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Pullback measure attractors for non-autonomous stochastic lattice systems

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The aim of this article is to study the asymptotic behaviour of non-autonomous stochastic lattice systems. We first show the existence and uniqueness of a pullback measure attractor. Moreover, when deterministic external forcing terms are periodic in time, we show the pullback measure attractors are periodic. We then study the upper semicontinuity of pullback measure attractors as the noise intensity goes to zero. Pullback asymptotic compact for a family of probability measures with respect to probability distributions of the solutions is demonstrated by using uniform a priori estimates for far-field values of solutions.

Keywords: dynamical behaviour; non-autonoumous; pullback measure attractor; stochastic lattice; asymptotic compact

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1. Introduction

In this article, we study the dynamical behaviour of the non-autonomous stochastic lattice system defined on the integer set $\mathbb{Z}$: for $\tau \in \mathbb{R}$,

$$
\begin{aligned}
&du_i(t) - \nu\left(u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)\right)dt + \lambda(t)u_i(t)\,dt \\
&\quad = \left(F(t, u_i(t)) + g_i(t)\right)dt + \varepsilon\sum_{k=1}^{\infty}\left(h_{i,k}(t) + \sigma_{i,k}(t, u_i(t))\right)dW_k(t), \quad t > \tau,
\end{aligned} \tag{1.1}
$$

with initial data

$$u_i\,(\tau) = \xi_i, \tag{1.2}$$

where $u = (u_i)_{i\in\mathbb{Z}}$ is an unknown sequence, $\xi = (\xi_i)_{i\in\mathbb{Z}} \in l^2$ is given, $0 < \varepsilon \le 1$, $\nu > 0$, $\lambda(t) > 0$, $g(t) = (g_i(t))_{i\in\mathbb{Z}}$ and $h(t) = (h_{i,k}(t))_{i\in\mathbb{Z},k\in\mathbb{N}}$ are given time dependent sequence, $F, \sigma_{i,k} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are nonlinearity satisfying certain structural conditions for every $i \in \mathbb{Z}$ and $k \in \mathbb{N}$, and $(W_k)_{k\in\mathbb{N}}$ is a sequence of independent standard two-side Wiener processes on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, P)$ satisfying the usual condition.

It seems that measure attractors for autonomous stochastic equations was first studied in [15] where Schmalfuss considered the existence of measure attractors of a stochastic Navier–Stokes equation with additive noise and upper semicontinuity of measure attractors as the noise intensity goes to zero. Related equations with nonlinear noise were investigated for the existence results of measure attractors in [12–14]. The relation between measure attractor and random attractor was studied in [7, 16] for stochastic equations with additive noise. In [9], Li and Wang extended the notion of measure attractors to pullback measure attractors in order to capture the dynamic behaviours of non-autonomous stochastic differential equations.

Lattice differential equations arise naturally in a wide variety of applications where the spatial structure has a discrete character. Such systems also arise in numerical simulations when discretizing PDEs (Partial Differential Equations) defined on unbounded domains. When random influences are taken into account, stochastic lattice systems have been extensively investigated. For random attractors, we refer the readers to [1, 3–5, 8, 17] for autonomous case and [2, 21, 22] for non-autonomous case. The invariant measures or periodic measures for stochastic lattice systems have been investigated by [6, 10, 11, 19, 20].

Here we prove the existence, uniqueness, periodicity, and upper semicontinuity of pullback measure attractors for the non-autonomous stochastic lattice differential equations (1.1)–(1.2). Note that the stochastic lattice system (1.1) shares some similar property with stochastic PDEs defined on the entire space $\mathbb{R}$. The main difficulty is how to establish the asymptotic tightness for a family of probability distributions of solutions. The uniform estimates on the tails of solutions are employed to prove the asymptotic tightness.

The rest of this article is organized as follows. In §2, we recall some fundamental results on the existence, uniqueness, and periodicity of a pullback measure attractor for non-autonomous dynamical systems defined on the space of probability measures of Banach spaces. Section 3 is devoted to the existence and uniqueness of solutions to the non-autonomous stochastic lattice system (1.1)–(1.2). In §4, we derive the uniform moment estimates of solutions as $t \to \infty$. These estimates are necessary for proving the existence of absorbing sets and the pullback asymptotic compactness of the non-autonomous dynamical systems with respect to the Markov semigroup generated by (1.1)–(1.2). In the last two sections, we establish the existence, uniqueness, and periodicity of pullback measure attractors for (1.1)–(1.2) and prove the convergence of pullback measure attractors of system (1.1)–(1.2) as $\varepsilon \to 0$.

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2. Preliminaries

In this section, for the readers convenience, we recall some results regarding pull-back measure attractors for non-autonomous dynamical systems on the space of probability measures(see, e.g., [9]).

In what follows, we denote by X a separable Banach space with norm $\|\cdot\|_X$. Let $C_b(X)$ be the space of bounded continuous functions $\varphi : X \to \mathbb{R}$ endowed with the norm

$$\|\varphi\|_\infty = \sup_{x\in X} |\varphi(x)| .$$

Denote by $L_b(X)$ the space of bounded Lipschitz functions on X which consists of all functions $\varphi \in C_b(X)$ such that

$$\operatorname{Lip}(\varphi) := \sup_{x_1,x_2\in X, x_1\neq x_2} \frac{|\varphi(x_1) - \varphi(x_2)|}{\|x_1 - x_2\|_X} < \infty.$$

The space $L_b(X)$ is endowed with the norm

$$\|\varphi\|_L = \|\varphi\|_\infty + \operatorname{Lip}(\varphi) .$$

Let $\mathcal{P}(X)$ be the set of probability measures on $(X, \mathcal{B}(X))$, where $\mathcal{B}(X)$ is the Borel σ-algebra of X. Given $\varphi \in C_b(X)$ and $\mu \in \mathcal{P}(X)$, we write

$$(\varphi, \mu) = \int_X \varphi(x)\, \mu(dx).$$

Recall that a sequence $\{\mu_n\}_{n=1}^\infty \subseteq \mathcal{P}(X)$ is weakly convergent to $\mu \in \mathcal{P}(X)$ if for every $\varphi \in C_b(X)$,

$$\lim_{n\to\infty} (\varphi, \mu_n) = (\varphi, \mu) .$$

Define a metric on $\mathcal{P}(X)$ by

$$d_{\mathcal{P}(X)}(\mu_1, \mu_2) = \sup_{\substack{\varphi\in L_b(X)\\ \|\varphi\|_L\le 1}} |(\varphi, \mu_1) - (\varphi, \mu_2)| , \quad \forall\, \mu_1, \mu_2 \in \mathcal{P}(X).$$

Then $(\mathcal{P}(X), d_{\mathcal{P}(X)})$ is a polish space. Moreover, a sequence $\{\mu_n\}_{n=1}^\infty \subseteq \mathcal{P}(X)$ converges to μ in $(\mathcal{P}(X), d_{\mathcal{P}(X)})$ if and only if $\{\mu_n\}_{n=1}^\infty$ converges to μ weakly.

Given $p > 0$, let $\mathcal{P}_p(X)$ be the subset of $\mathcal{P}(X)$ defined by

$$\mathcal{P}_p(X) = \left\{ \mu \in \mathcal{P}(X) : \int_X \|x\|_X^p \mu(dx) < \infty \right\}.$$

Then $(\mathcal{P}_p(X), d_{\mathcal{P}(X)})$ is also a metric space. Without confusion, we denote $(\mathcal{P}_p(X), d_{\mathcal{P}(X)})$ by $(\mathcal{P}_p(X), d_{\mathcal{P}_p(X)})$. Given $r > 0$, denote by

$$B_{\mathcal{P}_p(X)}(r) = \left\{ \mu \in \mathcal{P}_p(X) : \left(\int_X \|x\|_X^p \mu(dx) \right)^{\frac{1}{p}} \le r \right\}.$$

Recall that the Hausdorff semi-metric between subsets of $\mathcal{P}_p(X)$ is given by

$$d_{\mathcal{P}_p(X)}(Y,Z)=\sup_{y\in Y}\inf_{z\in Z} d\left(y,z\right), \quad Y,Z\subseteq \mathcal{P}_p(X), \quad Y,Z\neq\emptyset.$$

If $\epsilon > 0$ and $B\subseteq \mathcal{P}_p\left(X\right)$, then the open ϵ-neighbourhood of B in $\mathcal{P}_p\left(X\right)$ is defined by

$$\mathcal{N}_\epsilon(B)=\left\{\mu\in\mathcal{P}_p(X): d_{\mathcal{P}_p(X)}\left(\mu,B\right)<\epsilon\right\}.$$

Definition 2.1. *A family $S=\{S(t,\tau): t\in\mathbb{R}^+,\ \tau\in\mathbb{R}\}$ of mappings from $\mathcal{P}_p(X)$ to $\mathcal{P}_p(X)$ is called a continuous non-autonomous dynamical system on $\mathcal{P}_p(X)$, if for all $\tau\in\mathbb{R}$ and $t,s\in\mathbb{R}^+$, the following conditions are satisfied:*

(a) $S(0,\tau)=I_{\mathcal{P}_p(X)}$, where $I_{\mathcal{P}_p(X)}$ is the identity operator on $\mathcal{P}_p(X)$;

(b) $S(t+s,\tau)=S(t,s+\tau)\circ S(s,\tau)$;

(c) $S(t,\tau):\mathcal{P}_p(X)\to\mathcal{P}_p(X)$ is continuous.

