n} v|. \tag{2.6}$$

Using Hölder inequality in the right side of (2.6), we find

$$\Bigl(\int_{\mathbb{H}^n} |\eta\nabla_{\mathbb{H}^n} v|^2\Bigr)^{\frac12} \le \frac{2|q|}{2|q|+1}\Bigl(\int_{\mathbb{H}^n} |v\nabla_{\mathbb{H}^n}\eta|^2\Bigr)^{\frac12}. \tag{2.7}$$

Using Lemma 3.2 in [CDG] for $1 \le a < b \le 2$, we can choose $\eta \in C_0^\infty(B_{2R})$ with $\eta \equiv 1$ on B_{aR} and $|\nabla_{\mathbb{H}^n}\eta| \le \frac{C}{(b-a)R}$. Now apply the Sobolev embedding theorem (Theorem 2.3 in [CDG]) to (2.7) to get

$$\Bigl(\int_{B_{aR}} |v|^{2\kappa}\, dx\, dy\, dt\Bigr)^{\frac{1}{2\kappa}} \le \frac{C}{(b-a)}|B_{2R}|^{\frac{1}{2\kappa}-\frac12}\Bigl(\int_{B_{bR}} v^2\, dx\, dy\, dt\Bigr)^{\frac12} \tag{2.8}$$

for $1 \le \kappa \le \frac{Q}{Q-2}$. Let $\kappa = \frac{Q}{Q-2}$. Note $v = \overline{u}^q$ and $q < 0$,

$$\Bigl(\int_{B_{aR}} \overline{u}^{2\kappa q}\, dx\, dy\, dt\Bigr)^{\frac{1}{2\kappa q}} \ge C^{\frac1q}(b-a)^{-\frac1q}|B_{2R}|^{-\frac{1}{qQ}}\Bigl(\int_{B_{bR}} \overline{u}^{2q}\, dx\, dy\, dt\Bigr)^{\frac{1}{2q}}. \tag{2.9}$$

Now as in [CDG, p. 1791], by Moser's iteration procedure, we get

$$\operatorname{essinf}_{B_R}\overline{u} \ge C\Bigl(\frac{1}{|B_{2R}|}\int_{B_{2R}} |\overline{u}|^{2q}\, dx\, dy\, dt\Bigr)^{\frac{1}{2q}}. \tag{2.10}$$

Where $2q$ ranges $(-\infty, 0)$ by the choices of β, q. Now choose $v = \log \overline{u}$, $\phi = \eta^2(\overline{u})^{-1}$ for $\eta \in C_0^\infty(B_{2R})$ and $|\nabla_{\mathbb{H}^n}\eta| \leq \frac{C}{R}$. Then

(2.11)
$$0 \leq \int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n}\overline{u}\nabla_{\mathbb{H}^n}\phi \, dx\, dy\, dt = \int_{\mathbb{H}^n} 2\eta|\overline{u}|^{-1}\nabla_{\mathbb{H}^n}\overline{u}\nabla_{\mathbb{H}^n}\eta - \eta^2|\overline{u}|^{-2}\nabla_{\mathbb{H}^n}\overline{u}\nabla_{\mathbb{H}^n}\overline{u}$$

and

$$\int_{\mathbb{H}^n} |\eta\nabla_{\mathbb{H}^n}v|^2 \leq 2\int_{\mathbb{H}^n} \eta\nabla_{\mathbb{H}^n}\eta\nabla_{\mathbb{H}^n}v. \tag{2.12}$$

Using the Hölder inequality on the right side of (2.12), we find

$$\left(\int_{\mathbb{H}^n} |\eta\nabla_{\mathbb{H}^n}v|^2\right)^{\frac{1}{2}} \leq 2\left(\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n}\eta|^2\right)^{\frac{1}{2}} \leq C|B_{2R}|^{\frac{1}{2}}R^{-1}. \tag{2.13}$$

Then as in [CDG, p. 1788], by using Poincaré inequality and John-Nirenberg theorem for space of homogeneous type, we infer that there exist $p_0 > 0$ and $C > 0$ such that

$$\left(\frac{1}{|B_{2R}|}\int_{B_{2R}} |\overline{u}|^{p_0}\right)^{\frac{1}{p_0}} \leq C\left(\frac{1}{|B_{2R}|}\int_{B_{2R}} |\overline{u}|^{-p_0}\right)^{-\frac{1}{p_0}}. \tag{2.14}$$

Then, by (2.10),

$$\left(\frac{1}{|B_{2R}|}\int_{B_{2R}} |\overline{u}|^{p_0}\right)^{\frac{1}{p_0}} \leq C \operatorname{essinf}_{B_R} \overline{u}. \tag{2.15}$$

Note constant C in (2.15) is independent on k. Let $k \longrightarrow 0$. We get (2.4).

