Canad. Math. Bull.Vol. 36 (2), 1993 pp. 154–163

SEPARATORS IN CONTINUOUS IMAGES OF ORDERED CONTINUA AND HEREDITARILY LOCALLY CONNECTED CONTINUA

J. GRISPOLAKIS, J. NIKIEL, J. N. SIMONE AND E. D. TYMCHATYN

ABSTRACT. Let X be a Hausdorff space which is the continuous image of an ordered continuum. We prove that every irreducible separator of X is metrizable. This is a far reaching extension of the 1967 theorem of S. Mardešić which asserts that X has a basis of open sets with metrizable boundaries. Our first result is then used to show that, in particular, if Y is an hereditarily locally connected continuum, then for subsets of Y quasi-components coincide with components, and that the boundary of each connected open subset of Y is accessible by ordered continua. These results answer open problems in the literature due to the fourth and third authors, respectively.

1. **Introduction.** We consider Hausdorff spaces only. By a *continuum* is meant a compact connected (Hausdorff) space. A continuum is said to be *hereditarily locally connected* if each of its subcontinua is locally connected. Metric hereditarily locally connected continua have been extensively studied since the 1920's (see *e.g.* [20], [19], [3] and [6], where more references can be found). A number of characterizations of hereditarily locally connected metric continua had been extended to the general case in [14] and [18] but some open problems remained. We provide solutions to two of those problems. Namely, one of the implications in Theorem 2 solves a problem in [18, p. 1226], and Theorem 4 answers affirmatively a question posed in [15]. The arguments we give make use of the corresponding results in the metrizable case and they involve the recently developed theory of continuous images of (Hausdorff) arcs.

A collection $\mathbf{A}$ of subsets of a compact space X is said to be a *null-family* in X if, for every open covering $\mathbf{U}$ of X, the subcollection $\{B \in \mathbf{A} : B$ is not contained in any $V \in \mathbf{U}\}$ is finite.

The following Theorem A summarizes characterization results of hereditarily locally connected continua that were obtained in [14] and [18].

THEOREM A (SEE [14] AND [18]). *For a continuum X the following conditions are equivalent:*

(a) X is hereditarily locally connected,

The second named author was supported by an NSERC International Fellowship and the fourth named author was supported by NSERC grant A5616.

Received by the editors January 14, 1992; revised June 4, 1992.

AMS subject classification: Primary: 54F15; secondary: 54C05, 54F05, 54F50.

Key words and phrases: ordered continuum, continuous image, hereditarily locally connected continuum, component, quasi-component, accessible, irreducible separator, totally disconnected.

(b) *there are no non-empty disjoint closed sets K and L in X and subcontinua $X_1, X_2, \ldots$ of X such that $K \cap X_n \neq \emptyset \neq L \cap X_n$ and $X_n \cap \mathrm{cl}(\bigcup_{i \neq n} X_i) = \emptyset$ for $n = 1, 2, \ldots$,*
(c) *X contains no continuum of convergence,*
(d) *components of every closed subset of X form a null-family,*
(e) *components of every open subset of X form a null-family,*
(f) *components of each subset of X form a null-family,*
(g) *quasi-components of each subset of X form a null-family,*
(h) *every connected subset of X is locally connected.*

A continuum X is said to be an *arc* (or, equivalently, an ordered continuum) if it has exactly two non-separating points. It is well-known that the class of arcs coincides with the class of compact and connected linearly ordered topological spaces, and that each separable arc is homeomorphic to the closed interval $[0, 1]$ of real numbers.

THEOREM B, [9]. *Each hereditarily locally connected continuum is a continuous image of an arc.*

Recall that Theorem B together with the result of Theorem E were used in [9, Theorem 4.1] to prove that hereditarily locally connected continua are rim-countable. That result was further generalized in [11]. Recall also that locally connected rim-countable continua need not be continuous images of arcs, [12].

Let A be a subset of a locally connected continuum X. We let $\mathbf{K}(X - A)$ denote the set of all components of $X - A$. Furthermore, we shall say that A is a *T-set* in X if A is closed and each component of $X - A$ has a two-point boundary. For example, X is a T-set in itself, because $\mathbf{K}(X - X) = \emptyset$.