If, in addition, there exists a positive number T such that for every $t\in\mathbb{R}^+$ and $\tau\in\mathbb{R}$,

$$S(t,\tau+T)=S(t,\tau),$$

then S is called a continuous periodic non-autonomous dynamical system on $\mathcal{P}_p(X)$ with period T.

Definition 2.2. *A set $D\subseteq\mathcal{P}_p\left(X\right)$ is called a bounded subset if there is $r > 0$ such that $D\subseteq B_{\mathcal{P}_p(X)}(r)$.*

In the sequel, we denote by $\mathcal{D}$ a collection of some families of nonempty subsets of $\mathcal{P}_p(X)$ parametrized by $\tau\in\mathbb{R}$; that is,

$$\mathcal{D}=\left\{D=\left\{D\left(\tau\right)\subseteq\mathcal{P}_p\left(X\right): D\left(\tau\right)\neq\emptyset,\ \tau\in\mathbb{R}\right\}:\ D\text{ satisfies some conditions}\right\}.$$

Definition 2.3. *A collection $\mathcal{D}$ of some families of nonempty subsets of $\mathcal{P}_p(X)$ is said to be neighbourhood-closed if for each $D=\{D\left(\tau\right):\tau\in\mathbb{R}\}\in\mathcal{D}$, there exists a positive number ϵ depending on D such that the family*

$$\left\{B\left(\tau\right): B\left(\tau\right)\ \textit{is a nonempty subset of}\ \mathcal{N}_\epsilon(D(\tau)),\ \forall\tau\in\mathbb{R}\right\}$$

also belongs to $\mathcal{D}$.

Note that the neighbourhood closedness of $\mathcal{D}$ implies for each $D\in\mathcal{D}$,

$$\widetilde{D}=\left\{\widetilde{D}\left(\tau\right):\emptyset\neq\widetilde{D}\left(\tau\right)\subseteq D\left(\tau\right),\ \tau\in\mathbb{R}\right\}\in\mathcal{D}. \tag{2.1}$$

A collection $\mathcal{D}$ satisfying (2.1) is said to be inclusion-closed in the literature.

Definition 2.4. *A family $K=\{K\left(\tau\right):\tau\in\mathbb{R}\}\in\mathcal{D}$ is called a $\mathcal{D}$-pullback absorbing set for S if for each $\tau\in\mathbb{R}$ and every $D\in\mathcal{D}$, there exists $T=T(\tau,D)>0$ such that*

$$S\left(t,\tau-t\right)D(\tau-t)\subseteq K(\tau),\quad \textit{for all}\quad t\geq T.$$

If there exists a positive number T such that $K(\tau+T)=K(\tau)$ for every $\tau\in\mathbb{R}$, then K is said to be periodic with period T.

Definition 2.5. *The non-autonomous dynamical system S is said to be $\mathcal{D}$-pullback asymptotically compact in $\mathcal{P}_p(X)$ if for each $\tau \in \mathbb{R}$, $\{S(t_n, \tau - t_n)\mu_n\}_{n=1}^{\infty}$ has a convergent subsequence in $\mathcal{P}_p(X)$ whenever $t_n \to +\infty$ and $\mu_n \in D(\tau - t_n)$ with $D \in \mathcal{D}$.*

Definition 2.6. *A family $\mathcal{A} = \{\mathcal{A}(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback measure attractor for S if the following conditions are satisfied,*
(i) $\mathcal{A}(\tau)$ is compact in $\mathcal{P}_p(X)$ for each $\tau \in \mathbb{R}$;
(ii) $\mathcal{A}$ is invariant, that is, $S(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(\tau + t)$, for all $\tau \in \mathbb{R}$ and $t \in \mathbb{R}^+$;
(iii) $\mathcal{A}$ attracts every set in $\mathcal{D}$, that is, for each $D = \{D(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$,

$$\lim_{t\to\infty} d\left(S\left(t, \tau - t\right) D(\tau - t), \mathcal{A}(\tau)\right) = 0.$$

Definition 2.7. *A mapping $\psi : \mathbb{R} \times \mathbb{R} \to \mathcal{P}_p(X)$ is called a complete orbit of S if for every $s \in \mathbb{R}$, $t \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$, the following holds:*

$$S\left(t, s + \tau\right)\psi\left(s, \tau\right) = \psi\left(t + s, \tau\right). \tag{2.2}$$

If, in addition, there exists $D = \{D(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$ such that $\psi(t,\tau)$ belongs to $D(\tau + t)$ for every $t \in \mathbb{R}$ and $\tau \in \mathbb{R}$, then ψ is called a $\mathcal{D}$-complete orbit of S.

Definition 2.8. *A mapping $\xi : \mathbb{R} \to \mathcal{P}_p(X)$ is called a complete solution of S if for every $t \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$, the following holds:*

$$S\left(t, \tau\right)\xi\left(\tau\right) = \xi\left(t + \tau\right).$$

If, in addition, there exists $D = \{D(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$ such that $\xi(\tau)$ belongs to $D(\tau)$ for every $\tau \in \mathbb{R}$, then ξ is called a $\mathcal{D}$-complete solution of S.

Definition 2.9. *For each $D = \{D(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$ and $\tau \in \mathbb{R}$, the pullback ω-limit set of D at τ is defined by*

$$\omega\left(D, \tau\right) := \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S\left(t, \tau - t\right) D(\tau - t)},$$

that is,

$$\omega\left(D, \tau\right) = \Big\{\nu \in \mathcal{P}_p\left(X\right) : \textit{ there exist } t_n \to \infty,\ \mu_n \in D(\tau - t_n) \textit{ such that } \nu = \lim_{n\to\infty} S\left(t_n, \tau - t_n\right)\mu_n\Big\}.$$

Based on the above notation, from theorem 2.25 and proposition 3.6 in [18], we have the following criterion for the existence, uniqueness, and periodicity of $\mathcal{D}$-pullback measure attractors.

Proposition 2.10. *Let $\mathcal{D}$ be a neighbourhood-closed collection of families of subsets of $\mathcal{P}_p(X)$ and S be a continuous non-autonomous dynamical system on $\mathcal{P}_p(X)$. Then S has a unique $\mathcal{D}$-pullback measure attractor $\mathcal{A}$ in $\mathcal{P}_p(X)$ if and only if S has a* *closed $\mathcal{D}$-pullback absorbing set $K \in \mathcal{D}$ and S is $\mathcal{D}$-pullback asymptotically compact in $\mathcal{P}_p(X)$. The $\mathcal{D}$-pullback measure attractor $\mathcal{A}$ is given by, for each $\tau \in \mathbb{R}$,*

$$\begin{aligned}\mathcal{A}(\tau) = \omega(K,\tau) &= \{\psi(0,\tau) : \psi \text{ is a } \mathcal{D}\text{-complete orbit of } S\} \\ &= \{\xi(\tau) : \xi \text{ is a } \mathcal{D}\text{-complete solution of } S\}.\end{aligned}$$

If, in addition, both S and K are T-periodic for some $T > 0$, then so is the attractor $\mathcal{A}$, i.e., $\mathcal{A}(\tau) = \mathcal{A}(\tau + T)$, for all $\tau \in \mathbb{R}$.

Next, we give an abstract result for the upper semicontinuity of pullback measure attractors of a family of non-autonomous dynamical systems on $\mathcal{P}_p(X)$.

Suppose Λ is an interval of $\mathbb{R}$, and for each $\lambda \in \Lambda$, Φ_λ is a non-autonomous dynamical system on $\mathcal{P}_p(X)$. Suppose that for each $\lambda \in \Lambda$, Φ_λ has a $\mathcal{D}$-pullback measure attractor $\mathcal{A}_\lambda \in \mathcal{D}$. Assume there exists $\lambda_0 \in \Lambda$ such that for $\tau \in \mathbb{R}$ and $t \in \mathbb{R}^+$,

$$\lim_{n\to\infty} \sup_{\mu\in\mathcal{A}_{\lambda_n}} d_{\mathcal{P}_p(X)}\left(\Phi_{\lambda_n}(t,\tau)\mu, \Phi_{\lambda_0}(t,\tau)\mu\right) = 0, \tag{2.3}$$

for any $\lambda_n \to \lambda_0$.