Proposition 2.5 **([S, Corollary 7.D])** *Let V and H be two Hilbert spaces with V dense in H and assume the embedding $V \longrightarrow H$ is compact. The continuous sesquilinear form $a(u, v)$ is symmetric and satisfies*

$$a(v, v) + \lambda\|v\|_H^2 \geq c\|v\|_V^2, \quad v \in V$$

for some $\lambda \in \mathbb{R}$ and $c > 0$. Now define the operator A:

$$D(A) = \{u \in V \mid a(u, \cdot) \text{ is continuous with respect to norm } \|\cdot\|_H\},$$
$$a(u, v) = (Au, v)_H \quad \text{for } u \in D(A), v \in V$$

Then there is an orthonormal sequence of eigenfunctions of A which is a basis for H and the corresponding eigenvalues satisfy $-\lambda < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$.

Apply Proposition 2.5 to

$$V = W_0^{1,2}(D), \quad H = L^2(D), \quad a(u, v) = \int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n}u\nabla_{\mathbb{H}^n}v,$$

to get $A = -\Delta_{\mathbb{H}^n}$ and the corresponding eigenvalues being nonnegative. The only eigenfunction of eigenvalue 0 is the zero function by the maximum principle.

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3 The Proof of the Theorem

Set

$$S_\lambda = \inf\{\|\nabla_{\mathbb{H}^n} u\|_2^2 - \lambda\|u\|_2^2; u \in W_0^{1,2}(D), \|u\|_{p+1} = 1\}. \tag{3.1}$$

As in the Euclidean case [BN], S_0 is independent on D and only on n. This follows from the fact that the ratio $\|\nabla_{\mathbb{H}^n} u\|_2/\|u\|_{p+1}$ is invariant under scaling, *i.e.*,

$$\frac{\|\nabla_{\mathbb{H}^n} u_k\|_2}{\|u_k\|_{p+1}} = \frac{\|\nabla_{\mathbb{H}^n} u\|_2}{\|u\|_{p+1}}, \tag{3.2}$$

where $u_k(x, y, t) = u(kx, ky, k^2 t)$, $k \in \mathbb{Z}$. Hence $S_0 = S$. To check (3.2), note

$$\Big(\frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}\Big)\big(u(kx, ky, k^2t)\big) = k\Big(\frac{\partial}{\partial x_j'} + 2y_j'\frac{\partial}{\partial t'}\Big)u(x', y', t')$$

$$\Big(\frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}\Big)\big(u(kx, ky, k^2t)\big) = k\Big(\frac{\partial}{\partial y_j'} - 2x_j'\frac{\partial}{\partial t'}\Big)u(x', y', t'),$$

for $j = 1, \dots, n$ and $(x', y', t') = (kx, ky, k^2t)$. Then

$$\Big(\int_{\mathbb{H}^n}\sum_{j=1}^n(|X_j u_k|^2 + |Y_j u_k|^2)\,dx\,dy\,dt\Big)^{\frac12} = k^{-n}\Big(\int_{\mathbb{H}^n}\sum_{j=1}^n(|X_j' u|^2 + |Y_j' u|^2)\,dx'\,dy'\,dt'\Big)^{\frac12}$$

by dilation transform $(x', y', t') = (kx, ky, k^2t)$, where $X_j' = \frac{\partial}{\partial x_j'} + 2y_j'\frac{\partial}{\partial t'}$, $Y_j' = \frac{\partial}{\partial y_j'} - 2x_j'\frac{\partial}{\partial t'}$, $j = 1, \dots, n$. Under the same dilation transformation,

$$\Big(\int_{\mathbb{H}^n}|u_k|^{p+1}\,dx\,dy\,dt\Big)^{\frac{1}{p+1}} = k^{\frac{-2n-2}{p+1}}\Big(\int_{\mathbb{H}^n}|u|^{p+1}\,dx'\,dy'\,dt'\Big)^{\frac{1}{p+1}}. \tag{3.3}$$

Then, (3.2) is verified by $p + 1 = \frac{2Q}{Q-2} = \frac{2(n+1)}{n}$.

