Canad. J. Math. 2025, pp. 1–26
http://dx.doi.org/10.4153/S0008414X25101806


# Signature maps from positive cones on algebras with involution

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*Abstract.* We introduced positive cones in an earlier paper as a notion of ordering on central simple algebras with involution that corresponds to signatures of hermitian forms. In the current article, we describe signatures of hermitian forms directly out of positive cones, and also use this approach to rectify a problem that affected some results in the previously mentioned paper.

## 1 Introduction

In [4] we introduced the notion of positive cones for central simple algebras with involution, inspired by the classical real algebra of ordered fields. They are linked to signatures of hermitian forms, whose investigation we started in [1], inspired by [6]. We also gave a complete description of the kernels of the signatures maps in [2].

In the current article, after providing the necessary background in Section 2, we show in Section 3 how to directly obtain such a kernel out of a given positive cone. This construction also allows us to rectify a recently discovered mistake in [4]. Specifically, the mistake occurs in the proof of [4, Lemma 5.5], and the lemma itself is likely incorrect. Hence, the lemmas in the remainder of [4, Section 5] (and their consequences) are potentially incorrect. These lemmas are used to prove [4, Proposition 5.8] which is the only result in [4, Section 5] that is used in the remainder of [4]. In this article, we provide in particular an entirely different proof of [4, Proposition 5.8], so that all the results in [4] now have correct proofs, except for [4, Lemmas 5.5–5.7] which are no longer needed.

Note that in the process of reproving [4, Proposition 5.8], we provide more direct proofs of results in [4] that were originally obtained as consequences of [4, Proposition 5.8]. Therefore, we clearly indicate in each statement in Sections 4 and 5 if it already appeared in [4].

## 2 Preliminaries

All fields in this article are assumed to have characteristic different from 2. Let $F$ be such a field. We denote by $W(F)$ the Witt ring of $F$, by $X_F$ the space of orderings

Received by the editors May 29, 2025; revised August 28, 2025; accepted September 3, 2025.
Published online on Cambridge Core November 12, 2025.
AMS subject classification: **13J30, 16W10, 06F25, 16K20, 11E39.**
Keywords: Real algebra, algebras with involution, orderings, hermitian forms.

of $F$, and by $F_P$ a real closure of $F$ at an ordering $P \in X_F$. We often denote the unique ordering on $F_P$ by $\widetilde{P}$.

By an *F-algebra with involution* we mean a pair $(A, \sigma)$, where $A$ is a finite-dimensional simple $F$-algebra with centre a field $K = Z(A)$, equipped with an involution $\sigma : A \to A$, such that $F = K \cap \mathrm{Sym}(A, \sigma)$, where $\mathrm{Sym}(A, \sigma) := \{a \in A \mid \sigma(a) = a\}$. More generally, we let $\mathrm{Sym}_\varepsilon(A, \sigma) := \{a \in A \mid \sigma(a) = \varepsilon a\}$ for $\varepsilon \in \{-1, +1\}$. If $A$ is a division algebra, we call $(A, \sigma)$ an *F-division algebra with involution*.

Observe that $[K : F] \leqslant 2$. We say that $\sigma$ is *of the first kind* if $K = F$ and *of the second kind* (or *of unitary type*) otherwise. Involutions of the first kind can be further subdivided into those of *orthogonal type* and those of *symplectic type*, depending on the dimension of $\mathrm{Sym}(A, \sigma)$ (cf. [13, Sections 2.A and 2.B]). We let $\iota = \sigma|_K$ and note that $\iota = \mathrm{id}_F$ if $\sigma$ is of the first kind.

We denote by $W(A, \sigma)$ the Witt group of Witt equivalence classes of nonsingular hermitian forms over $(A, \sigma)$ defined on finitely generated right $A$-modules. Note that $W(A, \sigma)$ is a $W(F)$-module. We denote isometry of forms by $\simeq$. For $a_1, \dots, a_k$ in $\mathrm{Sym}(A, \sigma)$ the notation $\langle a_1, \dots, a_\ell \rangle_\sigma$ stands for the diagonal hermitian form

$$\big((x_1, \dots, x_\ell), (y_1, \dots, y_\ell)\big) \in A^\ell \times A^\ell \mapsto \sum_{i=1}^{\ell} \sigma(x_i) a_i y_i \in A.$$

We often identify nonsingular quadratic and hermitian forms with their Witt classes if no confusion is possible. If $h$ is a hermitian form over $(A, \sigma)$, we denote the set of elements represented by $h$ by $D_{(A,\sigma)}(h)$.

We denote by $\mathrm{Int}(u)$ the inner automorphism determined by $u \in A^\times$, i.e., $\mathrm{Int}(u)(x) := uxu^{-1}$ for $x \in A$. We also write $\sigma^t$ for the involution $(a_{ij}) \mapsto (\sigma(a_{ji}))$ on $M_n(A)$.

## 2.1 Signatures of hermitian forms over quadratic field extensions and quaternions

Let $k$ be a field and let $D_k \in \{k, k(\sqrt{d}), (a, b)_k\}$, where $a, b, d \in k$, $k(\sqrt{d}) \neq k$ and $(a, b)_k$ is a quaternion division algebra over $k$. Let $\vartheta_k$ denote the canonical involution on $D_k$, i.e., the identity map on $k$ or conjugation in the remaining cases.

If $h$ is a hermitian form over $(D_k, \vartheta_k)$, then $b_h(x) := h(x, x)$ is a quadratic form over $k$. A straightforward computation shows that if $h \simeq \langle a_1, \dots, a_n \rangle_{\vartheta_k}$ with $a_1, \dots, a_n \in \mathrm{Sym}(D_k, \vartheta_k) = k$, then

$$b_h \simeq \begin{cases} \langle a_1, \dots, a_n \rangle & \text{if } D_k = k \\ \langle 1, -d \rangle \otimes \langle a_1, \dots, a_n \rangle & \text{if } D_k = k(\sqrt{d}) \\ \langle 1, -a, -b, ab \rangle \otimes \langle a_1, \dots, a_n \rangle & \text{if } D_k = (a, b)_k \end{cases}. \tag{2.1}$$

By Jacobson's theorem (cf. [18, Chapter 10, Theorems 1.1 and 1.7, Remark 1.3]), the map

$$W(D_k, \vartheta_k) \to W(k),\ h \mapsto b_h$$

is injective.

Let $P \in X_k$. The preceding paragraph motivates defining the signature of $h$ at $P$ in terms of the Sylvester signature $\operatorname{sign}_P b_h$ as follows:

$$\operatorname{sign}_P h := \begin{cases} \operatorname{sign}_P b_h & \text{if } D_k = k \\ \frac{1}{2}\operatorname{sign}_P b_h & \text{if } D_k = k(\sqrt{d}) \text{ with } d <_P 0 \\ \frac{1}{4}\operatorname{sign}_P b_h & \text{if } D_k = (a,b)_k \text{ with } a, b <_P 0 \\ 0 & \text{in all remaining cases.} \end{cases} \tag{2.2}$$

***Remark 2.1*** If $D_k = k(\sqrt{d})$, skew-hermitian forms over $(D_k, \vartheta_k)$ are equivalent to hermitian forms( cf. [1, Lemma 2.1(iii)]).

If $D_k \in \{k, (a,b)_k\}$ and $h$ is a skew-hermitian form over $(D_k, \vartheta_k)$, or a hermitian or skew-hermitian form over $(D_k \times D_k^{\mathrm{op}}, \widehat{\ })$ with $\widehat{(x, y^{\mathrm{op}})} := (y, x^{\mathrm{op}})$ the exchange involution, then (the nonsingular part of) $h$ is torsion in the Witt group and we let

$$\operatorname{sign}_P h := 0, \tag{2.3}$$

cf. [1, Section 3.1 and Lemma 2.1].

## 2.2 Signatures of hermitian forms over $F$-algebras with involution

Returning to the general case of an $F$-algebra with involution $(A, \sigma)$, let $P \in X_F$ and let

$$(D_P, \vartheta_P) := (D_{F_P}, \vartheta_{F_P}) \in \{(F_P, \mathrm{id}), (F_P(\sqrt{-1}), -), ((-1,-1)_{F_P}, -)\},$$

using the notation from Section 2.1. We define the signature of a hermitian form $h$ over $(A, \sigma)$ by extending scalars to $F_P$. Write $Z(A) = F(\sqrt{d})$ with $d \in F$. We consider two cases:

(1) If $\sigma$ is of the second kind and $d >_P 0$, then $Z(A) \otimes_F F_P \cong F_P \times F_P$ and we obtain

$$(A \otimes_F F_P, \sigma \otimes \mathrm{id}) \cong (M_{n_P}(D_P) \times M_{n_P}(D_P)^{\mathrm{op}}, \widehat{\ }), \tag{2.4}$$

cf. [13, Proposition 2.14]. Since (the nonsingular part of) $h$ is zero in the Witt group (cf. [1, Lemma 2.1(iv)]), we will define the signature of $h$ at $P$ to be zero in this case (cf. (2.7) below).

(2) If $\sigma$ is of the second kind and $d <_P 0$, or if $\sigma$ is of the first kind, then by the Skolem–Noether theorem, we obtain an isomorphism of $F_P$-algebras with involution

$$(A \otimes_F F_P, \sigma \otimes \mathrm{id}) \cong (M_{n_P}(D_P), \mathrm{Int}(\Phi_P) \circ {\vartheta_P}^t), \tag{2.5}$$

where $\Phi_P \in \mathrm{Sym}_\varepsilon(M_{n_P}(D_P), {\vartheta_P}^t)$ is invertible and $\varepsilon = 1$ if $\sigma$ and $\vartheta_P$ are of the same type and $\varepsilon = -1$ otherwise (cf. [13, Propositions 2.7 and 2.18]).

The $F_P$-algebra with involution $(M_{n_P}(D_P), \mathrm{Int}(\Phi_P) \circ {\vartheta_P}^t)$ is hermitian Morita equivalent to $(M_{n_P}(D_P), {\vartheta_P}^t)$ (via scaling by $\Phi_P^{-1}$), which in turn is hermitian Morita equivalent to $(D_P, \vartheta_P)$ (cf. [1, Section 2.4]. We denote the composition of these equivalences, and its induced map on hermitian forms, by $\mathfrak{m}_P$.

***Remark 2.2*** Observe that if $\sigma$ is orthogonal and $D_P = (-1,-1)_{F_P}$ or if $\sigma$ is symplectic and $D_P = F_P$, then $\varepsilon = -1$ and $\mathfrak{m}_P(h \otimes_F F_P)$ is skew-hermitian over $(D_P, \vartheta_P)$.

Therefore, in accordance with Remark 2.1, we will define the signature of $h$ at $P$ to be zero in this case (cf. (2.7) below).

***Definition 2.3*** (See also (2.8) below) We say that $P$ is a *nil-ordering of* $(A,\sigma)$ if (2.4) holds or if one of the cases described in Remark 2.2 occurs. We denote the set of nil-orderings of $(A,\sigma)$ by $\mathrm{Nil}[A,\sigma]$, where the square brackets indicate that this set depends only on the Brauer class of $A$ and the type of $\sigma$.

Assume now that $P \in X_F \backslash \mathrm{Nil}[A,\sigma]$. As already mentioned, the idea is to define the signature of $h$ at $P$ as $\mathrm{sign}_{\widetilde{P}}\, \mathfrak{m}_P(h \otimes_F F_P)$ via (2.2), where $\widetilde{P}$ denotes the unique ordering on $F_P$. There is, however, a problem: while a different choice of real closure does not affect this definition (cf. [1, Proposition 3.3]) there is no canonical choice of Morita equivalence, and different choices can result in sign changes (cf. [1, Proposition 3.4]). This problem can be addressed as follows: we showed in [1, Theorem 6.4] and [2, Sections 2 and 3] that there exists a hermitian form $\mu$ over $(A,\sigma)$, called a reference form for $(A,\sigma)$, with the property that the signature of the hermitian form $\mathfrak{m}_Q(\mu \otimes F_Q)$ over $(D_Q, \vartheta_Q)$ is nonzero at all $Q \in X_F \backslash \mathrm{Nil}[A,\sigma]$. Let $s_P \in \{-1, 1\}$ denote the sign of $\mathrm{sign}_{\widetilde{P}}\, \mathfrak{m}_P(\mu \otimes F_P)$. The $\mu$-signature of $h$ at $P$ is then defined as

$$\mathrm{sign}^{\mu}_P\, h := s_P \cdot \mathrm{sign}_{\widetilde{P}}\, \mathfrak{m}_P(h \otimes F_P). \tag{2.6}$$

This definition ensures that the use of different Morita equivalences does not change the result (cf. [1, Lemma 3.8]. The choice of a different reference form may result in $\mathrm{sign}^{\mu}_P\, h$ changing sign continuously at all $P \in X_F \backslash \mathrm{Nil}[A,\sigma]$ (cf. [2, Proposition 3.3(iii)].

Finally, if $P \in \mathrm{Nil}[A,\sigma]$, we define

$$\mathrm{sign}^{\mu}_P\, h := 0. \tag{2.7}$$

Note that by [1, Theorem 6.4], we actually have

$$P \in \mathrm{Nil}[A,\sigma] \Leftrightarrow \mathrm{sign}^{\mu}_P = 0. \tag{2.8}$$

***Remark 2.4*** Observe that if $P \in X_F \backslash \mathrm{Nil}[A,\sigma]$, then there exists $a \in \mathrm{Sym}(A,\sigma) \cap A^{\times}$ such that $\mathrm{sign}^{\mu}_P \langle a \rangle_\sigma \neq 0$. Indeed, if $h$ is such that $\mathrm{sign}^{\mu}_P\, h \neq 0$, this follows from "weak diagonalization" (cf. [4, Lemma 2.2]) and the fact that $\mathrm{sign}^{\mu}_P$ is additive.

***Remark 2.5*** The definition of $\mu$-signature implies that

$$\mathrm{sign}^{\mu}_P\, h = \mathrm{sign}^{\mu \otimes F_P}_{\widetilde{P}} (h \otimes F_P).$$

Furthermore, if $(A,\sigma) = (D_F, \vartheta_F)$ (with notation as in Section 2.1), then $\mu := \langle 1 \rangle_\sigma$ is a reference form for $(A,\sigma)$ (since $\mathrm{sign}_P \langle 1 \rangle_\sigma = 1$ for all $P \in X_F \backslash \mathrm{Nil}[A,\sigma]$) and

$$\mathrm{sign}^{\mu}_P = \mathrm{sign}_P\,.$$

**Assumption for the remainder of this article:** $(A,\sigma)$ is an $F$-algebra with involution and $\mu$ is a reference form for $(A,\sigma)$.