We also assume that

$$K = \left\{K(\tau) = \bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda(\tau) : \tau \in \mathbb{R}\right\} \in \mathcal{D}. \tag{2.4}$$

We now present the upper semicontinuity of $\mathcal{A}_\lambda$ as $\lambda \to \lambda_0$.

THEOREM 2.11 *Suppose (2.3)–(2.4) hold. Then for $\tau \in \mathbb{R}$,*

$$\lim_{\lambda\to\lambda_0} d_{\mathcal{P}_p(X)}\big(\mathcal{A}_\lambda(\tau), \mathcal{A}_{\lambda_0}(\tau)\big) = 0.$$

Proof. Since $K \in \mathcal{D}$ from (2.4), for $\tau \in \mathbb{R}$ and $\eta > 0$, there exists a $T = T(\tau,\eta) > 0$ such that for all $t \geq T$,

$$d_{\mathcal{P}_p(X)}(\Phi_{\lambda_0}(t,\tau-t)K(\tau-t), \mathcal{A}_{\lambda_0}(\tau)) < \eta. \tag{2.6}$$

Now let $\mu_n \in \mathcal{A}_{\lambda_n}(\tau)$, $n \in \mathbb{N}$. Since the measure attractor $\mathcal{A}_{\lambda_n}$ is invariant under Φ_{λ_n}, there exists a $\nu_n \in \mathcal{A}_{\lambda_n}(\tau - T)$ such that

$$\mu_n = \Phi_{\lambda_n}(T,\tau-T)\nu_n. \tag{2.7}$$

By (2.3), we obtain

$$\lim_{n\to\infty} \sup_{\nu_n\in\mathcal{A}_{\lambda_n}(\tau-T)} \|\Phi_{\lambda_n}(T,\tau-T)\nu_n - \Phi_{\lambda_0}(T,\tau-T)\nu_n\|_X = 0. \tag{2.8}$$

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It follows from (2.6)–(2.8) that for large enough n,

$$\begin{aligned}
&d_{\mathcal{P}_p(X)}\big(\mathcal{A}_{\lambda_n}(\tau), \mathcal{A}_{\lambda_0}(\tau)\big) = \sup_{\mu_n \in \mathcal{A}_{\lambda_n}(\tau)} d_{\mathcal{P}_p(X)}\big(\mu_n, \mathcal{A}_{\lambda_0}(\tau)\big) \\
&\le \sup_{\mu_n \in \mathcal{A}_{\lambda_n}(\tau)} (d_{\mathcal{P}_p(X)}(\mu_n, \Phi_{\lambda_0}(T, \tau - T)\nu_n) + d_{\mathcal{P}_p(X)}(\Phi_{\lambda_0}(T, \tau - T)\nu_n, \mathcal{A}_{\lambda_0}(\tau))) \\
&\le \sup_{\nu_n \in \mathcal{A}_{\lambda_n}(\tau - T)} d_{\mathcal{P}_p(X)}(\Phi_{\lambda_n}(T, \tau - T)\nu_n, \Phi_{\lambda_0}(T, \tau - T)\nu_n) \\
&+ \sup_{\nu_n \in \mathcal{A}_{\lambda_n}(\tau - T)} d_{\mathcal{P}_p(X)}(\Phi_{\lambda_0}(T, \tau - T)\nu_n, \mathcal{A}_{\lambda_0}(\tau)) \\
&\le \sup_{\nu_n \in \mathcal{A}_{\lambda_n}(\tau - T)} d_{\mathcal{P}_p(X)}(\Phi_{\lambda_n}(T, \tau - T)\nu_n, \Phi_{\lambda_0}(T, \tau - T)\nu_n) \\
&+ d_{\mathcal{P}_p(X)}(\Phi_{\lambda_0}(T, \tau - T)K(\tau - T), \mathcal{A}_{\lambda_0}(\tau)) \\
&< 2\eta.
\end{aligned}$$

This completes the proof. □

3. Existence and uniqueness of solutions

In this section, we prove the existence and uniqueness of solutions to system (1.1)–(1.2). We first discuss the assumptions on the nonlinear drift and diffusion terms in (1.1).

Throughout this article, suppose $g, h : \mathbb{R} \to l^2$, $g(t) = (g_i(t))_{i \in \mathbb{Z}}$ and $h(t) = (h_{i,k}(t))_{i \in \mathbb{Z}, k \in \mathbb{N}}$ are both continuous in $t \in \mathbb{R}$, which implies that for every $t \in \mathbb{R}$,

$$\|g(t)\|^2 = \sum_{i \in \mathbb{Z}} |g_i(t)|^2 < \infty \quad \text{and} \quad \|h(t)\|^2 = \sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{N}} |h_{i,k}(t)|^2 < \infty, \tag{3.1}$$

where $\|\cdot\|$ is the norm of l^2. The inner product of l^2 will be denoted by $(\cdot, \cdot)$ throughout this article.

Assume that $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $F = F(t, s)$, is continuous in $(t, s) \in \mathbb{R} \times \mathbb{R}$, $\frac{\partial F(t,s)}{\partial s} \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exists a positive continuous function $\beta_0 : \mathbb{R} \to \mathbb{R}$ such that

$$F(t, 0) = 0 \quad \text{and} \quad \frac{\partial F(t, s)}{\partial s} \le -\beta_0(t), \text{ for all } t, s \in \mathbb{R}. \tag{3.2}$$

For the diffusion terms in (1.1), we assume $\sigma_{i,k} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\sigma_{i,k} = \sigma_{i,k}(t, s)$, is continuous in $(t, s) \in \mathbb{R} \times \mathbb{R}$ and globally Lipschitz in $s \in \mathbb{R}$ uniformly with respect to $i \in \mathbb{Z}$; more precisely, for every $k \in \mathbb{N}$, there exists a constant $L_k > 0$ such that for all $t, s, s^* \in \mathbb{R}$, and $i \in \mathbb{Z}$,

$$|\sigma_{i,k}(t, s) - \sigma_{i,k}(t, s^*)| \le L_k |s - s^*|, \tag{3.3}$$

where $L = (L_k)_{k \in \mathbb{N}} \in l^2$. In addition, we assume $\sigma_{i,k}(t, s)$ grows linearly in $s \in \mathbb{R}$; that is, for each $k \in \mathbb{N}$ and $i \in \mathbb{Z}$,

$$|\sigma_{i,k}(t, s)| \le \delta_{i,k}(t) + \beta_k(t)|s|, \quad \forall\ t,\ s \in \mathbb{R},\ k \in \mathbb{N} \quad \text{and} \quad i \in \mathbb{Z}, \tag{3.4}$$

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where $\delta(\cdot) = (\delta_{i,k}(\cdot))_{i\in\mathbb{Z},k\in\mathbb{N}} : \mathbb{R} \to l^2$ and $\beta(\cdot) = (\beta_k(\cdot))_{k\in\mathbb{N}} : \mathbb{R} \to l^2$ are positive continuous functions.

The following notation will be used throughout the article:

$$\|L\|^2 = \sum_{k\in\mathbb{N}} |L_k|^2,\ \|\beta(t)\|^2 = \sum_{k\in\mathbb{N}} |\beta_k(t)|^2,\ \|\delta\|^2 = \sum_{i\in\mathbb{Z}} \sum_{k\in\mathbb{N}} |\delta_{i,k}|^2.$$

For convenience, we set for all $i \in \mathbb{Z}$ and $s, t \in \mathbb{R}$,

$$f(t,s) = F(t,s) + \beta_0(t)\, s.$$

Then by (3.2) we obtain for all $s, t \in \mathbb{R}$,

$$f(t,0) = 0 \quad \text{and} \quad \frac{\partial f(t,s)}{\partial s} \le 0. \tag{3.5}$$

In addition, for $u = (u_i)_{i\in\mathbb{Z}} \in l^2$, we write $f(t,u) = (f(t,u_i))_{i\in\mathbb{Z}}$ and $\sigma_k(t,u) = (\sigma_{i,k}(t,u_i))_{i\in\mathbb{Z}}$. Since $\frac{\partial f(t,s)}{\partial s} \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ with $f(t,0) = 0$, one can verify that f is a locally Lipschitz mapping from l^2 to l^2; that is, for every bounded set E in l^2 and I in $\mathbb{R}$, there exists a constant $L_f = L_f(E, I) > 0$ such that

$$\|f(t,u_1) - f(t,u_2)\|^2 \le L_f \|u_1 - u_2\|^2, \quad \text{for all} \quad u_1, u_2 \in E \quad \text{and} \quad t \in I. \tag{3.6}$$