Proposition 3.1 *$S_\lambda < S$ for all $\lambda > 0$.*

Proof Since $\Delta_{\mathbb{H}^n}$ is invariant under left translation, we can assume $0 \in D$. Let's estimate the quotient

$$Q_\lambda(u) = \frac{\|\nabla_{\mathbb{H}^n} u\|_2^2 - \lambda\|u\|_2^2}{\|u\|_{p+1}^2} \tag{3.4}$$

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for $u_\varepsilon(a) = \phi(a) \cdot \delta_\varepsilon(a)$ with

$$\delta_\varepsilon(a) = \left(\frac{\varepsilon^4}{\varepsilon^4 t^2 + (\varepsilon^2 |z|^2 + 1)^2} \right)^{\frac{n}{2}},$$

where $a = (x, y, t) \in \mathbb{H}^n$, $\phi \geq 0$ on D, $\phi \in \mathfrak{D}(D)$ satisfying $\phi = 1$ on some neighbourhood of $(0, \dots, 0)$ contained in D. It can be directly checked that

$$\begin{aligned} X_j \delta_\varepsilon &= -\frac{n}{2} \varepsilon^{2n} \big(\varepsilon^4 t^2 + (\varepsilon^2 |z|^2 + 1)^2 \big)^{-\frac{n}{2}-1} \big(4\varepsilon^4 y_j t + 4\varepsilon^2 x_j (\varepsilon^2 |z|^2 + 1) \big) \\ Y_j \delta_\varepsilon &= -\frac{n}{2} \varepsilon^{2n} \big(\varepsilon^4 t + (\varepsilon^2 |z|^2 + 1)^2 \big)^{-\frac{n}{2}-1} \big(-4\varepsilon^4 x_j t + 4\varepsilon^2 y_j (\varepsilon^2 |z|^2 + 1) \big) \end{aligned} \tag{3.5}$$

for $j = 1, \dots, n$, then

$$|\nabla_{\mathbb{H}^n} \delta_\varepsilon|^2 = \frac{n^2}{4} \varepsilon^{4n} \frac{16\varepsilon^8 |z|^2 t^2 + 16\varepsilon^4 |z|^2 (\varepsilon^2 |z|^2 + 1)^2}{\big(\varepsilon^4 t^2 + (\varepsilon^2 |z|^2 + 1)^2 \big)^{n+2}}. \tag{3.6}$$

Since $\nabla_{\mathbb{H}^n} u_\varepsilon = \nabla_{\mathbb{H}^n} \phi \cdot \delta_\varepsilon + \phi \cdot \nabla_{\mathbb{H}^n} \delta_\varepsilon$, and

$$\begin{aligned} |\nabla_{\mathbb{H}^n} \delta_\varepsilon|^2 &\leq 4n^2 \varepsilon^{4n} \left(\frac{\varepsilon^8 |z|^2 t^2}{(\varepsilon^4 t^2 + \varepsilon^4 |z|^4)^{n+2}} + \frac{\varepsilon^4 |z|^2}{(\varepsilon^4 t^2 + \varepsilon^4 |z|^4)^{n+1}} \right) \\ &\leq 4n^2 \left(\frac{|z|^2 t^2}{(t^2 + |z|^4)^{n+2}} + \frac{|z|^2}{(t^2 + |z|^4)^{n+1}} \right) \end{aligned}$$

is bounded if (x, y, t) is not in a neighbourhood of $(0, \dots, 0)$, and similarly, $\delta_\varepsilon(x, y, t)$ is also bounded if (x, y, t) is not in a neighbourhood of $(0, \dots, 0)$, we find

$$\begin{aligned} &\int_D |\nabla_{\mathbb{H}^n} u_\varepsilon|^2 \, dx \, dy \, dt \\ &\quad = \int_D |\nabla_{\mathbb{H}^n} \delta_\varepsilon|^2 \, dx \, dy \, dt + O(1) \\ &\quad = \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} \delta_\varepsilon|^2 \, dx \, dy \, dt + O(1) \\ &\quad = \int_{\mathbb{H}^n} 4n^2 \varepsilon^{4n} \frac{\varepsilon^8 |z|^2 t^2 + \varepsilon^4 |z|^2 (\varepsilon^2 |z|^2 + 1)^2}{\big(\varepsilon^4 t^2 + (\varepsilon^2 |z|^2 + 1)^2 \big)^{n+2}} \, dx \, dy \, dt + O(1) \\ &\quad = \varepsilon^{2n} \int_{\mathbb{H}^n} 4n^2 \frac{|z|^2 t^2 + |z|^2 (|z|^2 + 1)^2}{\big(t^2 + (|z|^2 + 1)^2 \big)^{n+2}} \, dx \, dy \, dt + O(1) \\ &\quad = \varepsilon^{Q-2} \|\nabla_{\mathbb{H}^n} \delta_0\|_2^2 + O(1) \end{aligned} \tag{3.7}$$