## 2.3 Signatures under ordered field embeddings

We recall the following consequence of [2, Lemma 4.1].

***Lemma 2.6*** *Let $F_P \subseteq L$ be a field extension with $L$ real closed. We denote by $\mathfrak{m}$ the hermitian Morita equivalence between $(A \otimes_F F_P, \sigma \otimes \mathrm{id})$ and $(D_P, \vartheta_P)$ as well as the induced isomorphism of Witt groups. Then, $\mathfrak{m}$ extends to a hermitian Morita equivalence $\mathfrak{m}'$ between $(A \otimes_F F_P \otimes_{F_P} L, \sigma \otimes \mathrm{id} \otimes \mathrm{id}) = (A \otimes_F L, \sigma \otimes \mathrm{id})$ and $(D_P \otimes_{F_P} L, \vartheta_P \otimes \mathrm{id})$ such that (denoting the induced isomorphism of Witt groups also by $\mathfrak{m}'$), the following diagram commutes:*

$$\begin{array}{ccc} W(A \otimes_F F_P, \sigma \otimes \mathrm{id}) & \xrightarrow{\quad \mathfrak{m} \quad} & W(D_P, \vartheta_P) \\ \downarrow & & \downarrow \\ W(A \otimes_F F_P \otimes_{F_P} L, \sigma \otimes \mathrm{id} \otimes \mathrm{id}) & \xrightarrow{\mathfrak{m}'} & W(D_P \otimes_{F_P} L, \vartheta_P \otimes \mathrm{id}). \end{array}$$

***Lemma 2.7*** *Let $(D, \vartheta) \in \{(F, \mathrm{id}), (F(\sqrt{-1}), -), ((-1,-1)_F, -)\}$ with $F$ real closed. Let $(F, P) \subseteq (L, Q)$ be an extension of ordered fields with $L$ real closed, and let $h$ be a nonsingular hermitian form over $(D, \vartheta)$. Then*

$$\mathrm{sign}_P\, h = \mathrm{sign}_Q(h \otimes L).$$

**Proof** Since $D$ is a division algebra, $h$ can be diagonalized with entries from $\mathrm{Sym}(D, \vartheta) = F$. Since $h$ is nonsingular and $F$ is real closed, we have $h \simeq r \times \langle 1 \rangle_\vartheta \perp s \times \langle -1 \rangle_\vartheta$, and so $\mathrm{sign}_P\, h = r - s$. Then, $h \otimes L \simeq r \times \langle 1 \rangle_{\vartheta \otimes \mathrm{id}} \perp s \times \langle -1 \rangle_{\vartheta \otimes \mathrm{id}}$, so that $\mathrm{sign}_Q(h \otimes L) = r - s$. ∎

***Lemma 2.8*** *Let $P \in X_F$ and let $\lambda : (F_P, \widetilde{P}) \to (L, Q)$ be an embedding of ordered fields with $(L, Q)$ real closed. Let $h$ be a nonsingular hermitian form over $(A, \sigma)$. Then*

$$\mathrm{sign}^\mu_P\, h = \mathrm{sign}^{\mu \otimes_\lambda L}_Q (h \otimes_\lambda L).$$

**Proof** The proof has two parts.

*Part 1:* Assume that $\lambda$ is an inclusion. Observe that by Lemma 2.6, $\mathfrak{m}(h \otimes F_P) \otimes_{F_P} L = \mathfrak{m}'(h \otimes_F F_P \otimes_{F_P} L) = \mathfrak{m}'(h \otimes L)$ in $W(D_P \otimes_{F_P} L, \vartheta_P \otimes \mathrm{id}) \cong W(D_L, \vartheta_L)$. Let $s_P$ and $s_Q \in \{-1, +1\}$ denote the sign of $\mathrm{sign}_{\widetilde{P}}\, \mathfrak{m}(\mu \otimes F_P)$ and $\mathrm{sign}_Q\, \mathfrak{m}'((\mu \otimes F_P) \otimes L)$, respectively. Observe that $s_P = s_Q$ by Lemma 2.6. It follows that:

$$\begin{aligned} \mathrm{sign}^\mu_P\, h &= s_P \cdot \mathrm{sign}_{\widetilde{P}}\, \mathfrak{m}(h \otimes F_P) \text{ by (2.6)} \\ &= s_P \cdot \mathrm{sign}_Q(\mathfrak{m}(h \otimes F_P) \otimes L) \text{ by Lemma 2.7} \\ &= s_Q \cdot \mathrm{sign}_Q\, \mathfrak{m}'((h \otimes F_P) \otimes L) \text{ by Lemma 2.6} \\ &= \mathrm{sign}^{\mu \otimes L}_Q (h \otimes L) \text{ by (2.6)}. \end{aligned}$$

*Part 2:* Returning to the general case of a morphism $\lambda : (F_P, \widetilde{P}) \to (L, Q)$, we have

$$\mathrm{sign}^\mu_P\, h = \mathrm{sign}^{\mu \otimes_\lambda L}_Q (h \otimes_\lambda L)$$

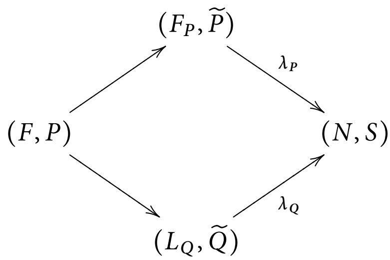
since both $h \otimes_\lambda L$ and $\mu \otimes_\lambda L$ are obtained by applying the isomorphism $\lambda : F_P \to \lambda(F_P)$, which preserves signatures by [2, Theorem 4.2], followed by the inclusion $\lambda(F_P) \subseteq L$, which also preserves signatures by the argument above. ∎

*Theorem 2.9* *Let $h$ be a hermitian form over $(A, \sigma)$ and let $P \in X_F$. Let $\lambda : (F, P) \to (L, Q)$ be an embedding of ordered fields. Then*

$$\operatorname{sign}_P^\mu h = \operatorname{sign}_Q^{\mu \otimes_\lambda L}(h \otimes_\lambda L).$$

**Proof** We may assume that $h$ is nonsingular since otherwise we can write $h \simeq h^{\mathrm{ns}} \perp h_0$, where $h^{\mathrm{ns}}$ is nonsingular and $h_0$ is a zero form of appropriate rank (cf. [3, Proposition A.3]), and thus $\operatorname{sign}_P^\mu h = \operatorname{sign}_P^\mu h^{\mathrm{ns}}$.

*Part 1:* Assume that $\lambda$ is an inclusion. Let $(L_Q, \widetilde{Q})$ be a real closure of $(L, Q)$. By [19, Exercise 1.4.3(b)] there is a real closed field $(N, S)$ and embeddings of ordered fields $\lambda_P$ and $\lambda_Q$ such that the following diagram commutes:

(2.9)



(This can also be obtained as a consequence of elimination of quantifiers for real closed fields by [9, Proposition 3.5.19].) By definition,

$$\operatorname{sign}_P^\mu h = \operatorname{sign}_{\widetilde{P}}^{\mu \otimes F_P}(h \otimes F_P)$$

and

$$\operatorname{sign}_Q^{\mu \otimes L}(h \otimes L) = \operatorname{sign}_{\widetilde{Q}}^{(\mu \otimes L) \otimes L_Q}((h \otimes L) \otimes L_Q).$$

By Lemma 2.8, we have

$$\operatorname{sign}_{\widetilde{P}}^{\mu \otimes F_P}(h \otimes F_P) = \operatorname{sign}_S^{(\mu \otimes F_P) \otimes_{\lambda_P} N}((h \otimes F_P) \otimes_{\lambda_P} N)$$

and

$$\operatorname{sign}_{\widetilde{Q}}^{\mu \otimes L \otimes L_Q}(h \otimes L \otimes L_Q) = \operatorname{sign}_S^{(\mu \otimes L \otimes L_Q) \otimes_{\lambda_Q} N}((h \otimes L \otimes L_Q) \otimes_{\lambda_Q} N).$$

The result follows, since $(h \otimes F_P) \otimes_{\lambda_P} N \cong (h \otimes L \otimes L_Q) \otimes_{\lambda_Q} N$ and $(\mu \otimes F_P) \otimes_{\lambda_P} N \cong (\mu \otimes L \otimes L_Q) \otimes_{\lambda_Q} N$ by commutativity of diagram (2.9).

*Part 2:* Assume that $\lambda$ is any embedding. We conclude as in Part 2 of the proof of Lemma 2.8. ∎

## 2.4 Positive cones

Positive cones on algebras with involution were introduced in [4] as an attempt to define a notion of ordering that corresponds to signatures of hermitian forms and that has good real-algebraic properties.

***Definition 2.10*** [4, Definition 3.1] A *prepositive cone* $\mathscr{P}$ *on* $(A,\sigma)$ is a subset $\mathscr{P}$ of $\operatorname{Sym}(A,\sigma)$ such that

(P1) $\mathscr{P} \neq \varnothing$;
(P2) $\mathscr{P} + \mathscr{P} \subseteq \mathscr{P}$;
(P3) $\sigma(a)\cdot\mathscr{P}\cdot a \subseteq \mathscr{P}$ for every $a \in A$;
(P4) $\mathscr{P}_F := \{u \in F \mid u\mathscr{P} \subseteq \mathscr{P}\}$ is an ordering on $F$;
(P5) $\mathscr{P} \cap -\mathscr{P} = \{0\}$ (we say that $\mathscr{P}$ is *proper*).

A prepositive cone $\mathscr{P}$ is *over* $P \in X_F$ if $\mathscr{P}_F = P$, and a *positive cone* is a prepositive cone that is maximal with respect to inclusion. We denote the set of all positive cones on $(A,\sigma)$ by $X_{(A,\sigma)}$.

Note that $\mathscr{P}$ is a (pre)positive cone over $P$ if and only if $-\mathscr{P}$ is a (pre)positive cone over $P$.

***Example 2.11*** The simplest non-trivial example of a positive cone is given by the set of positive semidefinite matrices in any of the following central simple algebras with involution:

$$(M_n(\mathbb{R}),t),\ (M_n(\mathbb{R}(\sqrt{-1})),{}^{-t}),\ (M_n((-1,-1)_{\mathbb{R}}),{}^{-t})$$

(see [4, Example 3.11 and Remark 4.11] for the case of $(M_n(\mathbb{R}),t)$; the exact same argument works for the other two cases, using the principal axis theorem, which also holds for matrices over quaternions by [20, Corollary 6.2]).

***Definition 2.12*** Let $S \subseteq \operatorname{Sym}(A,\sigma)$ and let $P \in X_F$. We define

$$\mathscr{C}_P(S) := \Big\{ \sum_{i=1}^{k} u_i\sigma(x_i)s_ix_i \,\Big|\, k \in \mathbb{N},\ u_i \in P,\ x_i \in A,\ s_i \in S \Big\},$$

and for $a \in \operatorname{Sym}(A,\sigma)$ and $\mathscr{P}$ a prepositive cone on $(A,\sigma)$ over $P$,

$$\mathscr{P}[a] := \Big\{ p + \sum_{i=1}^{k} u_i\sigma(x_i)ax_i \,\Big|\, p \in \mathscr{P},\ k \in \mathbb{N},\ u_i \in P,\ x_i \in A \Big\}.$$

It is clear that $\mathscr{C}_P(S)$ and $\mathscr{P}[a]$ both satisfy properties (P1), (P2), and (P3). Moreover, they are prepositive cones if and only if they are proper, i.e., satisfy (P5) (since they will both satisfy (P4) if they satisfy (P5)).

***Definition 2.13*** We define, for $P \in X_F$,

$$m_P(A,\sigma) := \max\{\operatorname{sign}^{\mu}_P\langle a\rangle_\sigma \mid a \in \operatorname{Sym}(A,\sigma)\cap A^\times\}$$

and, for $P \in X_F \backslash \operatorname{Nil}[A,\sigma]$,

$$\mathscr{M}^{\mu}_P(A,\sigma) := \{a \in \operatorname{Sym}(A,\sigma)\cap A^\times \mid \operatorname{sign}^{\mu}_P\langle a\rangle_\sigma = m_P(A,\sigma)\} \cup \{0\}.$$

Observe that if $P \in X_F\backslash\operatorname{Nil}[A,\sigma]$ then $m_P(A,\sigma) > 0$ and so $\mathscr{M}^{\mu}_P(A,\sigma) \neq \{0\}$, by Remark 2.4.