It follows from (3.5) that for all $t \in \mathbb{R}$ and $u_1, u_2 \in l^2$,

$$(f(t,u_1) - f(t,u_2), u_1 - u_2) \le 0. \tag{3.7}$$

Similarly, by (3.3)–(3.4), we have for all $t \in \mathbb{R}$ and u_1, u_2,

$$\sum_{k\in\mathbb{N}} \|\sigma_k(t,u_1) - \sigma_k(t,u_2)\|^2 \le \|L\|^2 \|u_1 - u_2\|^2 \tag{3.8}$$

and

$$\sum_{k\in\mathbb{N}} \|\sigma_k(t,u_1)\|^2 \le 2\|\delta(t)\|^2 + 2\|\beta(t)\|^2 \|u_1\|^2. \tag{3.9}$$

For simplicity, define linear operators $A, B, : l^2 \to l^2$ by

$$(Au)_i = -u_{i-1} + 2u_i - u_{i+1}, \quad (Bu)_i = u_{i+1} - u_i, \quad i \in \mathbb{Z},\ u = (u_i)_{i\in\mathbb{Z}} \in l^2.$$

Then, system (1.1)–(1.2) can be put into the following form in l^2 for $t > \tau$:

$$\begin{aligned} du(t) + \nu Au(t)\,dt + (\lambda(t) + \beta_0(t))u(t)\,dt &= (f(t,u(t)) + g(t))\,dt \\ &\quad + \varepsilon \sum_{k=1}^{\infty} (h_k(t) + \sigma_k(t,u(t)))dW_k(t), \end{aligned} \tag{3.10}$$

with initial condition

$$u(\tau) = \xi, \tag{3.11}$$

where $h_k(t) = (h_{i,k}(t))_{i\in\mathbb{Z}} \in l^2$ for each $k \in \mathbb{N}$.

We use $L^2_{\mathcal{F}_\tau}\left(\Omega, l^2\right)$ to denote the space of all $\mathcal{F}_\tau$-measurable, l^2-valued random variables φ with $\mathbb{E}(\|\varphi\|^2) < \infty$, where $\mathbb{E}$ means the mathematical expectation. Similar to [19], under conditions (3.1)–(3.4), we can show that for any $\xi \in L^2_{\mathcal{F}_\tau}(\Omega, l^2)$, system (3.10)–(3.11) has a unique solution, which is written as $u(t)$. In particular, $u(t)$, $t \geq \tau$, is a continuous l^2-valued $\mathcal{F}_t$-adapted stochastic process such that

$$u \in L^2\left(\Omega, C\left(\left[\tau, \tau+T\right], l^2\right)\right), \tag{3.12}$$

for every $T > 0$. To highlight the initial time and initial values, we denote by $u(t, \tau, \xi)$ the solution of (3.10)–(3.11) with initial conditions $u(\tau) = \xi \in L^2_{\mathcal{F}_\tau}(\Omega, l^2)$.

Give a subset E of $\mathcal{P}_2\left(l^2\right)$, define

$$\|E\|_{\mathcal{P}_2\left(l^2\right)} = \inf\left\{r > 0 : \sup_{\mu \in E}\left(\int_{l^2} \|z\|^2 \mu(dz)\right)^{\frac{1}{2}} \leq r\right\},$$

with the convention that $\inf \emptyset = \infty$. If E is a bounded subset of $\mathcal{P}_2\left(l^2\right)$, then $\|E\|_{\mathcal{P}_2\left(l^2\right)} < \infty$. Let $\mathcal{D}$ be the collection of families of bounded nonempty subsets of $\mathcal{P}_2\left(l^2\right)$ as given by

$$\mathcal{D} = \Big\{D = \left\{D\left(\tau\right) \subseteq \mathcal{P}_2\left(l^2\right) : \emptyset \neq D\left(\tau\right) \text{ bounded in } \mathcal{P}_2\left(l^2\right),\ \tau \in \mathbb{R}\right\} : \\ \lim_{\tau \to -\infty} e^{\gamma\tau} \left\|D\left(\tau\right)\right\|^2_{\mathcal{P}_2(l^2)} = 0\Big\},$$

where $\gamma > 0$ defined later.

Throughout this article, we assume

$$\int_{-\infty}^{\tau} e^{\gamma t}(\|g(t)\|^2 + \|h(t)\|^2 + \|\delta(t)\|^2)dt < \infty, \quad \forall \tau \in \mathbb{R}. \tag{3.13}$$

4. Uniform moment estimates

In this section, we derive uniform moment estimates of the solution of problem (3.10)–(3.11) which are necessary for establishing the existence of pullback measure attractors. In the sequel, we use $\mathcal{L}(\xi)$ to denote the distribution law of a random variable ξ. We assume that

$$\gamma = \inf_{t \in \mathbb{R}}\left\{\lambda\left(t\right) + \beta_0\left(t\right) - \frac{1}{2} - 2\left\|\beta\left(t\right)\right\|^2\right\} > 0. \tag{4.1}$$

We first discuss uniform estimates of solutions of problem (3.10)–(3.11) in $L^2(\Omega, l^2)$.

LEMMA 4.1. *Suppose (3.1)–(3.4), (3.13), and (4.1) hold. Then for every $\tau \in \mathbb{R}$ and $D = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$, there exists $T = T(\tau, D) > 0$, independent of ε,* *such that for all* $t \geq T$, $\xi \in L^2_{\mathcal{F}_{\tau-t}}(\Omega, l^2)$ *with* $\mathcal{L}(\xi) \in D(\tau - t)$, *and* $0 < \varepsilon \leq 1$, *the solution* u *of (3.10)–(3.11) satisfies*

$$\mathbb{E}\left(\|u(\tau, \tau - t, \xi)\|^2\right) + \int_{\tau-t}^{\tau} e^{-\gamma(\tau-s)}\mathbb{E}\left(\|u(s, \tau - t, \xi)\|^2\right)ds \leq R(\tau), \tag{4.2}$$

where

$$R(\tau) = M_1 + M_1 \int_{-\infty}^{\tau} e^{-\gamma(\tau-s)} \left(\|g(s)\|^2 + \|h(s)\|^2 + \|\delta(s)\|^2\right)ds,$$

with $M_1 > 0$ *being a constant independent of* τ, ε, *and* D.

Proof. By (3.10) and Ito's formula, we have for $t \geq \tau$

$$\begin{aligned} &\mathbb{E}\left(\|u(t)\|^2\right) + 2\nu \int_{\tau}^{t} \mathbb{E}(\|Bu(s)\|^2)ds + 2\int_{\tau}^{t} (\lambda(s) + \beta_0(s))\mathbb{E}(\|u(s)\|^2)ds \\ &= \mathbb{E}\left(\|\xi\|^2\right) + 2\int_{\tau}^{t} \mathbb{E}\left(u(s), f(s, u(s))\right) ds \\ &+ 2\int_{\tau}^{t} \mathbb{E}\left(u(s), g(s)\right)ds + \varepsilon^2 \sum_{k=1}^{\infty} \int_{\tau}^{t} \mathbb{E}\left(\|h_k(s) + \sigma_k(s, u(s))\|^2\right) ds. \end{aligned} \tag{4.3}$$

It should be noted that $u \in C([\tau, +\infty); L^2(\Omega; l^2))$ due to the fact that $u \in L^2(\Omega; C([\tau, \tau + T]; l^2))$ for all $T > 0$ and the Lebesgue Dominated Theorem. Thus, by (3.7), (3.9), and (4.1), we obtain from (4.3) for $t > \tau$

$$\begin{aligned} \frac{d}{dt}\mathbb{E}\left(\|u(t)\|^2\right) &\leq -\varpi(t)\mathbb{E}\left(\|u(t)\|^2\right) + 4(\|g(t)\|^2 + \|h(t)\|^2 + \|\delta(t)\|^2) \\ &\leq -2\gamma\mathbb{E}\left(\|u(t)\|^2\right) + 4(\|g(t)\|^2 + \|h(t)\|^2 + \|\delta(t)\|^2), \end{aligned} \tag{4.4}$$

where $\varpi(t) = 2(\lambda(t) + \beta_0(t)) - 1 - 4\|\beta(t)\|^2$.