THEOREM C, [7, LEMMA 3.2]. *If A is a T-set in a locally connected continuum X then the collection $\{\mathrm{cl}(J) : J \in \mathbf{K}(X - A)\}$ is a null-family in X.*

THEOREM D, [16, THEOREM 6]. *Let X be a locally connected continuum and A a T-set in X. There exists an upper semi-continuous decomposition G_A of X into closed sets such that if X_A denotes the quotient space and $f: X \longrightarrow X_A$ is the quotient map, then:*
(i) *$f|_A$ is one-to-one and $f(A)$ is a T-set in X_A,*
(ii) *each $Z \in \mathbf{K}\big(X_A - f(A)\big)$ is homeomorphic to $]0, 1[$,*
(iii) *for each $Z \in \mathbf{K}\big(X_A - f(A)\big)$ there exists a unique $P_Z \in \mathbf{K}(X - A)$ such that $f(P_Z) \subset \mathrm{cl}(Z)$, and each component of $X - A$ is a P_Z for some $Z \in \mathbf{K}\big(X_A - f(A)\big)$.*

We remark that the space X_A is uniquely determined by X and A. If the set A is metrizable it follows, by local connectedness of X, that $\mathbf{K}(X - A)$ is countable, [7, Lemma 4.1]; and therefore X_A is metrizable, by [7, Lemma 4.2].

We shall need some simple concepts from cyclic element theory (see [3] or [20] for the classical theory in the metrizable case and [1], [10] or [13] for its generalization to Hausdorff continua).

Let X be a locally connected continuum. A subset Y of X is said to be a *cyclic element* of X if Y is connected and maximal with respect to having no separating point of itself (*i.e.*, $Y - \{y\}$ is also connected for each $y \in Y$). In particular, X is said to be *cyclic* if it is the only cyclic element of itself, *i.e.*, if X has no separating point. It is well-known that each cyclic element of a locally connected continuum is itself a locally connected subcontinuum. Let $x, y \in X$ and

$$E_{xy} = \{x, y\} \cup \big\{z \in X : x \text{ and } y \text{ are in distinct components of } X - \{z\}\big\}.$$

We shall say that X is a *cyclic chain* from x to y if

$$X = E_{xy} \cup \bigcup \{Y : Y \text{ is a cyclic element of } X \text{ and } |Y \cap E_{xy}| = 2\}.$$

The following simple remark is very useful. Let X be a cyclic locally connected continuum, A be a T-set in X, and $J \in \mathbf{K}(X - A)$ with $\mathrm{bd}(J) = \{a, b\}$. Then $\mathrm{cl}(J)$ is a locally connected continuum and, since X is cyclic, $\mathrm{cl}(J)$ is a cyclic chain from a to b.

Let Y be a cyclic locally connected continuum. We say that a sequence $(A_n)_{n=1}^{\infty}$ of T-subsets of Y *T-approximates* Y if

(1) A_1 is metrizable,

(2) $A_n \subset A_{n+1}$,

(3) if $Z \in \mathbf{K}(Y - A_n)$, then $\{x \in Z : x \text{ separates } Z\} \subset A_{n+1}$,

(4) if $Z \in \mathbf{K}(Y - A_n)$ and C is a non-degenerate cyclic element of $\mathrm{cl}(Z)$, then $C \cap A_{n+1}$ is a metrizable set which contains at least three points.

Note that the conditions of the above definition imply that $\bigcup_{n=1}^{\infty} A_n$ is dense in Y (see [7, Lemma 3.4]).

In [7] several characterizations of continuous Hausdorff images of ordered continua are given. We shall need the following:

THEOREM E (SEE [7, THEOREM 1.1 AND COROLLARY 4.10]). *Let X be a locally connected continuum. Then the following conditions are equivalent:*

(a) X is a continuous image of an arc,

(b) if B is a closed metrizable subset of a cyclic element Y of X, then there exists a metrizable T-set A in Y such that $B \subset A$,

(c) if Y is a non-degenerate cyclic element of X, then there exists a sequence $(A_n)_{n=1}^{\infty}$ of T-sets in Y which T-approximates Y. Moreover, one may assume that A_1 contains any three given points of Y.

THEOREM F, [13, THEOREM 4.2]. *Let X be a continuous image of an arc and A be a T-set in Y. If A is separable and $\mathbf{K}(X - A)$ is countable, then A is metrizable.*

Further properties of continuous images of arcs and ordered compacta can be found in [4], [17], [10] and [13].