***Proposition 2.14*** *Let $P \in X_F\backslash\operatorname{Nil}[A,\sigma]$. If $A$ is an $F$-division algebra, then $\mathscr{M}^{\mu}_P(A,\sigma)$ is a prepositive cone on $(A,\sigma)$ over $P$. Otherwise, $\mathscr{C}_P(\mathscr{M}^{\mu}_P(A,\sigma))$ is a prepositive cone over $P$.*

**Proof** The first statement is [4, Example 3.13]. For the second statement, it suffices to check that $\mathscr{C}_P(\mathscr{M}_P^\mu(A,\sigma))$ is proper, since properties (P1) to (P4) are clear. Assume that this is not the case. Then, $\mathscr{C}_P(\mathscr{M}_P^\mu(A,\sigma)) = \mathrm{Sym}(A,\sigma)$ by [4, Proposition 3.5]. In particular, there are elements $a_1,\dots,a_r,b_1,\dots,b_s \in \mathscr{M}_P^\mu(A,\sigma)\setminus\{0\}$ such that $1 \in D_{(A,\sigma)}\langle a_1,\dots,a_r\rangle_\sigma$ and $-1 \in D_{(A,\sigma)}\langle b_1,\dots,b_s\rangle_\sigma$. Since both 1 and −1 are invertible, a standard argument shows that $\langle 1\rangle_\sigma \perp \varphi \simeq \langle a_1,\dots,a_r\rangle_\sigma$ and $\langle -1\rangle_\sigma \perp \psi \simeq \langle b_1,\dots,b_s\rangle_\sigma$ for some nonsingular hermitian forms $\varphi$ and $\psi$ over $(A,\sigma)$. Therefore, $\langle 1,-1\rangle_\sigma \perp \varphi \perp \psi \simeq \langle a_1,\dots,a_r,b_1,\dots,b_s\rangle_\sigma$. By "weak diagonalization", cf. [4, Lemma 2.2], we have

$$\ell \times \langle 1,-1\rangle_\sigma \perp \langle c_1,\dots,c_k\rangle_\sigma \simeq \ell \times \langle a_1,\dots,a_r,b_1,\dots,b_s\rangle_\sigma$$

for some $\ell \in \mathbb{N}$, $c_1,\dots,c_k \in \mathrm{Sym}(A,\sigma)\cap A^\times$, and $2\ell + k = \ell(r+s)$. Comparing signatures at $P$, we obtain that the right-hand side has signature $\ell(r+s)\cdot m_P(A,\sigma)$ (with $m_P(A,\sigma) > 0$ since $P \notin \mathrm{Nil}[A,\sigma]$), which is the maximal value that can be obtained by the signature of a diagonal form of dimension $\ell(r+s)$. But the left-hand side can only have signature at most $k\cdot m_P(A,\sigma)$, which is smaller than $\ell(r+s)\cdot m_P(A,\sigma)$, contradiction. ∎

## 2.5 Reduction to diagonal forms

We recall from [5, Section 4.4] that there exists a pairing $*$ of hermitian forms over $(A,\sigma)$ (first studied in detail by Garrel in [10]) such that $\varphi * \psi$ is a hermitian form over $(Z(A),\iota)$, where $\iota := \sigma|_{Z(A)}$, and which preserves orthogonal sums, isometries, and nonsingularity (cf. [5, Corollary 4.8]. Furthermore, $*$ satisfies the following "pivot property"

$$(\varphi * \psi)\otimes_{Z(A)} \chi \simeq (\chi * \psi)\otimes_{Z(A)} \varphi, \tag{2.10}$$

cf. [5, Theorem 4.9]. We also note that if $a,b \in \mathrm{Sym}(A,\sigma)\cap A^\times$, then by [10, Proposition 4.9] or [5, Lemma 4.11] we have

$$\langle a\rangle_\sigma * \langle b\rangle_\sigma \simeq \varphi_{a,b,\sigma},$$

where $\varphi_{a,b,\sigma}(x,y) := \mathrm{Trd}_A(\sigma(x)ayb)$.

Observe that by [4, Lemma 3.6] there exists an invertible element $a$ in $\mathscr{P}$.

***Proposition 2.15*** *Let $\mathscr{P}$ be a positive cone on $(A,\sigma)$ over $P \in X_F$ and let $a \in \mathscr{P}\cap A^\times$. Then, $\mathrm{sign}_P^\mu\langle a\rangle_\sigma \neq 0$ and $\mathrm{sign}_P(\langle a\rangle_\sigma * \langle a\rangle_\sigma) \neq 0$.*

**Proof** Assume for the sake of contradiction that $\mathrm{sign}_P^\mu\langle a\rangle_\sigma = 0$. By continuity of the total signature map $\mathrm{sign}_\bullet^\mu\langle a\rangle_\sigma$ (cf. [1, Theorem 7.2]), there exist $u_1,\dots,u_k \in F^\times$ such that $P$ belongs to the Harrison set $H(u_1,\dots,u_k)$ and $\mathrm{sign}_\bullet^\mu\langle a\rangle_\sigma = 0$ on $H(u_1,\dots,u_k)$. Consider the Pfister form $\langle\langle u_1,\dots,u_k\rangle\rangle := \langle 1,u_1\rangle\otimes\cdots\otimes\langle 1,u_k\rangle$. Then, we have $\mathrm{sign}_Q^\mu \langle\langle u_1,\dots,u_k\rangle\rangle\otimes\langle a\rangle_\sigma = 0$, for all $Q \in X_F$. It then follows from Pfister's local-global principle (cf. [15, Theorem 4.1] or [7, Theorem 6.5]) that there exists $n \in \mathbb{N}$ such that $2^n \times \langle\langle u_1,\dots,u_k\rangle\rangle\otimes\langle a\rangle_\sigma$ is hyperbolic. Since this form is a diagonal form with $2^{k+n}$ entries we can write it as a sum of hyperbolic planes as follows:

$$2^n \times \langle\langle u_1,\dots,u_k\rangle\rangle\otimes\langle a\rangle_\sigma \simeq 2^{k+n-1}\times\langle -a,a\rangle_\sigma.$$

In particular, $-a$ is represented by the form on the left-hand side and so $a \in -\mathscr{P}$, which contradicts that $\mathscr{P}$ is proper.

Next we prove the second statement. Since $\sigma(a^{-1})aa^{-1} = a^{-1}aa^{-1} = a^{-1}$, we have $\langle a \rangle_\sigma \simeq \langle a^{-1} \rangle_\sigma$. Therefore,

$$\langle a \rangle_\sigma * \langle a \rangle_\sigma \simeq \langle a \rangle_\sigma * \langle a^{-1} \rangle_\sigma \simeq \varphi_{a,a^{-1},\sigma} = T_{(A,\sigma_a)},$$

where $\sigma_a := \operatorname{Int}(a^{-1}) \circ \sigma$ and

$$T_{(A,\sigma_a)}(x, y) := \operatorname{Trd}_A(\sigma_a(x)y) = \operatorname{Trd}_A(a^{-1}\sigma(x)ay).$$

It then follows from [3, Equation (4.1) and Proposition 4.4(i)] that

$$\operatorname{sign}_P(\langle a \rangle_\sigma * \langle a \rangle_\sigma) = \operatorname{sign}_P T_{(A,\sigma_a)} = (\operatorname{sign}_P \sigma_a)^2 = \lambda_P{}^2 (\operatorname{sign}_P^\mu \langle a \rangle_\sigma)^2,$$

where $\lambda_P \neq 0$. (We can actually be more precise and observe that $\lambda_P \in \{1, 2\}$: If $P \in \operatorname{Nil}[A, \sigma]$ we can take $\lambda_P = 1$, cf. [3, Proposition 4.4(i)] and the observation after [3, Equation (4.2)], while if $P \notin \operatorname{Nil}[A, \sigma]$, then $\lambda_P := \deg D_P$ and so $\lambda_P = 1$ if $(D_P, \vartheta_P) \in \{(F_P, \mathrm{id}), (F_P(\sqrt{-1}), -)\}$ and $\lambda_P = 2$ if $(D_P, \vartheta_P) = ((-1, -1)_{F_P}, -)$, cf. [1, Lemma 4.5].) ∎

***Proposition 2.16*** *Let $\mathscr{P}$ be a positive cone on $(A, \sigma)$ over $P \in X_F$ and let $a \in \mathscr{P} \cap A^\times$. Let $\varphi$ be a nonsingular hermitian form over $(A, \sigma)$. Then there exist $u_1, \dots, u_r, v_1, \dots, v_s \in P \setminus \{0\}$ such that*

$$(\langle a \rangle_\sigma * \langle a \rangle_\sigma) \otimes_{Z(A)} \varphi \simeq (\langle u_1, \dots, u_r \rangle \perp \langle -v_1, \dots, -v_s \rangle) \otimes_F \langle a \rangle_\sigma$$

*and there exists a nonsingular quadratic form $q$ over $F$ such that $\operatorname{sign}_P q \neq 0$ and*

$$q \otimes_F \varphi \simeq (\langle u_1, \dots, u_r \rangle \perp \langle -v_1, \dots, -v_s \rangle) \otimes_F \langle a \rangle_\sigma.$$

**Proof** Using (2.10), we have

$$(\langle a \rangle_\sigma * \langle a \rangle_\sigma) \otimes_{Z(A)} \varphi \simeq (\varphi * \langle a \rangle_\sigma) \otimes_{Z(A)} \langle a \rangle_\sigma. \tag{2.11}$$

The forms $\langle a \rangle_\sigma * \langle a \rangle_\sigma$ and $\varphi * \langle a \rangle_\sigma$ are both nonsingular hermitian over $(Z(A), \iota)$, and are thus diagonalizable with coefficients in $\operatorname{Sym}(Z(A), \iota) \cap Z(A)^\times = F^\times$. Hence, there exist $w_1, \dots, w_t \in F^\times$ such that

$$\langle a \rangle_\sigma * \langle a \rangle_\sigma \simeq \langle w_1, \dots, w_t \rangle_\iota$$

and there exist $u_1, \dots, u_r, v_1, \dots, v_s \in P \setminus \{0\}$ such that

$$\varphi * \langle a \rangle_\sigma \simeq \langle u_1, \dots, u_r \rangle_\iota \perp \langle -v_1, \dots, -v_s \rangle_\iota. \tag{2.12}$$

The first part of the proposition follows from (2.11) and (2.12). For the second part, applying [5, Lemma 2.1] to (2.11), we obtain

$$\begin{aligned}\langle w_1, \dots, w_t \rangle \otimes_F \varphi &\simeq \langle w_1, \dots, w_t \rangle_\iota \otimes_{Z(A)} \varphi \\ &\simeq (\langle u_1, \dots, u_r \rangle_\iota \perp \langle -v_1, \dots, -v_s \rangle_\iota) \otimes_{Z(A)} \langle a \rangle_\sigma \\ &\simeq (\langle u_1, \dots, u_r \rangle \perp \langle -v_1, \dots, -v_s \rangle) \otimes_F \langle a \rangle_\sigma.\end{aligned}$$

It follows from Proposition 2.15 that $\operatorname{sign}_P\langle w_1,\dots,w_t\rangle_\iota \neq 0$, and thus that $\operatorname{sign}_P\langle w_1,\dots,w_t\rangle \neq 0$ by (2.1) and (2.2). ∎

***Lemma 2.17*** *Let $\mathscr{P}$ be a positive cone on $(A,\sigma)$ over $P \in X_F$. In an isometry of diagonal hermitian forms with coefficients in $\mathscr{P} \cap A^\times$ and $-\mathscr{P} \cap A^\times$, if there are as many elements in $\mathscr{P}$ as in $-\mathscr{P}$ on one side, it must be the same on the other side.*

**Proof** Assume that, for some $a_i, b_i, c_i, d_i \in \mathscr{P} \cap A^\times$, we have

$$\langle a_1,\dots,a_r\rangle_\sigma \perp \langle -b_1,\dots,-b_r\rangle_\sigma \simeq \langle c_1,\dots,c_s\rangle_\sigma \perp \langle -d_1,\dots,-d_t\rangle_\sigma,$$

with, for instance, $s > t$. Then, $s > r > t$ and

$$\langle c_1,\dots,c_s\rangle_\sigma \perp \langle b_1,\dots,b_t\rangle_\sigma \simeq \langle a_1,\dots,a_r\rangle_\sigma \perp \langle d_1,\dots,d_t\rangle_\sigma \perp \langle -b_{t+1},\dots,-b_r\rangle_\sigma.$$

Since the entries on the left-hand side are all in $\mathscr{P} \cap A^\times$, the entries on the right-hand side must be in $\mathscr{P}$ (they are represented by the first form, and $\mathscr{P}$ is closed under the operations presented in properties (P2) and (P3)). In particular, $-b_r \in \mathscr{P}$, which contradicts that $\mathscr{P}$ is proper. ∎

## 3 Signature maps from positive cones

Consider a positive cone $\mathscr{P}$ on $(A,\sigma)$ over $P \in X_F$. In this section, we will define the signature map $\operatorname{sign}_P^\mu W(A,\sigma) \to \mathbb{Z}$ directly out of $\mathscr{P}$ via the concept of prime m-ideals, that was introduced in [2], and that we recall now.

***Definition 3.1*** [2, Definition 5.1] We say that a pair $(I,N)$ is an *m-ideal* of $W(A,\sigma)$ if:

(1) $I$ is an ideal of $W(F)$ and $N$ is a $W(F)$-submodule of $W(A,\sigma)$;
(2) $I \cdot W(A,\sigma) \subseteq N$.

In addition, we say that the m-ideal $(I,N)$ is *prime* if $I$ is a proper prime ideal of $W(F)$, $N \neq W(A,\sigma)$ and, for every $q \in W(F)$ and every $h \in W(A,\sigma)$, $q \cdot h \in N$ implies that $q \in I$ or $h \in N$.

We recall the following proposition [2, Proposition 6.5].

***Proposition 3.2*** *Let $(I,N)$ be a prime m-ideal of $W(A,\sigma)$ such that $2 \notin I$ and $W(A,\sigma)/N$ is torsion-free. Then, there exists $P \in X_F$ such that $(I,N) = (\ker \operatorname{sign}_P, \ker \operatorname{sign}_P^\mu)$.*

We will define a prime m-ideal $(I_{\mathscr{P}}, N_{\mathscr{P}})$ such that $2 \notin I_{\mathscr{P}}$ and the quotient $W(A,\sigma)/N_{\mathscr{P}}$ is torsion-free directly from $\mathscr{P}$, thus recovering the signature map $\operatorname{sign}_P^\mu$ out of the positive cone $\mathscr{P}$.

Therefore, and since we will ultimately have $N_{\mathscr{P}} = \ker \operatorname{sign}_P^\mu$, we need to determine the nonsingular hermitian forms over $(A,\sigma)$ that are good candidates for having zero signature at $P$, and use their Witt classes as elements of $N_{\mathscr{P}}$.