Multiplying (4.4) by $e^{\gamma t}$ and then integrating the resulting inequality on $(\tau - t, \tau)$ with $t \in \mathbb{R}^+$, we obtain

$$\begin{aligned} &\mathbb{E}\left(\|u(\tau, \tau - t, \xi)\|^2\right) + \gamma \int_{\tau-t}^{\tau} e^{-\gamma(\tau-s)}\mathbb{E}\left(\|u(s, \tau - t, \xi)\|^2\right)ds \\ &\leq \mathbb{E}\left(\|\xi\|^2\right) e^{-\gamma t} + 4\int_{\tau-t}^{\tau} e^{-\gamma(\tau-s)} \left(\|g(s)\|^2 + \|h(s)\|^2 + \|\delta(s)\|^2\right)ds \\ &\leq \mathbb{E}\left(\|\xi\|^2\right) e^{-\gamma t} + 4\int_{-\infty}^{\tau} e^{-\gamma(\tau-s)} \left(\|g(s)\|^2 + \|h(s)\|^2 + \|\delta(s)\|^2\right)ds. \end{aligned} \tag{4.5}$$

Since $\mathcal{L}(\xi) \in D_1(\tau - t)$ we have

$$e^{-\gamma t}\mathbb{E}\left(\|\xi\|^2\right) \leq e^{-\gamma t}\|D(\tau - t)\|^2 \to 0, \quad \text{as } t \to \infty,$$

and hence there exists $T = T(\tau, D) > 0$ such that for all $t \geq T$,

$$e^{-\gamma t}\mathbb{E}\left(\|\xi\|^2\right) \leq \frac{4}{\gamma},$$

which along with (4.5) concludes the proof. □

Next, we derive uniform estimates on the tails of the solutions of (3.10)–(3.11) which are crucial for establishing the $\mathcal{D}$-pullback asymptotic compact in $\mathcal{P}_2(l^2)$ of the family of probability distributions of the solutions.

LEMMA 4.2. *Suppose (3.1)–(3.4), (3.13), and (4.1) hold. Then for every $\eta > 0$, $\tau \in \mathbb{R}$ and $D = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$, there exist $T = T(D, \tau, \eta)$ and $N = N(D, \tau, \eta) \in \mathbb{N}$ such that for all $0 < \varepsilon \leq 1$, $t \geq T$ and $n \geq N$, the solution u of (3.10)–(3.11) satisfies,*

$$\sum_{|i| \geq n} \mathbb{E}(|u_i(\tau, \tau - t, \xi)|^2) \leq \eta,$$

when $\xi \in L^2_{\mathcal{F}_{\tau-t}}(\Omega, l^2)$ with $\mathcal{L}(\xi) \in D(\tau - t)$.

Proof. Let $\theta : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $0 \leq \theta(s) \leq 1$ for all $s \in \mathbb{R}$ and

$$\theta(s) = 0, \quad \text{for } |s| \leq 1, \text{ and } \quad \theta(s) = 1, \text{ for } |s| \geq 2. \tag{4.6}$$

Given $n \in \mathbb{N}$, denote by $\theta_n = (\theta(\frac{i}{n}))_{i \in \mathbb{Z}}$ and $\theta_n u = (\theta(\frac{i}{n})u_i)_{i \in \mathbb{Z}}$ for $u = (u_i)_{i \in \mathbb{Z}}$. By (3.10), we obtain

$$\begin{aligned} d(\theta_n u(t)) = &-\theta_n \nu A u(t) dt - (\lambda(t) + \beta_0(t))\theta_n u(t) dt + \theta_n f(t, u(t)) dt \\ &+ \theta_n g(t) dt + \varepsilon \sum_{k=1}^{\infty} (\theta_n h_k(t) + \theta_n \sigma_k(t, u(t))) dW_k(t), \quad t > \tau. \end{aligned} \tag{4.7}$$

By (4.7), Ito's formula and taking the expectation we obtain for all $t \geq \tau$ and $\Delta t \geq 0$,

$$\begin{aligned} \mathbb{E}\left(\|\theta_n u(t + \Delta t)\|^2\right) = &\,\mathbb{E}\Big(\|\theta_n u(t)\|^2\Big) - 2\nu \int_t^{t+\Delta t} \mathbb{E}\left(Bu(s), B\left(\theta_n^2 u(s)\right)\right) ds \\ &- 2\int_t^{t+\Delta t} (\lambda(s) + \beta_0(s))\mathbb{E}\left(\theta_n u(s), \theta_n u(s)\right) ds \\ &+ 2\int_t^{t+\Delta t} \mathbb{E}\left(\theta_n u(s), \theta_n f(s, u(s))\right) ds + 2\int_t^{t+\Delta t} \mathbb{E}\left(\theta_n u(s), \theta_n g(s)\right) ds \\ &+ \varepsilon^2 \sum_{k=1}^{\infty} \int_t^{t+\Delta t} \mathbb{E}\left(\|\theta_n h_k(s) + \theta_n \sigma_k(s, u(s))\|^2\right) ds. \end{aligned} \tag{4.8}$$

For the second term on the right-hand side of (4.8), we have

$$\begin{aligned}
&-2\nu\int_t^{t+\triangle t}\mathbb{E}\left(Bu(s),B\left(\theta_n^2u(s)\right)\right)ds\\
&=-2\nu\int_t^{t+\triangle t}\mathbb{E}\left(\sum_{i\in\mathbb{Z}}(u_{i+1}-u_i)\left(\theta^2\left(\frac{i+1}{n}\right)u_{i+1}-\theta^2\left(\frac{i}{n}\right)u_i\right)\right)ds\\
&\le 4\nu\int_t^{t+\triangle t}\mathbb{E}\left(\sum_{i\in\mathbb{Z}}\left|\theta\left(\frac{i+1}{n}\right)-\theta\left(\frac{i}{n}\right)\right||u_{i+1}-u_i||u_i|\right)ds\\
&\le\frac{c}{n}\int_t^{t+\triangle t}\mathbb{E}\left(\sum_{i\in\mathbb{Z}}|u_{i+1}-u_i||u_i|\right)ds\le\frac{2c}{n}\int_t^{t+\triangle t}\mathbb{E}\left(\|u(s)\|^2\right)ds,
\end{aligned}\tag{4.9}$$

where $c>0$ depends only on θ.

For the fourth term on the right-hand side of (4.8), by (3.5), we obtain

$$2\int_t^{t+\triangle t}\mathbb{E}\left(\theta_nu(s),\theta_nf(s,u(s))\right)ds\le 0.\tag{4.10}$$

On the other hand, by Young's inequality, we get

$$2\int_t^{t+\triangle t}\mathbb{E}\left(\theta_nu(s),\theta_ng(s)\right)ds\le\int_t^{t+\triangle t}\mathbb{E}\left(\|\theta_nu(\tau)\|^2\right)d\tau+\int_t^{t+\triangle t}\sum_{|i|\ge n}g_i^2(s)ds.\tag{4.11}$$

For the last term on the right-hand side of (4.8), by (3.9), we obtain

$$\begin{aligned}
&\varepsilon^2\sum_{k=1}^{\infty}\int_t^{t+\triangle t}\mathbb{E}\left(\|\theta_nh_k+\theta_n\sigma_k(u(\tau))\|^2\right)d\tau\\
&\qquad\le 4\int_t^{t+\triangle t}\|\beta(s)\|^2\,\mathbb{E}\left(\|\theta_nu(s)\|^2\right)ds+4\int_t^{t+\triangle t}\sum_{|i|\ge n}\sum_{k=1}^{\infty}\left(h_{i,k}^2(s)+\delta_{i,k}^2(s)\right)ds.
\end{aligned}\tag{4.12}$$

We obtain from (4.1) and (4.8)–(4.12) that

$$\begin{aligned}
D^+\mathbb{E}\left(\|\theta_nu(t)\|^2\right)&\le-\left(2(\lambda(t)+\beta_0(t))-1-4\|\beta(t)\|^2\right)\mathbb{E}\left(\|\theta_nu(t)\|^2\right)\\
&\quad+\frac{2c}{n}\mathbb{E}(\|u(t)\|^2)+\sum_{|i|\ge n}g_i^2(t)+4\sum_{|i|\ge n}\sum_{k=1}^{\infty}\left(h_{i,k}^2(t)+\delta_{i,k}^2(t)\right)\\
&\le-\gamma\mathbb{E}\left(\|\theta_nu(t)\|^2\right)+\frac{2c}{n}\mathbb{E}(\|u(t)\|^2)+\sum_{|i|\ge n}g_i^2(t)+4\sum_{|i|\ge n}\sum_{k=1}^{\infty}\left(h_{i,k}^2(t)+\delta_{i,k}^2(t)\right),
\end{aligned}\tag{4.13}$$