2. **Separators in continuous images of ordered continua.**

LEMMA 1. *Let Y be a cyclic locally connected continuum, $(B_n)_{n=1}^{\infty}$ a sequence of subsets of Y and $B = \mathrm{cl}(\bigcup_{n=1}^{\infty} B_n)$. Suppose that there is a sequence $(A_n)_{n=1}^{\infty}$ of T-sets in Y such that:*

(i) $(A_n)_{n=1}^{\infty}$ T-approximates Y,

(ii) $B_1 = A_1$ and $B_n \subset B_{n+1}$ for $n = 1, 2, \ldots$, and

(iii) if $Z \in \mathbf{K}(Y - A_n)$ for some n, then $B \cap Z \cap \{x : x \text{ separates } Z\}$ is a finite set which is contained in B_{n+1}, and $C \cap A_{n+1} = C \cap B_{n+1}$ for each non-degenerate cyclic element C of $\mathrm{cl}(Z)$ such that $C \cap B \not\subset \mathrm{bd}(Z)$.

Then B is a metrizable T-set in Y.

PROOF. Our argument somewhat resembles a part of the proof of [8, Theorem 4]. A simple inductive proof involving (ii) and (iii) shows that $B_n = B \cap A_n$ and each B_n is a T-set in Y. Moreover, by the first part of (iii), it follows that if $Z \in \mathbf{K}(Y - A_n)$ for some n, then B meets only finitely many non-degenerate cyclic elements of $\mathrm{cl}(Z)$. An easy inductive argument shows that each B_n is metrizable (because of the condition (4) in the definition of T-approximation; recall that a compact space that admits a countable cover by closed metrizable subsets is metrizable again—see *e.g.* [2, Theorem 3.1.19]). Since $(B_n)_{n=1}^{\infty}$ is an increasing sequence of T-sets in Y, it follows that B is a T-set in Y, [16, Theorem 6]. Since each B_n is metrizable, B is separable. By Theorem F, in order to prove that B is metrizable it suffices to show that $Y - B$ has countably many components.

Let $\mathbf{L}_n = \{J \in \mathbf{K}(Y - B_n) : J \cap B = \emptyset\}$ for $n = 1, 2, \ldots$. Since B_n is metrizable, [7, Lemma 4.1] implies that $\mathbf{K}(Y - B_n)$ and $\mathbf{L}_n$ are countable. Observe that $\mathbf{L}_1 \subset \mathbf{L}_2 \subset \cdots$. For each n let $D_n = B_n \cup \bigcup \mathbf{L}_n$. Then D_n is a T-set in Y. Note that $D_1 \subset D_2 \subset \cdots$ and if $J \in \mathbf{K}(Y - D_n)$ then the set of separating points of $\mathrm{cl}(J)$ is contained in D_{n+1}. Moreover, the second part of (iii) implies that if $J \in \mathbf{K}(Y - D_n)$ and C is a non-degenerate cyclic element of $\mathrm{cl}(J)$, then $C \cap D_{n+1}$ contains at least three points. By [7, Lemma 3.4], $\bigcup_{n=1}^{\infty} D_n$ is dense in Y. Therefore, $\mathbf{K}(Y - B) \subset \bigcup_{n=1}^{\infty} \mathbf{L}_n$. In fact, suppose that $J \in \mathbf{K}(Y - B)$ and $J \notin \bigcup_{n=1}^{\infty} L_n$. Then $J \cap \bigcup \bigcup_{n=1}^{\infty} L_n = \emptyset$, whence $J \cap \bigcup_{n=1}^{\infty} D_n = \emptyset$. Since J is a non-empty open subset of Y, it must meet the dense set $\bigcup_{n=1}^{\infty} D_n$, a contradiction.

Since $\mathbf{K}(Y - B) \subset \bigcup_{n=1}^{\infty} \mathbf{L}_n$, the collection $\mathbf{K}(Y - B)$ is countable. By Theorem F, B is metrizable.

Let X be a compact space and $Z \subset X$. Then Z is said to be an *irreducible separator* of X if Z separates X and no proper subset of Z separates X. Obviously, each irreducible separator of X is closed, and if X is locally connected then each separator of X contains an irreducible separator of X.

The following Theorem 1 strengthens some of the results of [5] and [11].

THEOREM 1. *If X is a continuous image of an arc, then each irreducible separator of X is metrizable.*

PROOF. Let Z be an irreducible separator of X. If Z contains a separating point z of X, then irreducibility of Z implies that $Z = \{z\}$. Hence, we may assume that Z contains no separating point of X. Since Z is irreducible, it follows that there is a non-degenerate cyclic element Y of X such that $Z \subset Y$. Then Z is an irreducible separator of Y.

