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PL LINK ISOTOPY, ESSENTIAL KNOTTING AND QUOTIENTS OF POLYNOMIALS

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ABSTRACT. Piecewise-linear (nonambient) isotopy of classical links may be regarded as link theory modulo knot theory. This note considers an adaptation of new (and old) polynomial link invariants to this theory, obtained simply by dividing a link's polynomial by the polynomials of the individual components. The resulting rational functions are effective in distinguishing isotopy classes of links, and in demonstrating that certain links are essentially knotted in the sense that every link in its isotopy class has a knotted component. We also establish geometric criteria for essential knotting of links.

In this note, a *link* means a piecewise-linear (PL) disjoint collection of embedded closed curves $L = L_1 \cup \cdots \cup L_n$ in 3-dimensional space R^3. Two links are *isotopic* if they are connected by a PL family of embeddings. (We work in the PL category throughout.) This relation is strictly weaker than the more customary one of ambient isotopy. As is well-known (see [R]), all knots (one-component links) are isotopic to each other, and in a link, local knots may be introduced or removed by isotopy. A local knot is defined by some 3-ball in R^3 which meets the link L in exactly one proper arc, which is knotted in that ball, thus expressing L as a connected sum $L = L'\#K$. The *Alexander trick* permits one to unknot the arc with an isotopy, supported in the 3-ball, showing that L and L' are isotopic. It may be, however, that a link cannot be isotoped to one having every component unknotted; it seems appropriate to call such a link *essentially knotted.*

EXAMPLE 1. The links labelled L and M in Figure 1 are essentially knotted, as will be seen shortly. However, N is not essentially knotted; the local trefoil can be pulled tight by an isotopy, taking N to the Hopf link of two linked round circles.

1. **Quotients of polynomials.** Let $L = L_1 \cup \cdots \cup L_n$ be a link, and let f_L denote any of the following ambient isotopy invariants of link theory:

$f_L = \Delta_L(t)$, the (symmetrized) Alexander polynomial, or equivalently,

$f_L = \nabla_L(z)$, the Conway potential function [C], or

$f_L = V_L(t)$, the Jones polynomial, [J], or

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$f_L = P_L(x, y, z)$ or $P_L(l, m)$, the polynomial of [FYHLMO], or

$f_L = Q_L(x)$, the *unoriented* polynomial of [H] and [BLM], or

$f_L = F_L(a, x)$, the Kauffman polynomial, [K].

It is assumed (except in the case of Q_L, where it is unnecessary) that the components of L have been oriented. Isotopies are then required to respect orientation. Define the rational function $\tilde{f}_L$ to be the link polynomial, divided by the product of the corresponding polynomials of the component knots:

$$\tilde{f}_L = \frac{f_L}{f_{L_1} \cdots f_{L_n}}.$$

We shall call $\tilde{\Delta}_L = \Delta_L / (\Delta_{L_1} \cdots \Delta_{L_n})$ the *reduced* Alexander polynomial, and similarly refer to reduced Jones polynomial, and so on, although they are not, in general, ordinary polynomials.

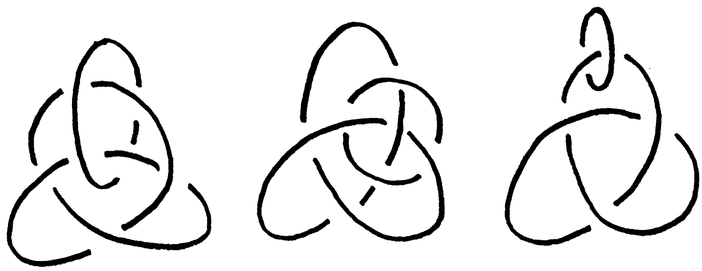

FIGURE 1: L and M are essentially knotted, while N is not.

THEOREM 1. *$\tilde{f}_L$ is unchanged under (PL) isotopy of L.*

PROOF. It was shown in [R] that isotopy is generated by ambient isotopy together with the introduction (or deletion) of local knots. But introducing a local knot K multiplies both the numerator and denominator by the same factor, as each of the invariants listed above is multiplicative: $f_{L\#K} = f_L \cdot f_K$, proving the theorem.

As expected of a PL isotopy invariant, if L has just one component, $\tilde{f}_L$ reduces to unity. Surprisingly, $\tilde{f}$ can actually be a more sensitive link invariant than f itself! An example is given in [J-R] of two 2-componenent links L and L' with $\Delta_L = \Delta'_L$ but $\tilde{\Delta}_L$ and $\tilde{\Delta}'_L$ unequal. In the same example, the reduced Jones polynomials also distinguish the links, whereas V_L itself does not. It also follows that these links, which are *rotants*, or generalized mutants, are not related by any finite sequence of ordinary mutation, because of the following, whose verification is left to the reader.

REMARK. $\tilde{f}_L$ is unchanged under mutation of L.