***Definition 3.3*** For a hermitian form $h$ over $(A, \sigma)$, we define the following property:

$$(3.1)\qquad \begin{cases} \text{There exists a nonsingular quadratic form } q_h \text{ over } F \\ \text{such that} \\ \quad \bullet\ \operatorname{sign}_P q_h \neq 0 \text{ and} \\ \quad \bullet\ q_h \otimes h \simeq \langle a_1, \dots, a_r \rangle_\sigma \perp \langle -b_1, \dots, -b_r \rangle_\sigma \text{ for some} \\ \qquad r \in \mathbb{N} \text{ and } a_1, \dots, a_r, b_1, \dots, b_r \in \mathscr{P} \cap A^\times. \end{cases}$$

***Lemma 3.4*** *Property (3.1) is preserved under Witt equivalence (of nonsingular forms).*

**Proof** Let $h$ be a nonsingular hermitian form over $(A, \sigma)$ that satisfies Property (3.1). Let $h'$ be a nonsingular hermitian form over $(A, \sigma)$ such that

$$(3.2)\qquad h \perp H \simeq h' \perp H',$$

where $H$ and $H'$ are hyperbolic forms over $(A, \sigma)$. Let $a \in \mathscr{P} \cap A^\times$. By Proposition 2.16, there exist nonsingular quadratic forms $q', q_1, q_2$ over $F$ that all have nonzero signature at $P$ and such that $q' \otimes h'$, $q_1 \otimes H$ and $q_2 \otimes H'$ are diagonal hermitian forms of the form $\pi \otimes \langle a \rangle_\sigma \perp \nu \otimes \langle -a \rangle_\sigma$, for some diagonal quadratic forms $\pi$ and $\nu$ with coefficients in $P^\times$. Let $\chi := q_h \otimes q' \otimes q_1 \otimes q_2$. It then follows from (3.2) that

$$\chi \otimes h \perp \chi \otimes H \simeq \chi \otimes h' \perp \chi \otimes H'.$$

Observe that by taking signatures at $P$, the form $\chi \otimes H$ has as many entries in $\mathscr{P} \cap A^\times$ as in $-\mathscr{P} \cap A^\times$ since it is still a hyperbolic form and thus has signature zero. The same argument applies to $\chi \otimes H'$. We now consider

$$\chi \otimes h = (q_h \otimes q' \otimes q_1 \otimes q_2) \otimes h \simeq q' \otimes q_1 \otimes q_2 \otimes (q_h \otimes h).$$

Since the form $q_h \otimes h$ has as many entries in $\mathscr{P} \cap A^\times$ as in $-\mathscr{P} \cap A^\times$ by definition of $q_h$ (cf. Definition 3.3), the same holds for $\chi \otimes h$. It then follows from Lemma 2.17 that the form $\chi \otimes h'$ has as many entries in $\mathscr{P} \cap A^\times$ as in $-\mathscr{P} \cap A^\times$. Since $\operatorname{sign}_P \chi \neq 0$, we conclude that $h'$ satisfies Property (3.1) with $q_{h'} = \chi$. ■

***Definition 3.5*** Denoting Witt classes with square brackets, we define

$$N_{\mathscr{P}} := \{[h] \in W(A, \sigma) \mid \text{Property (3.1) holds for } h\}$$

and

$$I_{\mathscr{P}} := \{[q] \in W(F) \mid \operatorname{sign}_P q = 0\},$$

the ideal of $W(F)$ corresponding to the ordering $P$ (which is clearly generated by the classes in $W(F)$ of all elements of the form $\langle 1, -u \rangle$ for $u \in P$).

Recall again that by [4, Lemma 3.6] there exists an invertible element $a$ in $\mathscr{P}$. It follows that the form $\langle a, -a \rangle_\sigma$ satisfies Property (3.1), and in particular that $N_{\mathscr{P}} \neq \varnothing$.

***Proposition 3.6*** *The pair $(I_{\mathscr{P}}, N_{\mathscr{P}})$ is an m-ideal of $W(A, \sigma)$, and $N_{\mathscr{P}} \neq W(A, \sigma)$.*

**Proof** We have to check the following:

(1) $N_{\mathscr{P}} + N_{\mathscr{P}} \subseteq N_{\mathscr{P}}$;
(2) $W(F) \cdot N_{\mathscr{P}} \subseteq N_{\mathscr{P}}$;

(3) $I_{\mathscr{P}} \cdot W(A,\sigma) \subseteq N_{\mathscr{P}}$;
(4) $N_{\mathscr{P}} \neq W(A,\sigma)$.

We do it in order. For the verification of (1) and (2), we fix two hermitian forms $\varphi$ and $\psi$ over $(A,\sigma)$ that satisfy Property 3.1, so that $[\varphi], [\psi] \in N_{\mathscr{P}}$. Therefore, there are quadratic forms $q_\varphi, q_\psi$ over $F$ such that

$$q_\varphi \otimes \varphi \simeq \langle a_1,\dots,a_r\rangle_\sigma \perp \langle -b_1,\dots,-b_r\rangle_\sigma$$

and

$$q_\psi \otimes \psi \simeq \langle c_1,\dots,c_s\rangle_\sigma \perp \langle -d_1,\dots,-d_s\rangle_\sigma$$

for some $a_1,\dots,a_r,b_1,\dots,b_r,c_1,\dots,c_s,d_1,\dots,d_s \in \mathscr{P}\cap A^\times$, and where $\operatorname{sign}_P q_\varphi \neq 0$ and $\operatorname{sign}_P q_\psi \neq 0$.

(1) We show that $[\varphi]+[\psi] = [\varphi\perp\psi] \in N_{\mathscr{P}}$ by showing that $\varphi\perp\psi$ satisfies Property (3.1). We have

$$\begin{aligned}(q_\varphi\otimes q_\psi)\otimes(\varphi\perp\psi) \simeq q_\psi\otimes(&\langle a_1,\dots,a_r\rangle_\sigma \perp \langle -b_1,\dots,-b_r\rangle_\sigma)\perp\\ &q_\varphi\otimes(\langle c_1,\dots,c_s\rangle_\sigma\perp\langle -d_1,\dots,-d_s\rangle_\sigma).\end{aligned} \tag{3.3}$$

Writing $q_\varphi = q_+ \perp q_-$ and $q_\psi = q'_+ \perp q'_-$ with $q_+, q'_+$ positive definite at $P$ and $q_-, q'_-$ negative definite at $P$, we have that the number of entries in $\mathscr{P}\cap A^\times$ on the right-hand side of (3.3) is

$$(\dim q'_+)r + (\dim q'_-)r + (\dim q_+)s + (\dim q_-)s = (\dim q_\psi)r + (\dim q_\varphi)s,$$

and that the number of entries in $-\mathscr{P}\cap A^\times$ on the right-hand side of (3.3) is

$$(\dim q'_-)r + (\dim q'_+)r + (\dim q_-)s + (\dim q_+)s = (\dim q_\psi)r + (\dim q_\varphi)s.$$

Both are equal, so $\varphi\perp\psi$ satisfies Property (3.1).

(2) Since $W(F)$ is additively generated by classes of one-dimensional forms, it suffices to check that $[\langle u\rangle\otimes\varphi] \in N_{\mathscr{P}}$ for every $u\in F^\times$, which follows from the fact that the form $\langle u\rangle\otimes\varphi$ clearly satisfies Property (3.1).

(3) Let $\varphi$ be a nonsingular hermitian form over $(A,\sigma)$. Since $I_{\mathscr{P}}$ is additively generated by the classes of the forms $\langle 1,-u\rangle$ for $u\in P^\times$, it suffices to check that $\langle 1,-u\rangle\otimes\varphi$ satisfies Property (3.1) for every $u\in P^\times$. By Proposition 2.16, there is a nonsingular quadratic form $q_\varphi$ over $F$ such that $\operatorname{sign}_P q_\varphi \neq 0$ and

$$q_\varphi \otimes \varphi \simeq \langle a_1,\dots,a_r\rangle_\sigma \perp \langle -b_1,\dots,-b_s\rangle_\sigma$$

for some $a_1,\dots,a_r,-b_1,\dots,-b_s \in \mathscr{P}\cap A^\times$. Then

$$\begin{aligned}q_\varphi\otimes(\langle 1,-u\rangle\otimes\varphi) \simeq \langle a_1,\dots,a_r\rangle_\sigma &\perp \langle ub_1,\dots,ub_s\rangle_\sigma \perp\\ &\langle -ua_1,\dots,-ua_r\rangle_\sigma \perp \langle -b_1,\dots,-b_s\rangle_\sigma,\end{aligned}$$

which shows that $\varphi$ satisfies Property (3.1).

(4) Let $a\in\mathscr{P}\cap A^\times$. We show that $[\langle a\rangle_\sigma] \notin N_{\mathscr{P}}$. Assume that it is not the case. Then, there is a nonsingular hermitian form $h$ over $(A,\sigma)$ such that $h$ satisfies Property (3.1) and $[h] = [\langle a\rangle_\sigma]$. It follows from Lemma 3.4 that $\langle a\rangle_\sigma$ also satisfies Property (3.1),

and thus that

$$q_{\langle a\rangle_\sigma} \otimes \langle a\rangle_\sigma \simeq \langle a_1, \dots, a_r\rangle_\sigma \perp \langle -b_1, \dots, -b_r\rangle_\sigma, \tag{3.4}$$

with $a_1, \dots, a_r, b_1, \dots, b_r \in \mathscr{P} \cap A^\times$. We write

$$q_{\langle a\rangle_\sigma} \simeq \langle u_1, \dots, u_s\rangle \perp \langle -v_1, \dots, -v_t\rangle$$

with $u_1, \dots, u_s, v_1, \dots, v_t \in P^\times$. Since $\operatorname{sign}_P q_{\langle a\rangle_\sigma} \neq 0$ we have $s \neq t$. Equation (3.4) then becomes

$$\langle u_1 a, \dots, u_s a\rangle_\sigma \perp \langle -v_1 a, \dots, -v_t a\rangle_\sigma \simeq \langle a_1, \dots, a_r\rangle_\sigma \perp \langle -b_1, \dots, -b_r\rangle_\sigma.$$

By Lemma 2.17, since the right-hand side has the same number of elements in $\mathscr{P}$ as in $-\mathscr{P}$, we must have $s = t$, contradiction. ∎

***Proposition 3.7*** *The quotient $W(A,\sigma)/N_{\mathscr{P}}$ is torsion-free and $(I_{\mathscr{P}}, N_{\mathscr{P}})$ is a prime m-ideal.*

**Proof** Let $\ell[h] = [\ell \times h] \in N_{\mathscr{P}}$ for some $\ell \in \mathbb{N}$, where $h$ is a nonsingular hermitian form over $(A, \sigma)$. By Lemma 3.4, $\ell \times h$ satisfies Property (3.1). Then

$$q_{\ell\times h} \otimes (\ell \times h) \simeq \langle a_1, \dots, a_r\rangle_\sigma \perp \langle -b_1, \dots, -b_r\rangle_\sigma$$

for some $a_1, \dots, a_r, b_1, \dots, b_r \in \mathscr{P} \cap A^\times$, and so, clearly

$$(\ell \times q_{\ell\times h}) \otimes h \simeq \langle a_1, \dots, a_r\rangle_\sigma \perp \langle -b_1, \dots, -b_r\rangle_\sigma,$$

proving that $h$ satisfies Property (3.1) and thus $[h] \in N_{\mathscr{P}}$.

We now prove the second statement: Assume that $[qh] \in N_{\mathscr{P}}$ for some $[q] \in W(F)$ and $[h] \in W(A, \sigma)$. Since $I_{\mathscr{P}}$ is the kernel of $\operatorname{sign}_P : W(F) \to \mathbb{Z}$, there is $k \in \mathbb{Z}$ such that $[q] = k \bmod I_{\mathscr{P}}$. Thus (and using that $I_{\mathscr{P}} \cdot W(A,\sigma) \subseteq N_{\mathscr{P}}$), we obtain $k[h] \in N_{\mathscr{P}}$. It follows that $k = 0$ (and thus $[q] \in I_{\mathscr{P}}$), or that $[h] \in N_{\mathscr{P}}$ by the first part. ∎

***Theorem 3.8*** *We have $(I_{\mathscr{P}}, N_{\mathscr{P}}) = (\ker \operatorname{sign}_P, \ker \operatorname{sign}^\mu_P)$ and $P \notin \operatorname{Nil}[A, \sigma]$.*

**Proof** By definition, $I_{\mathscr{P}} = \ker \operatorname{sign}_P \not\ni 2$. By [2, Proposition 6.5], we obtain that $N_{\mathscr{P}} = \ker \operatorname{sign}^\mu_P$. Therefore, $P \notin \operatorname{Nil}[A, \sigma]$ since $N_{\mathscr{P}} \neq W(A, \sigma)$ by Proposition 3.6 and the equivalence in (2.8). ∎

***Corollary 3.9*** *Let $\mathscr{P}$ be a positive cone on $(A, \sigma)$ over $P \in X_F$. Then, for every $a, b \in \mathscr{P} \cap A^\times$, $\operatorname{sign}^\mu_P \langle a\rangle_\sigma = \operatorname{sign}^\mu_P \langle b\rangle_\sigma$.*

**Proof** The hermitian form $\langle a, -b\rangle_\sigma$ trivially satisfies Property (3.1). Therefore, $[\langle a, -b\rangle_\sigma] \in N_{\mathscr{P}} = \ker \operatorname{sign}^\mu_P$, so that $\operatorname{sign}^\mu_P \langle a\rangle_\sigma = \operatorname{sign}^\mu_P \langle b\rangle_\sigma$. ∎

***Remark 3.10*** We will actually show in Proposition 5.6 that $\operatorname{sign}^\mu_P \langle a\rangle_\sigma = \pm n_P(A, \sigma)$ for every $a \in \mathscr{P} \cap A^\times$.

## 4 Description of positive cones and the topology of $X_{(A,\sigma)}$

We use the previous results to describe positive cones in terms of $\operatorname{sign}^\mu_P$ and to establish some properties of the (Harrison) topology $\mathcal{T}_\sigma$ on the space of positive cones $X_{(A,\sigma)}$

of $(A, \sigma)$. Recall from [4, Section 9] that $\mathcal{T}_\sigma$ is the topology generated by the sets

$$H_\sigma(a_1, \ldots, a_k) := \{\mathscr{P} \in X_{(A,\sigma)} \mid a_1, \ldots, a_k \in \mathscr{P}\},$$

where $a_1, \ldots, a_k \in \mathrm{Sym}(A, \sigma)$.

Let $(D, \vartheta)$ be an $F$-division algebra with involution, and let $\eta$ be a reference form for $(D, \vartheta)$.