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where D^+ is the upper right Dini derivative. Given $t \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$, integrating the above over $(\tau - t, \tau)$, we obtain

$$\mathbb{E}\left(\|\theta_n u(\tau, \tau - t, \xi)\|^2\right) \leq \mathbb{E}\left(\|\theta_n \xi\|^2\right) e^{-\gamma t} + \frac{2c}{n} \int_{\tau-t}^{\tau} e^{-\gamma(\tau-s)} \mathbb{E}\left(\|u(s, \tau-t, \xi)\|^2\right) ds$$
$$+ 4 \int_{\tau-t}^{\tau} e^{-\gamma(\tau-s)} \left(\sum_{|i|\geq n} g_i^2(s) + \sum_{|i|\geq n} \sum_{k=1}^{\infty} \left(h_{i,k}^2(s) + \delta_{i,k}^2(s)\right)\right) ds. \tag{4.14}$$

For every $\eta > 0$, $\tau \in \mathbb{R}$ and $D = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$, there exists $T_1 = T_1(D, \tau, \eta) > 0$ such that

$$\mathbb{E}\left(\|\theta_n \xi\|^2\right) e^{-\gamma t} \leq e^{-\gamma \tau} \|D(\tau - t)\|_{\mathcal{P}_2(l^2)}^2 e^{\gamma(\tau-t)} < \eta.$$

By lemma 4.1 for every $\eta > 0$, $\tau \in \mathbb{R}$ and $D = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$, we find that there exist $T_2 = T_2(D, \tau, \eta) > 0$ and $N_1 = N_1(D, \tau, \eta) \in \mathbb{N}$ such that for all $t \geq T_2$ and $n \geq N_1$,

$$\frac{2c}{n} \int_{\tau-t}^{\tau} e^{-\gamma(\tau-s)} \mathbb{E}\left(\|u(s, \tau - t, \xi)\|^2\right) ds \leq \frac{2c}{n} R(\tau) < \eta, \tag{4.15}$$

where $R(\tau)$ is given by (4.2). By (3.13), we know for every $\eta > 0$ and $\tau \in \mathbb{R}$, there exists $N_2 = N_2(\tau, \eta) \in \mathbb{N}$ such that for $n > N_2$,

$$4 \int_{\tau-t}^{\tau} e^{-\gamma(\tau-s)} \left(\sum_{|i|\geq n} g_i^2(s) + \sum_{|i|\geq n} \sum_{k=1}^{\infty} \left(h_{i,k}^2(s) + \delta_{i,k}^2(s)\right)\right) ds < \eta. \tag{4.16}$$

Since $\mathcal{L}(\xi) \in D(\tau - t)$, there exists $T_2 = T_2(D, \tau, \eta) > T_1$, such that for $t > T_2$,

$$\mathbb{E}(\|\theta_n \xi\|^2) e^{-\gamma t} \leq \|D(\tau - t)\|_{\mathcal{P}_2(l^2)}^2 e^{-\gamma t} < \eta.$$

Combining (4.14), (4.15), and (4.16), we get for every $\eta > 0$, $\tau \in \mathbb{R}$ and $D = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$, there exist $T = \max\{T_1, T_2\}$ and $N = \max\{N_1, N_2\}$ such that for all $0 < \varepsilon \leq 1$, $t \geq T$ and $n \geq N$,

$$\mathbb{E}\left(\|\theta_n u(t)\|^2\right) \leq 3\eta,$$

when $\xi \in L^2_{\mathcal{F}_{\tau-t}}(\Omega, l^2)$ with $\mathcal{L}(\xi) \in D(\tau - t)$. This completes the proof. □

5. Existence of pullback measure attractors

This section is devoted to the existence, uniqueness and periodicity of $\mathcal{D}$-pullback measure attractors of (3.10)–(3.11) in $\mathcal{P}_2(l^2)$.

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As usual, if $\phi : l^2 \to \mathbb{R}$ is a bounded Borel function, then for $r \le t$ and $\xi \in l^2$, we set

$$(p(t,r)\phi)(\xi) = \mathbb{E}(\phi(u(t,r,\xi))),$$

and

$$p(r,\xi;t,\Gamma) = (p(t,r)1_\Gamma)(\xi),$$

where $\Gamma \in \mathcal{B}(l^2)$ and 1_Γ is the characteristic function of Γ.

The following properties of $\{p(t,r)\}_{r\le t}$ are standard (see, e.g., [19]) and the proof is omitted.

LEMMA 5.1. *Suppose (3.1)–(3.4), (3.13), and (4.1) hold. Then:*

(i) The family $\{p(t,r)\}_{r\le t}$ is Feller; that is, for any $r \le t$, the function $p(t,r)\phi \in C_b(l^2)$ is bounded and continuous if so is ϕ.

(ii) For every $r \in \mathbb{R}$ and $\xi \in l^2$, the process $\{u(t,r,\xi)\}_{t\ge r}$ is an l^2-valued Markov process.

We will also investigate the periodicity of pullback measure attractors of system (3.10)–(3.11) for which we assume that all given time-dependent functions are ϖ-periodic in t for some $\varpi > 0$; that is, for all $t \in \mathbb{R}$ and $k \in \mathbb{N}$,

$$\begin{aligned} &\lambda(t+\varpi) = \lambda(t), \quad \beta_0(t+\varpi) = \beta_0(t) \quad g(t+\varpi) = g(t), \\ &f(t+\varpi,\cdot,\cdot) = f(t,\cdot,\cdot), \quad h_k(t+\varpi) = h_k(t), \quad \sigma_k(t+\varpi,\cdot,\cdot) = \sigma_k(t,\cdot,\cdot). \end{aligned} \tag{5.1}$$

By the similar argument as that of lemma 4.1 in [10], we get the following lemma.

LEMMA 5.2. *Suppose (3.1)–(3.4), (3.13), (4.1), and (5.1) hold. Then we have the family $\{p(t,r)\}_{r\le t}$ is ϖ-periodic; that is, for all $t \ge r$,*

$$p(r,\xi;t,\cdot) = p(r+\varpi,\xi;t+\varpi,\cdot), \quad \forall \xi \in l^2.$$

Given $t \ge r$ and $\mu \in \mathcal{P}(l^2)$, define

$$p_*(t,r)\mu(\cdot) = \int_{l^2} p(r,\xi;t,\cdot)\mu(d\xi). \tag{5.2}$$

Then $p_*(t,r) : \mathcal{P}(l^2) \to \mathcal{P}(l^2)$ is the dual operator of $p(t,r)$. By (3.12), we find that for all $t \ge r$, $p_*(t,r)$ maps $\mathcal{P}_2(l^2)$ to $\mathcal{P}_2(l^2)$.

We now define a non-autonomous dynamical system $S(t,\tau)$, $t \ge \tau$, for the family of operators $p_*(t,\tau)$. Given $t \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$, let $S(t,\tau) : \mathcal{P}_2(l^2) \to \mathcal{P}_2(l^2)$ be the map given by

$$S(t,\tau)\mu = p_*(\tau+t,\tau)\mu, \quad \forall\ \mu \in \mathcal{P}_2(l^2).$$

LEMMA 5.3. *Suppose (3.1)–(3.4), (3.13), and (4.1) hold. Then $S(t,\tau)$, $t \ge \tau$, is a continuous non-autonomous dynamical system in $\mathcal{P}_2(l^2)$ generated by (3.10)–(3.11); more precisely, $S(t,\tau) : \mathcal{P}_2(l^2) \to \mathcal{P}_2(l^2)$ satisfies the following conditions*

(a) $S(0,\tau) = I_{\mathcal{P}_2(l^2)}$, for all $\tau \in \mathbb{R}$;

(b) $S(s+t,\tau) = S(t,\tau+s) \circ S(s,\tau)$, for any $\tau \in \mathbb{R}$ and $t,s \in \mathbb{R}^+$;

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(c) $S(t,\tau):\mathcal{P}_2(l^2)\to\mathcal{P}_2(l^2)$ *is continuous, for every* $\tau\in\mathbb{R}$ *and* $t\in\mathbb{R}^+$.

Proof. Note that (a) follows from the definition of S, and (b) follows the Markov property of the solutions of (3.10)–(3.11).

We now prove (c). Suppose $\mu_n\to\mu$ in $\mathcal{P}_2(l^2)$. We will show $S(t,\tau)\mu_n\to S(t,\tau)\mu$ in $\mathcal{P}_2(l^2)$ for every $\tau\in\mathbb{R}$ and $t\in\mathbb{R}^+$. Let $\varphi\in C_b(l^2)$. By lemma 5.1, we have $p(\tau+t,\tau)\varphi\in C_b(l^2)$ for all $\tau\in\mathbb{R}$ and $t\in\mathbb{R}^+$, and hence

$$\begin{aligned}&\lim_{n\to\infty}\left(\varphi,S\left(t,\tau\right)\mu_n\right)=\lim_{n\to\infty}\left(\varphi,p_*\left(\tau+t,\tau\right)\mu_n\right)\\&=\lim_{n\to\infty}\left(p(\tau+t,\tau)\varphi,\mu_n\right)=\left(p(\tau+t,\tau)\varphi,\mu\right)\\&=\left(\varphi,p_*\left(\tau+t,\tau\right)\mu\right)=\left(\varphi,S\left(t,\tau\right)\mu\right),\end{aligned}\tag{5.3}$$

as desired. □

By lemma 4.1, we obtain a $\mathcal{D}$-pullback absorbing set for S as stated below.