Below, all boundaries of sets are taken in Y.

Let $(A_n)_{n=1}^{\infty}$ be a sequence of T-subsets of Y which T-approximates Y. We are going to construct a metrizable T-set B in Y such that $Z \subset B$. Namely, we shall employ induction to get an increasing sequence $(B_n)_{n=1}^{\infty}$ of subsets of Y such that the hypotheses of Lemma 1 are satisfied.

Let $B_1 = A_1$. Suppose that the set B_n is already constructed for some n.

Let $J \in \mathbf{K}(Y - A_n)$. If $J \cap Z = \emptyset$, let $B^J = C^J = \emptyset$. Suppose that $J \cap Z \neq \emptyset$. Since Z is an irreducible separator of Y, it follows that $Z \cap \mathrm{bd}(J) = \emptyset$ and Z is an irreducible separator of $\mathrm{cl}(J)$. Hence, either $J \cap Z = \{z^J\}$ for some separating point z^J of J or there is a non-degenerate cyclic element S^J of $\mathrm{cl}(J)$ such that $Z \cap J \subset S^J - \mathrm{bd}(S^J)$ (recall that $\mathrm{bd}(S^J)$ consists of exactly two points which belong to the set $\mathrm{bd}(J) \cup \{x : x \text{ separates } J\}$). If $J \cap Z = \{z^J\}$ we let $B^J = C^J = \{z^J\}$, and otherwise we let $B^J = S^J \cap A_{n+1}$ and $C^J = S^J$.

Let $B_{n+1} = B_n \cup \bigcup\{B^J : J \in \mathbf{K}(Y - A_n)\}$. This finishes the inductive construction.

It is easy to see that the assumptions of Lemma 1 are satisfied. Hence, $B = \mathrm{cl}(\bigcup_{n=1}^{\infty} B_n)$ is a metrizable T-set in Y. Now, it remains to show that $Z \subset B$. Let $C_n = B_n \cup \bigcup\{C^J : J \in \mathbf{K}(Y - A_n)\}$ for $n = 1, 2, \ldots$. Then each C_n is a T-set in Y and, moreover, $C_{n+1} \subset C_n$, $B \subset C_n$ and $Z \subset C_n$. Let $C = \bigcap_{n=1}^{\infty} C_n$. Then $B \subset C$ and $Z \subset C$. We shall prove that $B = C$. As in the proof of Lemma 1, let $\mathbf{L}_n = \{J \in \mathbf{K}(Y - B_n) : J \cap B = \emptyset\}$. A straightforward argument shows that $\mathbf{K}(Y - C) = \bigcup_{n=1}^{\infty} \mathbf{L}_n$, *i.e.*, $Y - C = \bigcup \bigcup_{n=1}^{\infty} \mathbf{L}_n$. By the end of the proof of Lemma 1, $\mathbf{K}(Y - B) \subset \bigcup_{n=1}^{\infty} \mathbf{L}_n$. Therefore, $Y - B \subset Y - C$, whence $C \subset B$. Thus, $Z \subset B = C$. The proof is complete.

By [5, Lemma 8], we get immediately the following corollary:

COROLLARY. *If a Hausdorff space X is a continuous image of an ordered compactum, $x, y \in X$, $Z \subset X$ and Z separates X between x and y, then Z contains a compact metrizable set (possibly void) which also separates X between x and y.*

Corollary gives immediately the main results (Lemma 4 and Theorems 3, 4 and 5) of [11] which deal with dimension-theoretic properties of continuous images of arcs and ordered compacta.

3. **Components of subsets of hereditarily locally connected continua.**

We omit easy and straightforward proofs of the following two lemmas.

LEMMA 2. *Let X be a locally connected continuum, $C \subset X$ and Q be a quasi-component of C. If $Q \cap Y$ is connected (possibly void) for each cyclic element Y of X, then Q is connected.*

LEMMA 3. *Let X be a locally connected continuum, Y a cyclic element of X, $C \subset X$ and Q be a quasi-component of C. Then $Q \cap Y$ is a quasi-component of $C \cap Y$.*

THEOREM 2. *If X is a continuum then the following conditions are equivalent:*

(a) X is hereditarily locally connected;

(b) for each subset C of X each quasi-component of C has connected closure;

(c) for each subset C of X each quasi-component of C is connected.