EXAMPLE 1, CONTINUED. Referring again to Figure 1, we use the Conway polynomial. For the links L and M (oriented counterclockwise) we calculate

$$\tilde{\nabla}_L = (2z + z^3)/(1 + z^2) \text{ and } \tilde{\nabla}_M = z^3/(1 + z^2).$$

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As neither of these is a polynomial (in the ordinary sense), we conclude that neither L nor M is isotopic to a link with unknotted components.

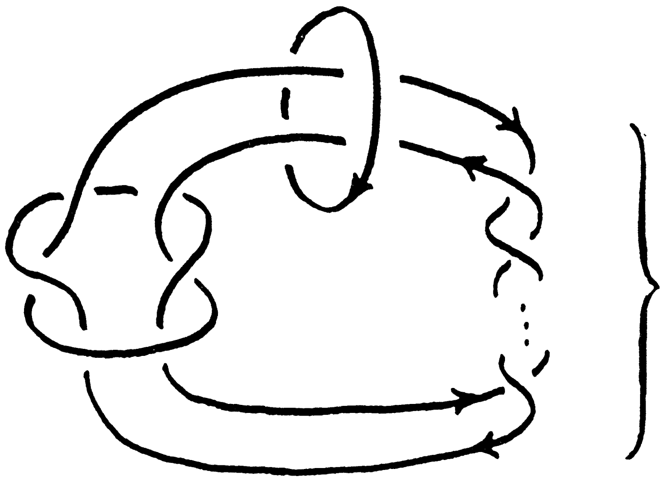

FIGURE 2: A family L_n of isotopically distinct links.

EXAMPLE 2. Consider the family of links $L_n = J \cup K_n$, $n \in Z$, depicted in Figure 2. The knotted component K_n has $2n$ crossovers, where indicated, in the left-hand screw sense (as oriented, these are of positive type), when $n \geq 0$. If n is negative, we take $2n$ twists in the opposite sense. Since the Alexander and Conway polynomials of these links all vanish, we turn to the Jones polynomial. To save notation, let $V_n = V_{L_n}(t)$ and $W_n = V_{K_n}(t)$ so that L_n has reduced Jones polynomial $\tilde{V}_n = V_n / W_n$. We will show the links L_n are isotopically distinct by showing that the $\tilde{V}_n$ are distinct. To see this, apply the general skein relation $t^{-1}V_+ - tV_- = (t^{1/2} - t^{-1/2})V_{split}$, and $\left((t^{-1} - t)/(t^{1/2} - t^{-1/2})\right)^{c-1} =$ Jones polynomial of the c-component unlink, to calculate:

$$V_{n+1} = t^2 V_n + t(t^{-1} - t)^2 / (t^{1/2} - t^{-1/2})$$

and

$$W_{n+1} = t^2 W_n + 1 - t^2.$$

One easily establishes inductively that, for all m,

$$W_{n+m} = t^{2m} W_n + 1 - t^{2m}.$$

Noting that $V_n(t^{1/2} - t^{-1/2})/(t^{-1} - t)$ obeys the same recursion relation as W_n, we get

$$V_{n+m} = t^{2m} V_n + (1 - t^{2m})(t^{-1} - t)/(t^{1/2} - t^{-1/2}).$$

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Now an assumption that $V_{n+m}/W_{n+m} = V_n/W_n$ for some n and some $m \neq 0$ would imply, using the above, that $V_n/W_n = (t^{-1} - t)/(t^{1/2} - t^{-1/2})$. In turn, this implies that V_n/W_n has that value for *all* n. However, an electronic calculation shows that

$$V_0(t) = t^{-9/2} - t^{-7/2} + t^{-5/2} - t^{-3/2} - t^{-1/2} - t^{1/2} - t^{3/2} + t^{5/2} - t^{7/2} + t^{9/2}$$

and

$$W_0(t) = -t^{-3} + t^{-2} - t^{-1} + 3 - t + t^2 - t^3,$$

and it easy to check that $V_0/W_0 \neq (t^{-1} - t)/(t^{1/2} - t^{-1/2})$. Therefore the $\tilde{V}_n$ are distinct, and the links L_n are non-isotopic, for different values of n. Note that they all have homeomorphic complements.

2. **Geometric criteria for essential knotting.** The fundamental theorem of [R] is that two links $L = L_1 \cup \cdots \cup L_n$ and $M = M_1 \cup \cdots \cup M_n$ are ambient isotopic if and only if they are isotopic and for each $i = 1, \ldots, n$ the knots L_i and M_i are ambient isotopic. From this we derive a criterion which guarantees essential knotting.