where we use the coordinate transformation $\varepsilon x \longrightarrow x$, $\varepsilon y \longrightarrow y$, $\varepsilon^2 t \longrightarrow t$. Similarly,

$$\begin{aligned}\|u_\varepsilon\|_{p+1}^{p+1} &= \int_D |\delta_\varepsilon|^{p+1} + O(1) \\ &= \int_{\mathbb{H}^n} \left(\frac{\varepsilon^4}{\varepsilon^4 t^2 + (\varepsilon^2|z|^2+1)^2}\right)^{n+1} dx\,dy\,dt + O(1) \\ &= \varepsilon^{2n+2}\|\delta_0\|_{p+1}^{p+1} + O(1).\end{aligned} \tag{3.8}$$

Then,

$$\|u_\varepsilon\|_{p+1}^2 = \varepsilon^{2n}\|\delta_0\|_{p+1}^2 + O(1).$$

Note $\int_{\mathbb{H}^n} \frac{1}{(t^2+(|z|^2+1)^2)^n}\, dx\,dy\,dt$ integrable if $n \geq 2$,

$$\|u_\varepsilon\|_2^2 = \int_{\mathbb{H}^n} \left(\frac{\varepsilon^4}{\varepsilon^4 t^2 + (\varepsilon^2|z|^2+1)^2}\right)^n + O(1) = \varepsilon^{2n-2}\|\delta_0\|_2^2 + O(1). \tag{3.9}$$

Then

$$Q_\lambda(u_\varepsilon) = \frac{\|\nabla_{\mathbb{H}^n}\delta_0\|_2^2}{\|\delta_0\|_{p+1}^2} - \lambda\varepsilon^{-2}\frac{\|\delta_0\|_2^2}{\|\delta_0\|_{p+1}^2} + O(\varepsilon^{-2n}) \tag{3.10}$$

by (3.7)–(3.9). Thus, $Q_\lambda(u_\varepsilon) < S_0 = S$ if ε sufficiently large.

Proposition 3.2 *The infimum of (3.1) is archived.*

Proof By using Proposition 3.1, the proof is exactly the same as the proof of Lemma 1.2 in [BN]. We omit the details.

Proof of the Theorem Let $u \in W_0^{1,2}(D)$ given by Proposition 3.2 and $\|u\|_{p+1} = 1$, $\|\nabla_{\mathbb{H}^n}u\|_2^2 - \lambda\|u\|_2^2 = S_\lambda$. We may assume $u \geq 0$ on D (otherwise we replace u by $|u|$ since $\big\|\nabla_{\mathbb{H}^n}|u|\big\|_2 \leq \|\nabla_{\mathbb{H}^n}u\|_2$ by the calculus of weak differentiation). Since u is a minimizer of (2.4), we obtain a Lagrange multiple $\mu \in \mathbb{R}$ such that

$$-\Delta_{\mathbb{H}^n}u - \lambda u = \mu u^p \quad \text{on } D. \tag{3.11}$$

$\mu = S_\lambda$. $S_\lambda > 0$ by $\lambda < \lambda_1$. Then ku satisfies (1.8) for $kS_\lambda = k^p$. Now $\Delta_{\mathbb{H}^n}(-u) \geq 0$. Hence $u > 0$ by Proposition 2.3, the maximum principle.

References

[BN] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equation involving critical Sobolev exponents.* Comm. Pure Appl. Math. **36**(1983), 437–477.

[CDG] L. Capogna, D. Danielli and N. Garofalo, *An embedding theorem and the Harnark inequality for nonlinear subelliptic equations.* Comm. Partial Differential Equations **18**(1993), 1765–1794.

[FS] G. B. Folland and E. M. Stein, *Estimates for the $\overline{\partial}_b$ complex and analysis on the Heisenberg group.* Comm. Pure Appl. Math. **27**(1974), 429–522.

[GL1] N. Garofalo and E. Lanconelli, *Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation.* Ann. Inst. Fourier Grenoble **40**(1990), 313–356.

[GL2] ———, *Existence and nonexistence results for semilinear equations on the Heisenberg group.* Indiana Univ. Math. J. (1) **41**(1992), 71–98.

[GT] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order.* Grundlehren Math. Wiss. **224**, Springer-Verlag, 1977.

[JL1] D. S. Jerison and L. M. Lee, *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem.* J. Amer. Math. Soc. **1**(1988), 1–13.

[JL2] ———, *Intrinsic CR coordinates and the CR Yamabe problem.* J. Differential Geom. **29**(1989), 303–343.

[S] R. E. Showalter, *Hilbert space methods for partial differential equations.* Pitman Publishing, London, 1977.

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