***Proposition 4.1*** *Let $P \in X_F \backslash \mathrm{Nil}[D, \vartheta]$. Then, $\mathscr{M}_P^\eta(D, \vartheta)$ is a positive cone on $(D, \vartheta)$ over $P$.*

**Proof** By [4, Example 3.13], it suffices to show that $\mathscr{M}_P^\eta(D, \vartheta)$ is maximal. Let $\mathscr{P}$ be a positive cone such that $\mathscr{M}_P^\eta(D, \vartheta) \subseteq \mathscr{P}$. By [4, Lemma 3.16], $\mathscr{P}$ is over $P$, and by Corollary 3.9, $\mathscr{P} = \mathscr{M}_P^\eta(D, \vartheta)$. ∎

***Lemma 4.2*** *Let $\mathscr{P}$ be a prepositive cone on $(D, \vartheta)$ over $P \in X_F$. Assume that $\mathrm{sign}_P^\eta \langle b \rangle_\vartheta > -m_P(D, \vartheta)$ for every $b \in \mathscr{P}$. Let $a \in \mathrm{Sym}(D, \vartheta) \cap D^\times$ be such that $\mathrm{sign}_P^\eta \langle a \rangle_\vartheta = m_P(D, \vartheta)$. Then:*

(1) *$\mathscr{P}[a]$ is a prepositive cone on $(D, \vartheta)$ over $P$.*
(2) *For every $x \in \mathscr{P}[a]$, $\mathrm{sign}_P^\eta \langle x \rangle_\vartheta > -m_P(D, \vartheta)$.*

**Proof** (1) Properties (P1)–(P4) are straightforward to check for $\mathscr{P}[a]$. We show that property (P5) holds, i.e., that $\mathscr{P}[a]$ is proper. Assume $\mathscr{P}[a]$ is not proper, and let $b \in \mathscr{P}[a] \cap -\mathscr{P}[a]$, $b \neq 0$. Then, there exist $p_1, p_2 \in \mathscr{P}$, $k, r \in \mathbb{N} \cup \{0\}$, $u_i, v_j \in P$, and $x_i, y_j \in D$ such that

$$b = p_1 + \underbrace{\sum_{i=1}^{k} u_i \vartheta(x_i) a x_i}_{\alpha} = -p_2 - \underbrace{\sum_{j=1}^{r} v_j \vartheta(y_j) a y_j}_{\beta}.$$

Observe that at least one of $\alpha$ or $\beta$ is nonzero, since $\mathscr{P}$ is proper. Furthermore, $\alpha + \beta \neq 0$. (Indeed, if $\alpha + \beta = 0$, then $\alpha = -\beta \neq 0$, contradicting that $\mathscr{M}_P^\eta(D, \vartheta)$ is proper since $a \in \mathscr{M}_P^\eta(D, \vartheta)$.) It follows that:

$$p_1 + p_2 = -\sum_{i=1}^{k} u_i \vartheta(x_i) a x_i - \sum_{j=1}^{r} v_j \vartheta(y_j) a y_j = -\alpha - \beta.$$

The right-hand side is a nonzero sum of elements that are in $-\mathscr{M}_P^\eta(D, \vartheta)$ (since $a \in \mathscr{M}_P^\eta(D, \vartheta)$), and thus belongs to $-\mathscr{M}_P^\eta(D, \vartheta)$. The left-hand side, being in $\mathscr{P}$, does not belong to $-\mathscr{M}_P^\eta(D, \vartheta)$ by hypothesis, contradiction.

(2) Let $x = p + \underbrace{\sum_{i=1}^{k} u_i \vartheta(x_i) a x_i}_{\alpha} \in \mathscr{P}[a]$. Observe that $\alpha \in \mathscr{M}_P^\eta(D, \vartheta)$. If $\alpha = 0$, then $x = p$ and $\mathrm{sign}_P^\eta \langle x \rangle_\vartheta > -m_P(D, \vartheta)$ by hypothesis. If $\alpha \neq 0$, assume that $\mathrm{sign}_P^\eta \langle x \rangle_\vartheta = -m_P(D, \vartheta)$. We have $x - \alpha = p$. The left-hand side is a nonzero element of $-\mathscr{M}_P^\eta(D, \vartheta)$ (the sum of two nonzero elements of a prepositive cone is nonzero by (P5)), while the right-hand side is not (by hypothesis), contradiction. ∎

Lemma 4.2 leads to a proof of the following theorem, which was stated in [4] and whose original proof relied on the incorrect [4, Lemma 5.5].

***Theorem 4.3*** [4, Proposition 7.1] *Let $\mathscr{P}$ be a prepositive cone on $(D,\vartheta)$ over $P \in X_F$. Then, $P \notin \mathrm{Nil}[D,\vartheta]$ and either $\mathscr{P} \subseteq \mathscr{M}^{\eta}_{P}(D,\vartheta)$, or $\mathscr{P} \subseteq -\mathscr{M}^{\eta}_{P}(D,\vartheta)$.*

*In particular, $\mathscr{M}^{\eta}_{P}(D,\vartheta)$ and $-\mathscr{M}^{\eta}_{P}(D,\vartheta)$ are the only positive cones on $(D,\vartheta)$ over $P$, i.e.,*

$$X_{(D,\vartheta)} = \{-\mathscr{M}^{\eta}_{P}(D,\vartheta), \mathscr{M}^{\eta}_{P}(D,\vartheta) \mid P \in X_F \backslash \mathrm{Nil}[D,\vartheta]\}.$$

**Proof** Since $\mathscr{P}$ is contained in a positive cone over $P$, we have $P \notin \mathrm{Nil}[D,\vartheta]$ by Theorem 3.8. We now consider two cases.

*Case 1:* There is $c \in \mathscr{P}$ such that $\mathrm{sign}^{\eta}_{P}\langle c\rangle_{\vartheta} = -m_P(D,\vartheta)$. Then, by Lemma 3.9, $\mathscr{P} \subseteq -\mathscr{M}^{\eta}_{P}(D,\vartheta)$.

*Case 2:* For every $c \in \mathscr{P}$, $\mathrm{sign}^{\eta}_{P}\langle c\rangle_{\vartheta} > -m_P(D,\vartheta)$. Then, using Lemma 4.2, we can add all elements of $\mathscr{M}^{\eta}_{P}(D,\vartheta)$ to $\mathscr{P}$ and we obtain in this way a prepositive cone $\mathscr{Q}$ containing both $\mathscr{P}$ and $\mathscr{M}^{\eta}_{P}(D,\vartheta)$. Since $\mathscr{M}^{\eta}_{P}(D,\vartheta)$ is a maximal prepositive cone (cf. Proposition 4.1), we obtain $\mathscr{Q} = \mathscr{M}^{\eta}_{P}(D,\vartheta)$ and thus $\mathscr{P} \subseteq \mathscr{M}^{\eta}_{P}(D,\vartheta)$. ■

Recall from [4, Equation (2.1)] (with $\varepsilon = 1$) that there is a hermitian Morita equivalence

$$g : \mathfrak{Herm}(M_\ell(D), \vartheta^t) \to \mathfrak{Herm}(D,\vartheta)$$

and that its inverse $g^{-1}$ sends any diagonal form $\langle a_1,\dots,a_\ell\rangle_\vartheta$ to the form $\langle \mathrm{diag}(a_1,\dots,a_\ell)\rangle_{\vartheta^t}$.

If $\mathscr{P}$ is a prepositive cone on $(D,\vartheta)$ over $P \in X_F$, we define

$$\mathrm{PSD}_\ell(\mathscr{P}) := \{B \in \mathrm{Sym}(M_\ell(D),\vartheta^t) \mid \forall X \in D^\ell \quad \vartheta(X)^t BX \in \mathscr{P}\},$$

cf. [4, Section 4.1].

***Lemma 4.4*** [4, Lemma 7.2] *We have*

$$\mathscr{C}_P(\mathscr{M}^{g^{-1}(\eta)}_{P}(M_\ell(D),\vartheta^t)) = \mathrm{PSD}_\ell(\mathscr{M}^{\eta}_{P}(D,\vartheta)).$$

**Proof** Let $B \in \mathrm{PSD}_\ell(\mathscr{M}^{\eta}_{P}(D,\vartheta))$. Then, by [4, Lemma 4.5], there is $G \in \mathrm{GL}_\ell(D)$ such that $\vartheta(G)^t BG = \mathrm{diag}(a_1,\dots,a_\ell)$ with $a_1,\dots,a_\ell \in \mathscr{M}^{\eta}_{P}(D,\vartheta)$, and we may assume that

$$\vartheta(G)^t BG = \mathrm{diag}(a_1,\dots,a_r,0,\dots,0)$$

with $a_1,\dots,a_r \in \mathscr{M}^{\eta}_{P}(D,\vartheta)\backslash\{0\}$.

Since $a_i I_\ell \in \mathscr{M}^{g^{-1}(\eta)}_{P}(M_\ell(D),\vartheta^t)$, it is now easy to represent $\vartheta(G)^t BG$, and thus $B$, as an element of $\mathscr{C}_P(\mathscr{M}^{g^{-1}(\eta)}_{P}(M_\ell(D),\vartheta^t))$, proving that

$$\mathrm{PSD}_\ell(\mathscr{M}^{\eta}_{P}(D,\vartheta)) \subseteq \mathscr{C}_P(\mathscr{M}^{g^{-1}(\eta)}_{P}(M_\ell(D),\vartheta^t)).$$

The equality follows since $\mathrm{PSD}_\ell(\mathscr{M}^{\eta}_{P}(D,\vartheta))$ is a positive cone by Theorem 4.3 and [4, Proposition 4.7], and $\mathscr{C}_P(\mathscr{M}^{g^{-1}(\eta)}_{P}(M_\ell(D),\vartheta^t))$ is a prepositive cone by Proposition 2.14. ■

Assume now that $(D,\vartheta)$ is the $F$-division algebra with involution that is Morita equivalent to $(A,\sigma)$. Observe that if $(A,\sigma)$ has at least one positive cone, we may assume that the involutions $\vartheta$ and $\sigma$ are of the same type, cf. [4, Assumption on

p. 8], and it follows from [12, Chapter I, Theorem 9.3.5] that there is a hermitian Morita equivalence $\mathfrak{m}$ between the categories of hermitian forms $\mathfrak{Herm}(A,\sigma)$ and $\mathfrak{Herm}(D,\vartheta)$. We let the reference form $\eta$ be equal to $\mathfrak{m}(\mu)$.

***Remark 4.5*** We note that [4, Lemma 7.4] is now valid with its original proof, after replacing the reference to [4, Lemma 7.2] by a reference to Lemma 4.4, since they both prove the same statement.

***Theorem 4.6*** [4, Theorem 7.5] *Let $\mathscr{P}$ be a prepositive cone on $(A,\sigma)$ over $P \in X_F$. Then either*

$$\mathscr{P} \subseteq \mathscr{C}_P(\mathscr{M}^\mu_P(A,\sigma)), \text{ or } \mathscr{P} \subseteq -\mathscr{C}_P(\mathscr{M}^\mu_P(A,\sigma)).$$

*In particular*

$$X_{(A,\sigma)} = \{-\mathscr{C}_P(\mathscr{M}^\mu_P(A,\sigma)), \mathscr{C}_P(\mathscr{M}^\mu_P(A,\sigma)) \mid P \in X_F \backslash \mathrm{Nil}[A,\sigma]\},$$

*and for each $\mathscr{P} \in X_{(A,\sigma)}$ there exists $\varepsilon \in \{-1,1\}$ such that $\mathscr{P} \cap A^\times = \varepsilon \mathscr{M}^\mu_P(A,\sigma)\backslash\{0\}$.*

**Proof** The original proof is still valid, but the references in it to the results in [4] stated after [4, Section 5] need to be replaced by the same results obtained in the current article, as follows:

| Original proof in [4]: | Corresponding statement in this article: |
|---|---|
| Proposition 6.6 | Theorem 3.8 |
| Proposition 7.1 | Theorem 4.3 |
| Lemma 7.2 | Lemma 4.4 |
| Proposition 7.1 | Theorem 4.3 |
| Lemma 7.4 | cf. Remark 4.5 |

■

***Corollary 4.7*** *Let $P \in X_F$. Then*

$$\mathscr{C}_P(\mathscr{M}^\mu_P(A,\sigma)) = \bigcup\{D_{(A,\sigma)}\langle a_1,\dots,a_k\rangle_\sigma \mid k \in \mathbb{N},\ a_1,\dots,a_k \in \mathscr{M}^\mu_P(A,\sigma)\}.$$

*In particular, if $\mathscr{P}$ is a positive cone on $(A,\sigma)$ over $P \in X_F$ such that $\mathscr{P} \cap A^\times = \mathscr{M}^\mu_P(A,\sigma)\backslash\{0\}$, then*

$$\mathscr{P} = \bigcup\{D_{(A,\sigma)}\langle a_1,\dots,a_k\rangle_\sigma \mid k \in \mathbb{N},\ a_1,\dots,a_k \in \mathscr{M}^\mu_P(A,\sigma)\}.$$

**Proof** The first statement is clear by definition of $\mathscr{C}_P$ and since $\mathscr{M}^\mu_P(A,\sigma)$ is closed under multiplication by elements of $P\backslash\{0\}$. The second statement follows immediately from Theorem 4.6. ■

***Proposition 4.8*** [4, Proposition 9.11(3)] *The topology $\mathcal{T}_\sigma$ is compact (by which me mean quasicompact).*

**Proof** A positive cone $\mathscr{P}$ is a subset of $\mathrm{Sym}(A,\sigma)$, so can be identified with a map from $\mathrm{Sym}(A,\sigma)$ to $\{0,1\}$ (with $\mathscr{P}(a) = 1$ iff $a \in \mathscr{P}$). Thus, we can view $X_{(A,\sigma)}$ as a subset of $Z := \{0,1\}^{\mathrm{Sym}(A,\sigma)}$ and the topology $\mathcal{T}_\sigma$ as the topology induced by the product topology $T$ on $Z$ of the discrete topology on $\{0,1\}$. Since $T$ is compact, it suffices to show that $X_{(A,\sigma)}$ is a closed subset of $Z$. We slightly reformulate the prepositive cone properties (P1), (P4), and (P5) in order to make it easier to check them:

(P1) $0 \in \mathscr{P}$.

(P4) $\forall u \in F\ u \in \mathscr{P}_F \vee -u \in \mathscr{P}_F$. (This reformulation is equivalent to the original (P4) since $\mathscr{P}_F$ is always a preordering, so we only need to check that it is total.)

(P5) $\forall a \in \mathrm{Sym}(A,\sigma)\setminus\{0\}\ \neg(a \in \mathscr{P} \wedge -a \in \mathscr{P})$.