LEMMA 5.4. *Suppose (3.1)–(3.4), (3.13), and (4.1) hold. Given* $\tau\in\mathbb{R}$, *denote by*

$$K(\tau)=B_{\mathcal{P}_2(l^2)}\left(L_1^{\frac{1}{2}}(\tau)\right),\tag{5.4}$$

where

$$L_1(\tau)=M_1+M_1\int_{-\infty}^{\tau}e^{-\gamma(\tau-s)}\left(\|g\left(s\right)\|^2+\|h\left(s\right)\|^2+\|\delta(s)\|^2\right)ds,\tag{5.5}$$

and $M_1>0$ *is the same constant as in lemma 4.1, independent of* τ *and* ε. *Then* $K=\{K(\tau):\ \tau\in\mathbb{R}\}\in\mathcal{D}$ *is a closed* $\mathcal{D}$*-pullback absorbing set of* S.

Proof. By (5.4) and lemma 4.1, we see that for every $\tau\in\mathbb{R}$ and $D=\{D(t):t\in\mathbb{R}\}\in\mathcal{D}$, there exists $T=T(\tau,D)>0$, independent of ε, such that for all $t\geq T$ and $0<\varepsilon\leq 1$, S satisfies

$$S\left(t,\tau-t\right)D(\tau-t)\subseteq K(\tau).\tag{5.6}$$

We now prove $K=\{K(\tau):\ \tau\in\mathbb{R}\}\in\mathcal{D}$. By (5.4), (5.5), and (3.13), we have

$$\begin{aligned}e^{\gamma\tau}\|K(\tau)\|^2_{\mathcal{P}_2(l^2)}&=e^{\gamma\tau}L_1(\tau)\\&=e^{\gamma\tau}M_1+M_1\int_{-\infty}^{\tau}e^{\gamma s}\left(\|g\left(s\right)\|^2+\|h\left(s\right)\|^2+\|\delta(s)\|^2\right)ds\to 0,\\&\times\text{ as }\tau\to-\infty,\end{aligned}$$

and hence $K=\{K(\tau):\ \tau\in\mathbb{R}\}\in\mathcal{D}$, which along with (5.6) concludes the proof. □

We now present the $\mathcal{D}$-pullback asymptotically compact of S associated with (3.10)–(3.11).

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Lemma 5.5. *If (3.1)–(3.4), (3.13), and (4.1) hold, then S is $\mathcal{D}$-pullback asymptotically compact in $\mathcal{P}_2\left(l^2\right)$; that is, for every $\tau \in \mathbb{R}$, $\{S\left(t_n, \tau - t_n\right)\mu_n\}_{n=1}^{\infty}$ has a convergent subsequence in $\mathcal{P}_2\left(l^2\right)$ whenever $t_n \to +\infty$ and $\mu_n \in D(\tau - t_n)$ with $D \in \mathcal{D}$.*

Proof. To complete the proof, by Prohorov theorem, we need to prove that for each $\tau \in \mathbb{R}$, the sequence $\{\mathcal{L}(u(\tau, \tau - t_n, \xi_n))\}_{n=1}^{\infty}$ is tight. It follows from lemma 4.1 that for $\tau \in \mathbb{R}$ and $D = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$ there exists a $N_1 = N_1(\tau, D) \in \mathbb{N}$ such that for all $\xi_n \in L^2_{\mathcal{F}_{\tau - t_n}}(\Omega, l^2)$ with $\mathcal{L}(\xi_n) \in D(\tau - t_n)$ and $n > N_1$,

$$\mathbb{E}\left(\|u(\tau, \tau - t_n, \xi_n)\|^2\right) \leq M, \tag{5.7}$$

where $M > 0$ is a constant depending on τ, but independent of ε and D. By Chebyshev's inequality, we obtain from (5.7) that for all $\xi_n \in L^2_{\mathcal{F}_{\tau - t_n}}(\Omega, l^2)$ with $\mathcal{L}(\xi_n) \in D(\tau - t_n)$ and $n > N_1$,

$$P\left(\|u(\tau, \tau - t_n, \xi_n)\|^2 > R_1\right) \leq \frac{M}{R_1^2} \to 0 \quad \text{as} \quad R_1 \to \infty.$$

Hence for every $\tau \in \mathbb{R}$, $\eta > 0$ and $m \in \mathbb{N}$, there exists $R_2 = R_2(\tau, \eta, m) > 0$ such that for all $\xi_n \in L^2_{\mathcal{F}_{\tau - t_n}}(\Omega, l^2)$ with $\mathcal{L}(\xi_n) \in D(\tau - t_n)$ and $n > N_1$,

$$P\left\{\|u(\tau, \tau - t_n, \xi_n)\|^2 > R_2\right\} < \frac{\eta}{2^{m+1}}. \tag{5.8}$$

By lemma 4.2, we infer that for each $\tau \in \mathbb{R}$, $D = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$, $\eta > 0$ and $m \in \mathbb{N}$, there exist an integer $n_m = n_m(\tau, D, \eta, m)$ and $H_m = H_m(\tau, D, \eta, m) > N_1$ such that for all $\xi_n \in L^2_{\mathcal{F}_{\tau - t_n}}(\Omega, l^2)$ with $\mathcal{L}(\xi_n) \in D(\tau - t_n)$ and $n \geq H_m$,

$$\mathbb{E}\left(\sum_{|i| > n_m} \left|u_i\left(\tau, \tau - t_n, \xi_n\right)\right|^2\right) < \frac{\eta}{2^{2m+1}},$$

and hence for all $\xi_n \in L^2_{\mathcal{F}_{\tau - t_n}}(\Omega, l^2)$ with $\mathcal{L}(\xi_n) \in D(\tau - t_n)$ and $n \geq H_m$,

$$P\left(\left\{\sum_{|i| > n_m} \left|u_i\left(\tau, \tau - t_n, \xi_n\right)\right|^2 > \frac{1}{2^m}\right\}\right) \leq 2^m \mathbb{E}\left(\sum_{|i| > n_m} \left|u_i\left(\tau, \tau - t_n, \xi_n\right)\right|^2\right) \tag{5.9}$$

$$< \frac{\eta}{2^{m+1}}. \tag{5.9}$$

Given $m \in \mathbb{N}$, set

$$Y_{1,m} = \left\{v \in l^2 : \|v\|^2 \leq R_2\right\}, \tag{5.10}$$

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$$Y_{2,m} = \left\{ v \in l^2 : \sum_{|i|>n_m} |v_i|^2 \le \frac{1}{2^m} \right\}, \tag{5.11}$$

and

$$Y_m = Y_{1,m} \cap Y_{2,m}. \tag{5.12}$$

By (5.10), we see that the set $\left\{(v_i)_{|i|\le n_m} : v \in Y_m\right\}$ is bounded in the finite-dimensional space $\mathbb{R}^{2n_m+1}$ and hence precompact. Consequently, $\left\{(v_i)_{|i|\le n_m} : v \in Y_m\right\}$ has a finite open cover of balls with radius $\frac{1}{\sqrt{2}^m}$, which along with (5.11) implies that the set $\{v : v \in Y_m\}$ has a finite open cover of balls with radius $\frac{1}{\sqrt{2}^{m-1}}$ in l^2. For each $\tau \in \mathbb{R}$ and $m \in \mathbb{N}$, there exists a compact set $K_m = K_m(\tau)$ such that for all $n \le H_m$, $P\left(\{u\left(\tau, \tau - t_n, \xi_n\right) \in K_m\}\right) > 1 - \frac{\eta}{2^m}$. Then by (5.8) and (5.9), there exists a set $\mathcal{Y}_m = Y_m \cup K_m$, which has a finite open cover of balls with radius $\frac{1}{\sqrt{2}^{m-1}}$ in l^2, such that for all $n \in \mathbb{N}$, $P\left(\{u\left(\tau, \tau - t_n, \xi_n\right) \in \mathcal{Y}_m\}\right) > 1 - \frac{\eta}{2^m}$. Set $\mathcal{Y} = \bigcap\limits_{m=1}^{\infty} \mathcal{Y}_m$. Then $\mathcal{Y}$ is a closed and totally bounded subset of l^2, and hence is compact. For all $n \in \mathbb{N}$,

$$P\left(\{u\left(\tau, \tau - t_n, \xi_n\right) \in \mathcal{Y}\}\right) > 1 - \sum_{m=1}^{\infty} \frac{\eta}{2^m} = 1 - \eta,$$

as desired. □

Next, we establish the existence, uniqueness, and periodicity of $\mathcal{D}$-pullback measure attractors for (3.10)–(3.11) on $\mathcal{P}_2(l^2)$.