THEOREM 2. *Suppose the link $L = L_1 \cup \cdots \cup L_n$ has a component (say L_1) which is knotted, and that L contains no local knots in L_1. Then L is essentially knotted. In fact any link isotopic to L has knotted first component.*

PROOF. Suppose L were isotopic to $M = M_1 \cup \cdots \cup M_n$ with M_1 unknotted. Then, using isotopies, we may introduce a local knot of type L_1 in M_1 , and local knots (engulfed by disjoint 3-balls) in $L_2, \ldots, L_n$ and $M_2, \ldots, M_n$, if necessary, to produce a link L' isotopic to L and a link M' isotopic to M so that, for each $i = 1, \ldots, n$, L'_i and M'_i have the same knot type. Note that L' and M' are isotopic to each other, so by the above, they are ambient isotopic. It follows that L' contains a local knot in the first component, since M' does. So we see that by introducing little local knots in L on the components $L_2, \ldots, L_n$, but leaving L_1 untouched, a local knot has appeared in L_1. An easy geometric argument shows that the original L must have had a local knot in L_1 as well, a contradiction proving the theorem.

The next result gives a method for manufacturing essentially knotted links. First we recall the concept of a satellite. Let $V \subset R^3$ be a solid torus, standardly embedded, and let $L = L_1 \cup \cdots \cup L_n$ be a link lying in the interior of V. Consider a knot K with tubular neighbourhood $N(K) \subset R^3$ and let $h: V \longrightarrow N(K)$ be a homeomorphism. Then the link $h(L) \subset R^3$ is the *satellite* link determined by the above data. L is called the *pattern* link, and K is called a *companion* of the satellite. A subset of V is called *geometrically essential* if it does not lie in a 3-ball inside V .

THEOREM 3. *Using the notation of the preceding paragraph, suppose K is knotted and that both L_1 and $L_2 \cup \cdots \cup L_n$ are geometrically essential sets in V. Then the satellite link $h(L)$ is essentially knotted. In fact, $h(L)$ is not isotopic to a link with unknotted first component.*

PROOF. Without loss of generality, we may assume that no 3-ball inside V encloses a local knot of L in L_1. For such local knots could be removed by isotopy inside V (without

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altering L_1 being essential) inducing *via* h an isotopy of $h(L)$ removing the corresponding local knot in its first component. Now, since K is knotted, it follows from [S] that its satellite knot $h(L_1)$ is also knotted. So the proof reduces, by Theorem 2, to showing that $h(L)$ does not have any local knots in its first component, knowing there are no such local knots defined by a 3-ball within $h(V)$. To this end we'll show that if a 2-sphere S in R^3 meets $h(L)$ transversely in exactly two points of $h(L_1)$, then it may be moved by an ambient isotopy, fixed on $h(L)$, to be disjoint from the boundary $\partial h(V)$. First, make S transverse to $\partial h(V)$, so that $S \cap \partial h(V)$ is a finite collection of simple closed curves. By standard arguments, beginning innermost on S, using the essentiality of $L_2 \cup \cdots \cup L_n$ and the fact that K is knotted, any curves of $S \cap \partial h(V)$ that do not separate (in S) the two points of $S \cap h(L)$ can be removed. Now we're done, as no curve on $\partial h(V)$ can bound a disk inside $h(V)$ such that the disk meets $h(L)$ transversely once and yet avoids $L_2 \cup \cdots \cup L_n$.

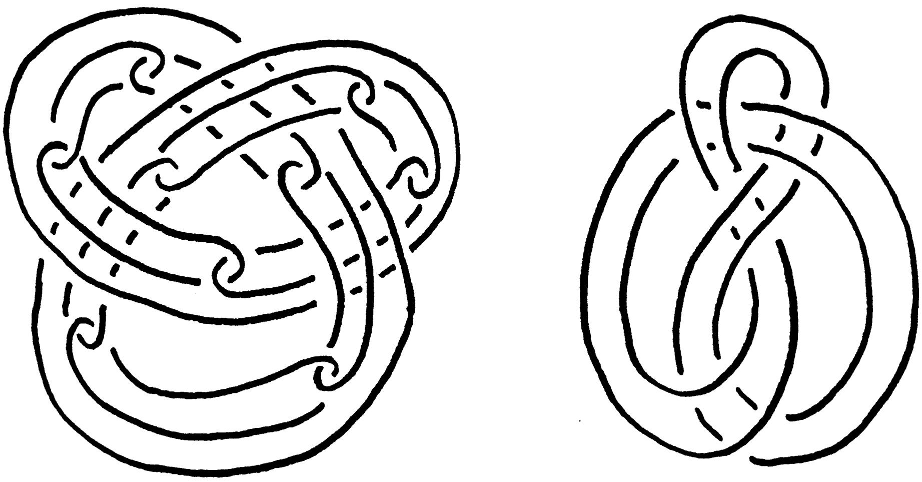

FIGURE 3: Both of these links are essentially knotted, by Theorem 3.

COROLLARY. *A link with two parallel knotted components is essentially knotted (parallel meaning they cobound an embedded annulus).*

COROLLARY. *Any link which is a cable of a nontrivial knot is essentially knotted (cable means satellite whose model is a geometrically essential torus link in V).*

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