We now show that the subset $S_i$ of $Z$ of subsets satisfying property (P$i$) is closed, for $i = 1,\dots,5$. The result follows since $X_{(A,\sigma)} = S_1 \cap \cdots \cap S_5$.

$$S_1 = \{\mathscr{P} \in Z \mid \mathscr{P}(0) = 1\},$$

which is closed in $T$.

$$\begin{aligned} S_2 &= \{\mathscr{P} \in Z \mid \forall a,b \in \mathrm{Sym}(A,\sigma)\ a,b \in \mathscr{P} \Rightarrow a+b \in \mathscr{P}\} \\ &= \{\mathscr{P} \in Z \mid \forall a,b \in \mathrm{Sym}(a,\sigma)\ \neg(a,b \in \mathscr{P}) \vee a+b \in \mathscr{P}\} \\ &= \{\mathscr{P} \in Z \mid \forall a,b \in \mathrm{Sym}(a,\sigma)\ a \notin \mathscr{P} \vee b \notin \mathscr{P} \vee a+b \in \mathscr{P}\} \\ &= \bigcap_{a,b\in\mathrm{Sym}(A,\sigma)} \{\mathscr{P} \in Z \mid \mathscr{P}(a) = 0 \vee \mathscr{P}(b) = 0 \vee \mathscr{P}(a+b) = 1\}, \end{aligned}$$

which is an intersection of closed sets in $T$, and therefore closed.

$$\begin{aligned} S_3 &= \{\mathscr{P} \in Z \mid \forall a \in \mathrm{Sym}(A,\sigma)\forall x \in A\ a \in \mathscr{P} \Rightarrow \sigma(x)ax \in \mathscr{P}\} \\ &= \{\mathscr{P} \in Z \mid \forall a \in \mathrm{Sym}(A,\sigma)\forall x \in A\ a \notin \mathscr{P} \vee \sigma(x)ax \in \mathscr{P}\} \\ &= \bigcap_{a\in\mathrm{Sym}(A,\sigma),\ x\in A} \{\mathscr{P} \in Z \mid \mathscr{P}(a) = 0 \vee \mathscr{P}(\sigma(x)ax) = 1\}, \end{aligned}$$

which is an intersection of closed sets in $T$, and therefore closed.

$$\begin{aligned} S_4 &= \{\mathscr{P} \in Z \mid \forall u \in F\ u \in \mathscr{P}_F \vee -u \in \mathscr{P}_F\} \\ &= \{\mathscr{P} \in Z \mid \forall u \in F\ (\forall a \in \mathscr{P}\ ua \in \mathscr{P}) \vee (\forall a \in \mathscr{P}\ -ua \in \mathscr{P})\} \\ &= \bigcap_{u\in F}\{\mathscr{P} \in Z \mid (\forall a \in \mathscr{P}\ ua \in \mathscr{P}) \vee (\forall a \in \mathscr{P}\ -ua \in \mathscr{P})\} \\ &= \bigcap_{u\in F}\{\mathscr{P} \in Z \mid \forall a \in \mathscr{P}\ ua \in \mathscr{P}\} \cup \{\mathscr{P} \in Z \mid \forall a \in \mathscr{P}\ -ua \in \mathscr{P}\} \\ &= \bigcap_{u\in F}\Big(\{\mathscr{P} \in Z \mid \forall a \in \mathrm{Sym}(A,\sigma)\ a \notin \mathscr{P} \vee ua \in \mathscr{P}\} \cup \\ &\qquad \{\mathscr{P} \in Z \mid \forall a \in \mathrm{Sym}(A,\sigma)\ a \notin \mathscr{P} \vee -ua \in \mathscr{P}\}\Big) \\ &= \bigcap_{u\in F}\Big(\bigcap_{a\in\mathrm{Sym}(A,\sigma)} \{\mathscr{P} \in Z \mid \mathscr{P}(a) = 0 \vee \mathscr{P}(ua) = 1\}\Big) \cup \\ &\qquad \Big(\bigcap_{a\in\mathrm{Sym}(A,\sigma)} \{\mathscr{P} \in Z \mid \mathscr{P}(a) = 0 \vee \mathscr{P}(-ua) = 1\}\Big), \end{aligned}$$

which is a closed set in $T$.

$$\begin{aligned} S_5 &= \{\mathscr{P} \in Z \mid \forall a \in \mathrm{Sym}(A,\sigma)\setminus\{0\}\ \neg(a \in \mathscr{P} \wedge -a \in \mathscr{P})\} \\ &= \{\mathscr{P} \in Z \mid \forall a \in \mathrm{Sym}(A,\sigma)\setminus\{0\}\ a \notin \mathscr{P} \vee -a \notin \mathscr{P}\} \\ &= \bigcap_{a\in\mathrm{Sym}(A,\sigma)\setminus\{0\}} \{\mathscr{P} \in Z \mid \mathscr{P}(a) = 0 \vee \mathscr{P}(-a) = 0\}, \end{aligned}$$

which is closed in $T$. ∎

***Proposition 4.9*** [4, Proposition 9.7(2)] *The map*

$$\pi : X_{(A,\sigma)} \to X_F,\ \mathscr{P} \mapsto \mathscr{P}_F$$

*is continuous, where $X_F$ is equipped with the usual Harrison topology.*

**Proof** The proof is the same as the proof of [4, Proposition 9.7(2)], except that we use an infinite union instead of a finite one in the final part: Let $u \in F \backslash \{0\}$. We show that $\pi^{-1}(H(u))$ is open. By definition,

$$\pi^{-1}(H(u)) = \{\mathscr{P} \in X_{(A,\sigma)} \mid u \in \mathscr{P}_F\}.$$

Observe that if $c \in \mathscr{P} \backslash \{0\}$, then $u \in \mathscr{P}_F$ if and only if $uc \in \mathscr{P}$ (indeed, $u \in \mathscr{P}_F$ or $u \in -\mathscr{P}_F$, and only one of them occurs by (P5); the first case corresponds to $uc \in \mathscr{P}$). Therefore, $\mathscr{P} \in \pi^{-1}(H(u))$ if and only if there is $c \in \mathrm{Sym}(A,\sigma)\backslash\{0\}$ such that $c \in \mathscr{P}$ and $uc \in \mathscr{P}$. Thus

$$\pi^{-1}(H(u)) = \bigcup_{c \in \mathrm{Sym}(A,\sigma)\backslash\{0\}} (H_\sigma(c) \cap H_\sigma(uc)),$$

which is open in $\mathcal{T}_\sigma$. ∎

***Corollary 4.10*** *The map $\pi$ is closed.*

**Proof** Since $X_{(A,\sigma)}$ is compact, $X_F$ is Hausdorff, and $\pi$ is continuous, the map $\pi$ is necessarily closed. ∎

***Proposition 4.11*** [4, Proposition 9.7(1)] *The map $\pi$ is open.*

**Proof** We show that $\pi(H_\sigma(a_1,\dots,a_k))$ is open for all $k \in \mathbb{N}$ and $a_1,\dots,a_k \in \mathrm{Sym}(A,\sigma)$. Letting $\widetilde{X}_F := X_F \backslash \mathrm{Nil}[A,\sigma]$, we first observe that

$$X_F \backslash \pi(H_\sigma(a_1,\dots,a_k)) = \mathrm{Nil}[A,\sigma] \cup (\widetilde{X}_F \backslash \pi(H_\sigma(a_1,\dots,a_k))) \tag{4.1}$$

since $\mathrm{Im}\,\pi \subseteq \widetilde{X}_F$, and we show

$$\widetilde{X}_F \backslash \pi(H_\sigma(a_1,\dots,a_k)) = \pi\big(X_{(A,\sigma)} \backslash (H_\sigma(a_1,\dots,a_k) \cup H_\sigma(-a_1,\dots,-a_k))\big). \tag{4.2}$$

"$\subseteq$": Let $P \in \widetilde{X}_F \backslash \pi(H_\sigma(a_1,\dots,a_k))$ and let $\mathscr{P}$ be a positive cone over $P$, so that $P = \pi(\mathscr{P}) = \pi(-\mathscr{P})$. We want to show that $\mathscr{P} \notin H_\sigma(a_1,\dots,a_k) \cup H_\sigma(-a_1,\dots,-a_k)$. If $\mathscr{P} \in H_\sigma(a_1,\dots,a_k)$, then $P \in \pi(H_\sigma(a_1,\dots,a_k))$, contradiction. If $\mathscr{P} \in H_\sigma(-a_1,\dots,-a_k)$, then $P = \pi(-\mathscr{P}) \in \pi(H_\sigma(a_1,\dots,a_k))$, contradiction again.

"$\supseteq$": Let $\mathscr{P} \in X_{(A,\sigma)} \backslash (H_\sigma(a_1,\dots,a_k) \cup H_\sigma(-a_1,\dots,-a_k))$ be over $P \in X_F$, so that $\pi(\mathscr{P}) = P$. If $P \in \pi(H_\sigma(a_1,\dots,a_k))$, then there is a positive cone $\mathscr{Q}$ over $P$ such that $\mathscr{Q} \in H_\sigma(a_1,\dots,a_k)$. In particular, $\mathscr{Q} = \mathscr{P}$ or $\mathscr{Q} = -\mathscr{P}$ by Theorem 4.6 since $\mathscr{Q}$, $\mathscr{P}$, and $-\mathscr{P}$ are all over $P$. Then, $\mathscr{P} \in H_\sigma(a_1,\dots,a_k)$ in the first case, and $\mathscr{P} \in H_\sigma(-a_1,\dots,-a_k)$ in the second case, which are both contradictions.

The right-hand side of (4.2) is $\pi$ of a closed set, so is closed by Corollary 4.10. Therefore, the left-hand side is closed, which shows that the set $\pi(H_\sigma(a_1,\dots,a_k))$ is open in $X_F$ by (4.1) and since $\mathrm{Nil}[A,\sigma]$ is clopen by [1, Corollary 6.5]. ∎

## 5 Maximum signatures and extension of positive cones

Let $P \in X_F \backslash \mathrm{Nil}[A, \sigma]$. Recall from Section 2.2 that

$$(A \otimes_F F_P, \sigma \otimes \mathrm{id}) \cong (M_{n_P}(D_P), \mathrm{Int}(\Phi_P) \circ \vartheta_P{}^t), \tag{5.1}$$

where $D_P \in \{F_P, F_P(\sqrt{-1}), (-1,-1)_{F_P}\}$, $\vartheta_P$ is the canonical involution on $D_P$, and $\Phi_P \in \mathrm{Sym}_\varepsilon(M_{n_P}(D_P), \vartheta_P{}^t) \cap M_{n_P}(D_P)^\times$. By Proposition 2.14, there exists a positive cone on $(A, \sigma)$ over $P$. Therefore, we may assume that $\varepsilon = 1$ by [4, Corollary 3.8].

We denote the integer $n_P$ that occurs in (5.1) by $n_P(A, \sigma)$ if we want to emphasize the dependence on $(A, \sigma)$.

***Proposition 5.1*** *Assume that $F$ is dense in $F_P$ (for the topology induced by the ordering $P$). Then, $m_P(A, \sigma) = n_P(A, \sigma)$.*

**Proof** Observe that by the definition of signatures, $m_P(A, \sigma) \leqslant n_P(A, \sigma)$ (cf. [3, Proposition 4.4(iii)]). For ease of notation, we assume that $(A \otimes_F F_P, \sigma \otimes \mathrm{id}) = (M_{n_P}(D_P), \mathrm{Int}(\Phi_P) \circ \vartheta_P{}^t)$. Let

$$\mathrm{PD}_{n_P}(D_P, \vartheta_P) := \{B \in \mathrm{Sym}(M_{n_P}(D_P), \vartheta_P{}^t) \mid \vartheta_P(X)^t B X > 0 \text{ in } F_P, \\ \text{for every } X \in (D_P)^{n_P} \backslash \{0\}\}.$$

Then, $\Phi_P \cdot \mathrm{PD}_{n_P}(D_P, \vartheta_P)$ is an open subset of $\mathrm{Sym}(A \otimes_F F_P, \sigma \otimes \mathrm{id})$ by [5, Lemma 2.21]. Since $F$ is dense in $F_P$, $A \otimes 1_{F_P}$ is dense in $A \otimes_F F_P$, and there is $a \in A$ such that $a \otimes 1_{F_P} \in \Phi_P \cdot \mathrm{PD}_{n_P}(D_P, \vartheta_P)$.

Let $\mathfrak{s}_P$ denote the hermitian Morita equivalence

$$\mathfrak{Herm}(M_{n_P}(D_P), \mathrm{Int}(\Phi_P) \circ \vartheta^t) \to \mathfrak{Herm}(M_{n_P}(D_P), \vartheta^t)$$

given by the scaling map $h \mapsto \Phi_P^{-1} h$. Denoting the unique ordering on $F_P$ by $\widetilde{P}$, we then have

$$\mathrm{sign}_P^\mu \langle a \rangle_\sigma = \mathrm{sign}_{\widetilde{P}}^{\mu \otimes 1} \langle a \otimes 1_{F_P} \rangle_{\sigma \otimes \mathrm{id}} = \mathrm{sign}_{\widetilde{P}}^{\mathfrak{s}_P(\mu \otimes 1)} \langle \Phi_P^{-1} \cdot a \otimes 1 \rangle_{\vartheta_P{}^t} = \pm n_P(A, \sigma),$$

where the second equality follows from [2, Theorem 4.2], the final equality holds since we are computing the signature of a positive definite $n_P \times n_P$ matrix, and where the $\pm$ is due to the reference form $\mathfrak{s}_P(\mu \otimes 1)$, which may induce a sign change in the result. Replacing $a$ by $-a$ if necessary, the conclusion follows. ∎

***Proposition 5.2*** *Let $F$ be a finitely generated extension of $\mathbb{Q}$. Then, the set of archimedean orderings on $F$ is dense in $X_F$.*

**Proof** Since $F$ is finitely generated over $\mathbb{Q}$, we can write $F = \mathbb{Q}(S)$ for some finite set $S$. By [11, Chapter IV, Theorem 8.6], $F$ has a transcendence basis $\{X_1, \dots, X_n\}$ over $\mathbb{Q}$ which is included in $S$. Thus, by the primitive element theorem, $F = \mathbb{Q}(X_1, \dots, X_n)(\alpha)$ with $\alpha$ algebraic over $\mathbb{Q}(X_1, \dots, X_n)$.