THEOREM 5.6 *If (3.1)–(3.4), (3.13), and (4.1) hold, then for every* $0 < \varepsilon \le 1$, *S associated with (3.10)–(3.11) has a unique $\mathcal{D}$-pullback measure attractor* $\mathcal{A} = \{\mathcal{A}(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$ *in* $\mathcal{P}_2(l^2)$, *which is given by, for each* $\tau \in \mathbb{R}$,

$$\begin{aligned} \mathcal{A}\left(\tau\right) &= \omega(K, \tau) = \{\psi(0, \tau) : \psi \textit{ is a } \mathcal{D}\textit{-complete orbit of } S\} \\ &= \{\xi(\tau) : \xi \textit{ is a } \mathcal{D}\textit{-complete solution of } S\}, \end{aligned}$$

where $K = \{K(\tau) : \tau \in \mathbb{R}\}$ *is the $\mathcal{D}$-pullback absorbing set of S as given by lemma 5.4.*

Proof. It follows from lemma 5.3 that S is a continuous non-autonomous dynamical system on $\mathcal{P}_2(l^2)$. Notice that S has a closed $\mathcal{D}$-pullback absorbing set K in $\mathcal{P}_2(l^2)$ by lemma 5.4 and is $\mathcal{D}$-pullback asymptotically compact in $\mathcal{P}_2(l^2)$ by lemma 5.5. Hence the existence and uniqueness of the $\mathcal{D}$-pullback measure attractor for S follows from proposition 2.10 immediately. □

We now consider the periodicity of the measure attractor $\mathcal{A}$. By (5.4) and (5.5), we find that K is ϖ-periodic. In addition, it follows from lemma 5.2 and (5.2), the non-autonomous dynamical system S associated with system (3.10)–(3.11) is also

ϖ-periodic. Thus, from proposition 2.10, the periodicity of the measure attractor $\mathcal{A}$ follows.

THEOREM 5.7 *If (3.1)–(3.4), (3.13), (4.1), and (5.1) hold, then for every* $0 < \varepsilon \leq 1$, *S associated with (3.10)–(3.11) has a unique ϖ-periodic $\mathcal{D}$-pullback measure attractor $\mathcal{A}$ in $\mathcal{P}_2(l^2)$.*

6. Upper semicontinuity of pullback measure attractors

In this section, we prove the upper semicontinuity of $\mathcal{D}$-pullback measure attractors for the non-autonomous stochastic lattice systems as the noise intensity ε tends to zero.

We apply theorem 2.11 to the non-autonomous stochastic lattice systems (3.10)–(3.11) with $\varepsilon \in [0,1]$. Note that all results in the previous sections are valid for $\varepsilon = 0$ in which case the proof is actually simpler. From now on, we write the solution of system (3.10)–(3.11) as $u^\varepsilon(t,\tau,\xi)$ at initial time τ with initial value $\xi \in L^2_{\mathcal{F}_\tau}(\Omega, l^2)$ to highlight the dependence of solutions on the parameter ε. Given $\varepsilon \in [0,1]$, let $p^\varepsilon(t,\tau)$ be the transition operator of $u^\varepsilon(t,\tau,\xi)$ and $p^\varepsilon_*(t,\tau)$ be the duality operator of p^ε. Given $t \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$, let $S^\varepsilon(t,\tau) : \mathcal{P}_2(l^2) \to \mathcal{P}_2(l^2)$ be the map given by

$$S^\varepsilon(t,\tau)\mu = p^\varepsilon_*(\tau+t,\tau)\mu, \quad \forall\ \mu \in \mathcal{P}_2(l^2).$$

Let $\mathcal{A}_\varepsilon$ be the $\mathcal{D}$-pullback measure attractor of S^ε.

Next, we establish the convergence of solutions of problem (3.10)–(3.11) when $\varepsilon \to 0$.

LEMMA 6.1. *Suppose (3.1)–(3.4) hold. Then given $\tau \in \mathbb{R}$ and a positive constant $\mathcal{K}(\tau)$, if $\xi \in L^2_{\mathcal{F}_\tau}(\Omega, l^2)$ with $\mathbb{E}(\|\xi\|^2) \leq \mathcal{K}^2(\tau)$, then we have for $t \in \mathbb{R}^+$,*

$$\lim_{\varepsilon \to 0} \sup_{\mu \in B_{\mathcal{P}_2(l^2)}(\mathcal{K}(\tau))} d_{\mathcal{P}_2(l^2)}\left(S^\varepsilon(t,\tau)\mu, S^0(t,\tau)\mu\right) = 0.$$

Proof. By the similar argument as that of lemma 6.2 in [11], we obtain for $\tau \in \mathbb{R}$, $\mathcal{K}(\tau)$ and $t \in \mathbb{R}^+$,

$$\sup_{\mathbb{E}(\|\xi\|^2) \leq \mathcal{K}^2(\tau)} \mathbb{E}\left(\left\|u^\varepsilon(\tau+t,\tau,\xi) - u^0(\tau+t,\tau,\xi)\right\|^2\right) \leq \chi(\varepsilon), \tag{6.1}$$

where $\chi(\varepsilon) \to 0$ as $\varepsilon \to 0$. Note that for all $t \in \mathbb{R}^+$ we have

$$\begin{aligned}
&\sup_{\mathbb{E}(\|\xi\|^2)\leq \mathcal{K}^2(\tau)} \sup_{\substack{\varphi\in L_b(l^2)\\ \|\varphi\|_L\leq 1}} \left|\mathbb{E}\left(\varphi\left(u^\varepsilon\left(\tau+t,\tau,\xi\right)\right)\right) - \mathbb{E}\left(\varphi\left(u^0\left(\tau+t,\tau,\xi\right)\right)\right)\right| \\
&\leq \sup_{\mathbb{E}(\|\xi^\varepsilon\|^2)\leq \mathcal{K}^2(\tau)} \sup_{\substack{\varphi\in L_b(l^2)\\ \|\varphi\|_L\leq 1}} \mathbb{E}\left(\left|\varphi\left(u^\varepsilon\left(\tau+t,\tau,\xi\right)\right) - \varphi\left(u^0\left(\tau+t,\tau,\xi\right)\right)\right|\right) \\
&\leq \sup_{\mathbb{E}(\|\xi\|^2)\leq \mathcal{K}^2(\tau)} \mathbb{E}\left(\left\|u^\varepsilon\left(\tau+t,\tau,\xi\right) - u^0\left(\tau+t,\tau,\xi\right)\right\|\right) \\
&\leq \left(\sup_{\mathbb{E}(\|\xi^\varepsilon\|^2)\leq \mathcal{K}^2(\tau)} \mathbb{E}\left(\left\|u^\varepsilon\left(\tau+t,\tau,\xi\right) - u^0\left(\tau+t,\tau,\xi\right)\right\|^2\right)\right)^{\frac{1}{2}}
\end{aligned}$$

which along with (6.1) implies that for all $t \in \mathbb{R}^+$,

$$\lim_{\varepsilon\to 0} \sup_{\mathbb{E}(\|\xi\|^2)\leq \mathcal{K}^2(\tau)} \sup_{\substack{\varphi\in L_b(l^2)\\ \|\varphi\|_L\leq 1}} \left|\mathbb{E}\left(\varphi\left(u^\varepsilon\left(\tau+t,\tau,\xi\right)\right)\right) - \mathbb{E}\left(\varphi\left(u^0\left(\tau+t,\tau,\xi\right)\right)\right)\right| = 0.$$

This completes the proof. □

By lemma 5.4, one can verify that

$$K = \left\{K(\tau) = \bigcup_{\varepsilon\in[0,1]} \mathcal{A}_\varepsilon(\tau) : \tau \in \mathbb{R}\right\} \in \mathcal{D}. \tag{6.2}$$

Then the main result of this section are given below.

THEOREM 6.2 *Suppose (3.1)–(3.4), (3.13), and (4.1) hold. Then for $\tau \in \mathbb{R}$,*

$$\lim_{\varepsilon\to 0} d_{\mathcal{P}_2(l^2)}\left(\mathcal{A}_\varepsilon(\tau), \mathcal{A}_0(\tau)\right) = 0. \tag{6.3}$$

Proof. Based on (6.2) and lemma 6.1, we obtain (6.3) immediately from theorem 2.11. □

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