Let $m_{\bar{X}}(X)$ be the minimal polynomial of $\alpha$ over $\mathbb{Q}(\bar{X})$, where $\bar{X} = (X_1, \dots, X_n)$. Let $P \in X_F$ and let $U$ be a basic open set containing $P$. The set $U$ is of the form

$$\{Q \in X_F \mid g_1(\bar{X}, \alpha) >_Q 0, \dots, g_r(\bar{X}, \alpha) >_Q 0\},$$

where the rational functions $g_1, \dots, g_r$ can be chosen to be polynomials in $\mathbb{Q}[X_1, \dots, X_n, \alpha]$.

***Claim*** There are $y_1, \ldots, y_n, \beta \in \mathbb{R}$ such that:

(1) $\{y_1, \ldots, y_n\}$ is algebraically independent over $\mathbb{Q}$;
(2) $g_j(y_1, \ldots, y_n, \beta) > 0$ for $j = 1, \ldots, r$ (the ordering is the one from $\mathbb{R}$);
(3) $\beta$ is a root of $m_{\bar{y}}(X)$, where $\bar{y} = (y_1, \ldots, y_n)$.

We will prove the Claim in the course of the next two lemmas, but we use it now. The map $\lambda : F = \mathbb{Q}(X_1, \ldots, X_n, \alpha) \to \mathbb{R}$ defined by $\lambda(X_i) = y_i$ and $\lambda(\alpha) = \beta$ is a morphism of fields, and $Q := \lambda^{-1}(\mathbb{R}_{\geqslant 0})$ is an ordering on $F$ such that $Q \in U$. Writing $F' := \operatorname{Im} \lambda = \mathbb{Q}(y_1, \ldots, y_m, \beta)$, the map $\lambda$ gives an isomorphism of ordered fields $(F, Q) \cong (F', F' \cap \mathbb{R}_{\geqslant 0})$, which is an ordered subfield of $(\mathbb{R}, \mathbb{R}_{\geqslant 0})$ and therefore archimedean. ∎

***Lemma 5.3*** *With notation as in the proof of Proposition 5.2, let*

$$S := \{\bar{x} := (x_1, \ldots, x_n) \in \mathbb{R}^n \mid m_{\bar{x}}(X) \textit{ has a root } \alpha \textit{ such that } g_i(\bar{x}, \alpha) > 0 \textit{ for } i = 1, \ldots, r\}.$$

*Then, $S$ contains a non-empty open subset of $\mathbb{R}^n$.*

**Proof** For $e \in \{1,2\}^r$, let $g^e := g_1^{e_1} \cdots g_r^{e_r}$ and

$$N_e(\bar{x}) := \sum_{c \in \mathbb{R}:\ m_{\bar{x}}(c) = 0} \operatorname{sgn} g^e(\bar{x}, c),$$

where sgn denotes the sign function. By [19, Proposition 1.3.36], we have

$$\left|\{c \in \mathbb{R} \mid m_{\bar{x}}(c) = 0,\ g_1(\bar{x}, c) > 0, \ldots,\ g_r(\bar{x}, c) > 0\}\right| = 2^{-r} \sum_{e \in \{1,2\}^r} N_e(\bar{x}),$$

for all $\bar{x} \in \mathbb{R}^n$, and thus,

$$S = \left\{\bar{x} \in \mathbb{R}^n \,\middle|\, \sum_{e \in \{1,2\}^r} N_e(\bar{x}) \geqslant 2^r\right\}.$$

Using this description of $S$, we show that $S$ contains an open subset. Listing all the possible values of $N_e(\bar{x})$, for $e \in \{1,2\}^r$, whose sum gives a result greater than or equal to $2^r$, we see that $S$ can be expressed as a finite union of finite intersections of sets of the form

$$S_{e,\ell} := \{\bar{x} \in \mathbb{R}^n \mid N_e(\bar{x}) = \ell\},$$

for some $e \in \{1,2\}^r$ and $\ell \in \mathbb{N} \cup \{0\}$. More precisely, we write

$$S = \bigcup_{i \in I} \bigcap_{(e,\ell) \in E_i} S_{e,\ell},$$

where $I$ and each $E_i$ are finite index sets. We will show that each set $S_{e,\ell} \cap Z$ is an open subset of $\mathbb{R}^n$ (for $i \in I$ and $(e,\ell) \in E_i$), where $Z$ is an open set that will be defined below, thus proving that $S \cap Z$ is open. We will then show that $S \cap Z$ is non-empty.

We first show that $S_{e,\ell} \cap Z$ is open (this will help us decide what $Z$ should be). For $i \in I$ and $(e,\ell) \in E_i$, let $(f_{e,0}, \ldots, f_{e,t_e}) \in \mathbb{Q}(x_1, \ldots, x_n)[X]$ be the Sturm sequence of $m$ and $g^e$, and

- let $p_e := (p_{e,1}, \ldots, p_{e,t_e})$ be the sequence of coefficients of the highest degree terms of $(f_{e,1}(X), \ldots, f_{e,t_e}(X))$;

- let $\tilde{p}_e := (\tilde{p}_{e,1}, \dots, \tilde{p}_{e,t_e})$ be the sequence of coefficients of the highest degree terms of $(f_{e,1}(-X), \dots, f_{e,t_e}(-X))$.

Let $\nu(m_{\bar{x}}, g^e)$ be the number of sign changes in the sequence $p_e$ and $\tilde{\nu}(m_{\bar{x}}, g^e)$ the number of sign changes in the sequence $\tilde{p}_e$. By [8, Corollary 1.2.12], $N_e(\bar{x})$ is equal to $\tilde{\nu}(m_{\bar{x}}, g^e) - \nu(m_{\bar{x}}, g^e)$, and thus

$$S_{e,\ell} = \{\bar{x} \in \mathbb{R}^n \mid \tilde{\nu}(m_{\bar{x}}, g^e) - \nu(m_{\bar{x}}, g^e) = \ell\}.$$

Listing all the possible values of $\tilde{\nu}(m_{\bar{x}}, g^e)$ and $\nu(m_{\bar{x}}, g^e)$ whose difference gives $\ell$, we see that $S_{e,\ell}$ is a finite union of sets of the form

$$T_1 := \{\bar{x} \in \mathbb{R}^n \mid \tilde{\nu}(m_{\bar{x}}, g^e) = k_1 \wedge \nu(m_{\bar{x}}, g^e) = k_2\},$$

with $k_1, k_2 \in \mathbb{N} \cup \{0\}$. We define

$$Z := \{\bar{x} \in \mathbb{R}^n \mid \text{all } p_{e,j} \text{ and all } \tilde{p}_{e,j} \text{ are} \neq 0, \text{ for all } i \in I \text{ and } (e, \ell) \in E_i\}.$$

The idea behind the introduction of $Z$ is that, if $\bar{x} \in Z$, then the various sequences $p_e$ and $\tilde{p}_e$ never contain a zero, which makes it easier to describe their sign changes.

The set $Z$ is clearly open. We show that $T_1 \cap Z$ is open, and for this it suffices to show that if

$$T_2 := \{\bar{x} \in \mathbb{R}^n \mid \nu(m_{\bar{x}}, g^e) = k\}$$

with $k \in \mathbb{N} \cup \{0\}$, then $T_2 \cap Z$ is open (since the other condition, on $\tilde{\nu}$, can be checked in the same way).

Since, for $\bar{x} \in Z$, the coefficients of $p_e$ are all non-zero, we can express that $\bar{x}$ is in $T_2 \cap Z$ by listing all the configurations $p_{e,i} < 0$ / $p_{e,i} > 0$ that enumerate all the different ways in which $k$ sign changes can be obtained in the sequence $(p_{e,1}, \dots, p_{e,t_e})$. But this clearly defines an open subset of $\mathbb{R}^n$.

We now check that $S \cap Z$ is non-empty using some basic model theory. Let $\varphi(\omega_1, \dots, \omega_n)$ be the following first-order formula in the language of rings:

$$\exists \alpha \; m_{\bar{\omega}}(\alpha) = 0 \wedge \bigwedge_{i=1}^{r} g_i(\omega_1, \dots, \omega_n, \alpha) > 0.$$

By choice of $P \in X_F$ in the proof of Proposition 5.2, we have

$$(F, P) \models \varphi(X_1, \dots, X_n).$$

Since $\{X_1, \dots, X_n\}$ is a transcendence basis of $F$ over $\mathbb{Q}$, if

$$\{r_j(x_1, \dots, x_n)\}_{j \in J}$$

is the finite list of all polynomials that appear in the definition of the set $Z$ above (the polynomials $p_{e,j}$ and $\tilde{p}_{e,j}$), we have

$$(F, P) \models \varphi(X_1, \dots, X_n) \wedge \bigwedge_{j \in J} r_j(X_1, \dots, X_n) \neq 0.$$

Since the formula $\varphi$ is existential, it follows that:

$$(F_P, \widetilde{P}) \models \varphi(X_1, \dots, X_n) \wedge \bigwedge_{j \in J} r_j(X_1, \dots, X_n) \neq 0.$$

In particular,

$$(F_P, \widetilde{P}) \models \exists x_1, \dots, x_n\ \varphi(x_1, \dots, x_n) \wedge \bigwedge_{j \in J} r_j(x_1, \dots, x_n) \neq 0,$$

and thus, by Tarski's transfer principle (cf. [16, Corollary 11.5.4]),

$$(\mathbb{R}, \mathbb{R}_{\geqslant 0}) \models \exists x_1, \dots, x_n\ \varphi(x_1, \dots, x_n) \wedge \bigwedge_{j \in J} r_j(x_1, \dots, x_n) \neq 0.$$

The open set $S \cap Z$ is thus non-empty. ∎

***Lemma 5.4*** *The set $S$ defined in Lemma 5.3 contains a tuple $(x_1, \dots, x_n)$ of elements that is algebraically independent over $\mathbb{Q}$. In particular, the Claim in the proof of Proposition 5.2 is verified.*

**Proof** By Lemma 5.3, there exist non-empty open intervals $I_1, \dots, I_n$ of $\mathbb{R}$ such that $I_1 \times \cdots \times I_n \subseteq S$. Let $x_1 \in I_1$ be transcendental over $\mathbb{Q}$. Assume that we have obtained $x_1 \in I_1, \dots, x_k \in I_k$ with $(x_1, \dots, x_k)$ algebraically independent over $\mathbb{Q}$, for some $k < n$. Since $I_{k+1}$ is uncountable, there is $x_{k+1} \in I_{k+1}$ such that $(x_1, \dots, x_{k+1})$ is algebraically independent over $\mathbb{Q}$, and we conclude by induction. ∎

Let $L$ be the language of rings $L_r$ together with $\{\underline{\sigma}, \underline{F}, \underline{P}, \underline{\mathscr{P}}\}$, where $\underline{\sigma}$ is a new unary function symbol and $\underline{F}, \underline{P}, \underline{\mathscr{P}}$ are new unary relation symbols.

If $\mathscr{P}$ is a prepositive cone on $(A, \sigma)$ over $P \in X_F$, we denote by $\mathfrak{A}$ the $L$-structure consisting of the algebra $A$ with the obvious interpretation of the symbols of $L$: $\underline{\sigma}$ is interpreted by $\sigma$, $\underline{F}$ by $F$, $\underline{P}$ by $P$, and $\underline{\mathscr{P}}$ by $\mathscr{P}$.

***Lemma 5.5*** *Let $P \in X_F$ and let $\mathscr{P}$ be a positive cone on $(A, \sigma)$ over $P$. Assume that $P$ belongs to the closure of the set of archimedean orderings of $F$. Then, there is an elementary extension $N$ of $F$ (in the language $L_r$) and an ordering $Q$ on $N$ extending $P$ such that*

(1) *$(N, Q)$ is dense in its real closure;*
(2) *There is a positive cone $\mathscr{Q}$ on $(A \otimes_F N, \sigma \otimes \mathrm{id})$ over $Q$ such that $\mathscr{P} \otimes 1 \subseteq \mathscr{Q}$.*

**Proof** Let $\Phi$ be the collection of formulas (without parameters) in the language of ordered fields expressing that an ordered field is dense in its real closure (cf. [14, Remark 4.4] and the references mentioned there), with quantifiers relativized to $\underline{F}$ (the formulas will be interpreted in an $L$-structure such as $\mathfrak{A}$, in which case we want them to be true if and only if they are true in the interpretation of $\underline{F}$).

Let $\Delta(F)$ be the complete theory (with parameters) of $F$ in the language $L_r$ (i.e., the set of all first-order $L_r$-formulas with parameters in $F$ that are true in $F$), and where the quantifiers are relativized to $\underline{F}$.

Fix an $F$-basis $\{e_1, \dots, e_m\}$ of $A$. We define the structure constants $f_{ijk} \in F$ of $A$ with respect to this basis by:

$$e_i e_j = \sum_{k=1}^{m} f_{ijk} e_k, \text{ for } 1 \leqslant i, j \leqslant m,$$

and the constants $f_{\sigma ik}$ defining $\sigma$ by

$$\sigma(e_i) = \sum_{k=1}^{m} f_{\sigma ik} e_k, \text{ for } 1 \leqslant i \leqslant m.$$

We consider the set of $L$-formulas:

$$\begin{aligned}\Omega := \Delta(F) \cup \Phi \cup \{\underline{P} \text{ ordering on } \underline{F}\} \cup \{\underline{\mathscr{P}} \text{ is a prepositive cone over } \underline{P}\}\\ \cup \{\{e_1, \dots, e_m\} \text{ is a basis over } \underline{F}\} \cup \Big\{ e_i e_j = \sum_{r=1}^{m} f_{ijk} e_k \,\Big|\, 1 \leqslant i, j \leqslant m \Big\}\\ \cup \Big\{ \underline{\sigma}(e_i) = \sum_{k=1}^{m} f_{\sigma ik} e_k \,\Big|\, 1 \leqslant i \leqslant m \Big\}\\ \cup \{u \in \underline{P} \mid u \in P\} \cup \{a \in \underline{\mathscr{P}} \mid a \in \mathscr{P}\}.\end{aligned}$$

Let $S$ be a finite subset of $\Omega$. Thus, $S$ is included in

$$\begin{aligned}S' := \Delta(F) \cup \Phi \cup \{\underline{P} \text{ ordering on } \underline{F}\} \cup \{\underline{\mathscr{P}} \text{ is a prepositive cone over } \underline{P}\}\\ \cup \{\{e_1, \dots, e_m\} \text{ is a basis over } \underline{F}\} \cup \Big\{ e_i e_j = \sum_{r=1}^{m} f_{ijk} e_k \,\Big|\, 1 \leqslant i, j \leqslant m \Big\}\\ \cup \Big\{ \underline{\sigma}(e_i) = \sum_{k=1}^{m} f_{\sigma ik} e_k \,\Big|\, 1 \leqslant i \leqslant m \Big\}\\ \cup \{u_1 \in \underline{P}, \dots, u_k \in \underline{P}\} \cup \{a_1 \in \underline{\mathscr{P}}, \dots, a_\ell \in \underline{\mathscr{P}}\},\end{aligned}$$

for some $u_1, \dots, u_k \in P$ and $a_1, \dots, a_\ell \in \mathscr{P}$.

Consider the open set $H(u_1, \dots, u_k)$ of $X_F$ and the open set $H_\sigma(a_1, \dots, a_\ell)$ of $X_{(A,\sigma)}$. Clearly, $P \in H(u_1, \dots, u_k)$ and $\mathscr{P} \in H_\sigma(a_1, \dots, a_\ell)$. Recall that the map $\pi : X_{(A,\sigma)} \to X_F$, $\pi(\mathscr{Q}) = \mathscr{Q}_F$ is open by Proposition 4.11. Therefore, $H(u_1, \dots, u_k) \cap \pi(H_\sigma(a_1, \dots, a_\ell))$ is an open subset of $X_F$ containing $P$, and thus contains an ordering $P'$ such that $(F, P')$ is archimedean by hypothesis. Since $P' \in \pi(H_\sigma(a_1, \dots, a_\ell))$, there is a positive cone $\mathscr{P}'$ on $(A, \sigma)$ over $P'$ such that $a_1, \dots, a_\ell \in \mathscr{P}'$.

In particular, the $L$-structure $(A, \sigma, F, P', \mathscr{P}')$ is a model of $S'$. (Recall that an archimedean ordered field is dense in its real closure. This follows directly from [17, Theorem 1.1.5].) Therefore, every finite subset of $\Omega$ has a model, so that $\Omega$ has a model $\mathcal{B} = (B, \tau, N, Q, \mathscr{S})$ by the compactness theorem.

By construction, $N$ is an elementary extension of $F$, $P \subseteq Q$, $\mathscr{P} \subseteq \mathscr{S}$, and $(N, Q)$ is dense in its real closure.

To prove statement (2), we first check that $(B, \tau) \cong (A \otimes_F N, \sigma \otimes \mathrm{id}_N)$: the structure constants of $B$ with respect to $\{e_1, \dots, e_m\}$ are by construction the same as those of $A$, and therefore as those of $A \otimes_F N$. Thus, since both $B$ and $A \otimes_F N$ are algebras over $N$ of the same dimension, they are isomorphic, and the isomorphism is induced by $e_i \mapsto e_i \otimes 1$ for $1 \leqslant i \leqslant m$. Similarly, the $N$-linear maps $\tau$ and $\sigma \otimes \mathrm{id}_N$ have the same matrix with respect to $\{e_1, \dots, e_m\}$ and $\{e_1 \otimes 1, \dots, e_m \otimes 1\}$, respectively, so that the algebras with involution $(B, \tau)$ and $(A \otimes_F N, \sigma \otimes \mathrm{id}_N)$ are isomorphic via

$$\xi : B \to A \otimes_F N,\ e_i \mapsto e_i \otimes 1.$$

The set $\xi(\mathscr{S})$ is a prepositive cone on $(A \otimes_F N, \sigma \otimes \mathrm{id}_N)$ over $Q$, so is included in a positive cone $\mathscr{Q}$ on $(A \otimes_F N, \sigma \otimes \mathrm{id}_N)$ over $Q$. We have $\mathscr{P} \subseteq \mathscr{S}$, so that $\xi(\mathscr{P}) \subseteq \mathscr{Q}$, i.e., $\mathscr{P} \otimes 1 \subseteq \mathscr{Q}$. ∎

***Proposition 5.6*** [4, Proposition 6.7] *Let $\mathscr{P}$ be a positive cone on $(A,\sigma)$ over $P \in X_F$. There is $\varepsilon \in \{-1,1\}$ such that for every $a \in \mathscr{P} \cap A^\times$, $\operatorname{sign}^\mu_P \langle a \rangle_\sigma = \varepsilon n_P(A,\sigma)$. In particular, $m_P(A,\sigma) = n_P(A,\sigma)$ for every $P \in X_F \backslash \operatorname{Nil}[A,\sigma]$.*

**Proof** By Theorem 4.6, there is $\varepsilon \in \{-1,1\}$ such that $\operatorname{sign}^\mu_P \langle a \rangle_\sigma = \varepsilon m_P(A,\sigma)$ for every $a \in \mathscr{P} \cap A^\times$. Let $a \in \mathscr{P} \cap A^\times$ (cf. [4, Lemma 3.6]). Since $m_P(A,\sigma) \leqslant n_P(A,\sigma)$ (cf. [3, Proposition 4.4(iii)]) and $\operatorname{sign}^\mu_P \langle -a \rangle_\sigma = -\operatorname{sign}^\mu_P \langle a \rangle_\sigma$, we prove the result by showing that $\operatorname{sign}^\mu_P \langle a \rangle_\sigma = \pm n_P(A,\sigma)$.

Fix a basis $\mathcal{B} = \{e_i\}_{i \in I}$ of $A$ over $F$, and let $F_0$ be the field obtained by adding the following elements to $\mathbb{Q}$, all determined with respect to the basis $\mathcal{B}$: the structure constants of $A$, the elements of the matrix of $\sigma$, and the coordinates of $a$. Let $A_0$ be the $F_0$-algebra determined by these structure constants for a given basis $\mathcal{B}_0 = \{e'_i\}_{i \in I}$ (i.e., we build the free $F_0$-algebra generated by the elements $e'_i$ and quotient out by the relations determined by the structure constants), let $\sigma_0$ be the $F_0$-linear map on $A_0$ with the same matrix as $\sigma$, and let $a_0 \in A_0$ be the element with the same coordinates as $a$ (all with respect to $\mathcal{B}_0$). Let $\xi : A_0 \otimes_{F_0} F \to A$, $e'_i \otimes 1 \mapsto e_i$. Since $A_0 \otimes_{F_0} F$ and $A$ are $F$-algebras with the same structure constants with respect to $\{e'_i \otimes 1\}_{i \in I}$ and $\{e_i\}_{i \in I}$, respectively, and the linear maps $\sigma_0 \otimes 1$ and $\sigma$ have the same matrices with respect to these same bases, the map $\xi$ is an isomorphism of $F$-algebras with involution from $(A_0 \otimes_{F_0} F, \sigma_0 \otimes \mathrm{id})$ to $(A,\sigma)$, and $\xi(a_0 \otimes 1) = a$. Therefore, we can assume for simplicity that $\xi$ is the identity map, so that $(A_0 \otimes_{F_0} F, \sigma_0 \otimes \mathrm{id}) = (A,\sigma)$, $a_0 \otimes 1 = a$, and $A_0 \subseteq A$.

Let $P_0 := F_0 \cap P$ and $\mathscr{P}_0 := A_0 \cap \mathscr{P}$. Note that $a_0 \in \mathscr{P}_0$. By construction, $F_0$ is finitely generated over $\mathbb{Q}$, and $(F,P)$ is an ordered extension of $(F_0,P_0)$. By Theorem 2.9, we have, for any reference form $\mu_0$ for $(A_0,\sigma_0)$:

$$\operatorname{sign}^{\mu_0}_{P_0} \langle a_0 \rangle_{\sigma_0} = \operatorname{sign}^{\mu_0 \otimes F}_P \langle a_0 \otimes 1 \rangle_{\sigma_0 \otimes 1} = \operatorname{sign}^{\mu_0 \otimes F}_P \langle a \rangle_\sigma = \pm \operatorname{sign}^\mu_P \langle a \rangle_\sigma,$$

where the final equality holds since a change of reference forms at most changes the sign of the signature (cf. [2, Proposition 3.3(iii)]). In particular, it suffices to prove the result for $(A_0,\sigma_0)$ and $P_0$, $\mathscr{P}_0$, $a_0$. Therefore, we simply use the original notation and assume that $F$ is finitely generated over $\mathbb{Q}$.

It follows by Proposition 5.2 that $P$ is in the closure of the set of archimedean orderings on $F$. Let $N$, $Q$, and $\mathscr{Q}$ be as in Lemma 5.5. By Proposition 5.1, and using that invertible elements in a given positive cone have signature equal to $\pm m_P(A,\sigma)$ by Theorem 4.6, we know that for every $b \in \mathscr{Q} \cap (A \otimes_F N)^\times$,

$$\operatorname{sign}^{\mu \otimes N}_Q \langle b \rangle_{\sigma \otimes \mathrm{id}} = \pm n_Q(A \otimes_F N, \sigma \otimes \mathrm{id}) = \pm n_P(A,\sigma),$$

where the final equality follows from the fact that $(N,Q)$ is an ordered extension of $(F,P)$ and Lemma 5.7 below. Since $a \otimes 1 \in (\mathscr{P} \otimes 1) \cap (A \otimes_F N)^\times \subseteq \mathscr{Q} \cap (A \otimes_F N)^\times$, it follows that:

$$\operatorname{sign}^\mu_P \langle a \rangle_\sigma = \operatorname{sign}^{\mu \otimes N}_Q \langle a \otimes 1 \rangle_{\sigma \otimes \mathrm{id}} = \pm n_P(A,\sigma). \qquad \blacksquare$$

***Lemma 5.7*** *Let $P \in X_F$ and let $(L,Q)$ be an ordered field extension of $(F,P)$. Then, $n_P(A,\sigma) = n_Q(A \otimes_F L, \sigma \otimes \mathrm{id})$.*

**Proof** Recall that $n_P(A,\sigma)$ is defined via the isomorphism $A \otimes_F F_P \cong M_{n_P}(D_{F_P})$, with notation as in Sections 2.1 and 2.2. Let $(L_Q, \widetilde{Q})$ be a real closure of $(L,Q)$.

Observe that we may assume that $F_P \subseteq L_Q$, and thus that $D_{F_P} \otimes_{F_P} L_Q \cong D_{L_Q}$, because $D_{F_P} \in \{F_P, F_P(\sqrt{-1}), (-1,-1)_{F_P}\}$. Therefore,

$$A \otimes_F L_Q \cong A \otimes_F F_P \otimes_{F_P} L_Q \cong M_{n_P}(D_{F_P}) \otimes_{F_P} L_Q \cong M_{n_P}(D_{L_Q}).$$

Hence, $n_{\widetilde{Q}}(A \otimes_F L_Q, \sigma \otimes \mathrm{id}) = n_P = n_P(A,\sigma)$. The result follows since $n_Q(A \otimes_F L, \sigma \otimes \mathrm{id}) = n_{\widetilde{Q}}(A \otimes_F L_Q, \sigma \otimes \mathrm{id})$ by definition. ∎

***Theorem 5.8*** [4, Proposition 5.8] *Let $P \in X_F$, let $(L,Q)$ be an ordered field extension of $(F,P)$ and let $\mathscr{P}$ be a prepositive cone on $(A,\sigma)$ over $P$. Then, $\mathscr{P} \otimes 1_L := \{a \otimes 1_L \mid a \in \mathscr{P}\}$ is contained in a prepositive cone on $(A \otimes_F L, \sigma \otimes \mathrm{id})$ over $Q$.*

**Proof** Up to replacing $\mathscr{P}$ by a positive cone that contains it, we may assume that $\mathscr{P}$ is a positive cone. By Theorem 4.6, we have $P \in X_F \backslash \mathrm{Nil}[A,\sigma]$, and so there is $a \in \mathrm{Sym}(A,\sigma)$ such that $\mathrm{sign}^{\mu}_P \langle a \rangle_\sigma \neq 0$ (cf. Remark 2.4). Therefore, by Theorem 2.9,

$$\mathrm{sign}^{\mu \otimes L}_Q \langle a \otimes 1 \rangle_{\sigma \otimes \mathrm{id}} = \mathrm{sign}^{\mu}_P \langle a \rangle_\sigma \neq 0,$$

and $Q \notin \mathrm{Nil}[A \otimes L, \sigma \otimes \mathrm{id}]$. In particular, there are positive cones on $(A \otimes L, \sigma \otimes \mathrm{id})$ and they are described by Theorem 4.6. By Corollary 4.7,

$$\mathscr{P} = \bigcup \{D_{(A,\sigma)} \langle a_1, \ldots, a_k \rangle_\sigma \mid k \in \mathbb{N},\ a_1, \ldots, a_k \in \mathscr{M}^{\mu}_P(A,\sigma)\}.$$

Therefore (using (a) and (b) below),

$$\begin{aligned} \mathscr{P} \otimes 1_L &\subseteq \bigcup \{D_{(A \otimes L, \sigma \otimes \mathrm{id})} \langle a_1 \otimes 1, \ldots, a_k \otimes 1 \rangle_\sigma \mid k \in \mathbb{N}, a_1, \ldots, a_k \in \mathscr{M}^{\mu}_P(A,\sigma)\} \\ &\subseteq \bigcup \{D_{(A \otimes L, \sigma \otimes \mathrm{id})} \langle b_1, \ldots, b_k \rangle_\sigma \mid k \in \mathbb{N}, b_1, \ldots, b_k \in \mathscr{M}^{\mu \otimes L}_Q(A \otimes L, \sigma \otimes \mathrm{id})\} \\ &= \mathscr{C}_Q(\mathscr{M}^{\mu \otimes L}_Q(A \otimes L, \sigma \otimes \mathrm{id})), \end{aligned}$$

which is a positive cone over $Q$ by Theorem 4.6, and where:

(a) The second inclusion uses the fact that $a \in \mathscr{M}^{\mu}_P(A,\sigma)$ implies $a \otimes 1 \in \mathscr{M}^{\mu \otimes L}_Q(A \otimes L, \sigma \otimes \mathrm{id})$ which follows from the fact that $m_P(A,\sigma) = n_P(A,\sigma) = n_Q(A \otimes_F L, \sigma \otimes \mathrm{id}) = m_Q(A \otimes_F L, \sigma \otimes \mathrm{id})$ by Proposition 5.6 and Lemma 5.7.
(b) The final equality follows from Corollary 4.7. ∎

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