PROOF. (c) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (a) Suppose that X is not hereditarily locally connected. By [14, Theorem 4], there exist open sets U and V in X with disjoint closures and a sequence $(C_i)_{i=1}^{\infty}$ of pairwise disjoint continua each of which meets both U and V and such that $C_j \cap \mathrm{cl}(\bigcup_{i\neq j} C_i) = \emptyset$ for each positive integer j. Let $(C_{i_\gamma})_{\gamma\in\Gamma}$ be a convergent subnet of $(C_i)_{i=1}^{\infty}$. Let $x \in \mathrm{Lim}_{\gamma\in\Gamma}\big(C_{i_\gamma} \cap \mathrm{bd}(U)\big)$ and $y \in \mathrm{Lim}_{\gamma\in\Gamma}\big(C_{i_\gamma} \cap \mathrm{bd}(V)\big)$. Then $\{x,y\}$ is a quasi-component of the set $\{x,y\} \cup \bigcup\{C_{i_\gamma} : \gamma \in \Gamma\}$.

(a) $\Rightarrow$ (c) Suppose that a continuum X is hereditarily locally connected and yet there exists $C \subset X$ such that some quasi-component I of C is not connected. By Lemma 2, there is a cyclic element Y of X such that $I \cap Y$ is not connected. By Lemma 3, $I \cap Y$ is a quasi-component of the subset $C \cap Y$ of Y. Hence, we may assume that X is cyclic.

Since I is not connected, there exists an irreducible separator B of X such that $B \cap I = \emptyset$ and I is not contained in a single component of $X - B$. By Theorem 1, B is metrizable. Let $x, y \in I$ be points that lie in distinct components of $X - B$. By Theorem E, there is a metrizable T-set A in X such that $B \cup \{x,y\} \subset A$.

Let X_A and $f: X \longrightarrow X_A$ be as in Theorem D. Then X_A is a metrizable locally connected continuum, by [7, Lemma 4.2]. An easy straightforward argument proves that X_A is hereditarily locally connected. To simplify the notation, we let $\mathbf{K} = \mathbf{K}\big(X_A - f(A)\big)$.

Let $Z \in \mathbf{K}$. We let $P_Z \in \mathbf{K}(X - A)$ be such that $f(P_Z) \subset \mathrm{cl}(Z)$ (recall that P_Z is unique). Also, we let $\mathrm{bd}(Z) = \{a_Z, b_Z\}$ and $\mathrm{bd}(P_Z) = \{p_Z, q_Z\}$ with $f(p_Z) = a_Z$ and $f(q_Z) = b_Z$. Furthermore, let $c_Z \in Z$ and M_Z and N_Z be components of $\mathrm{cl}(Z) - \{c_Z\}$ such that $a_Z \in M_Z$ and $b_Z \in N_Z$.

Let

$$\begin{aligned} E = f(C \cap A) &\cup \bigcup\{Z : Z \in \mathbf{K} \text{ and } \mathrm{bd}(P_Z) \text{ is contained in a quasi-component of } \mathrm{cl}(P_Z) \cap C\} \\ &\cup \bigcup\{M_Z : Z \in \mathbf{K} \text{ and } p_Z \in C\} \cup \bigcup\{N_Z : Z \in \mathbf{K} \text{ and } q_Z \in C\} \end{aligned}$$

and

$$\begin{aligned} K = f(I \cap A) &\cup \bigcup\{Z : Z \in \mathbf{K} \text{ and } \mathrm{bd}(P_Z) \subset I\} \\ &\cup \bigcup\{M_Z : Z \in \mathbf{K} \text{ and } p_Z \in I\} \cup \bigcup\{N_Z : Z \in \mathbf{K} \text{ and } q_Z \in I\}. \end{aligned}$$

Then $f(x), f(y) \in K \subset E$ and a straightforward argument shows that K is a quasi-component of E because I is a quasi-component of C. Observe that $f(B)$ is a separator of X_A, $f(B) \cap E = \emptyset$, and $f(x)$ and $f(y)$ are in distinct components of $X_A - f(B)$. Thus, X_A is a metrizable hereditarily locally connected continuum which contains a subset E such that some quasi-component K of E is not connected. This contradicts [20, V.2.4, p. 91].

Recall that a space Z is *hereditarily disconnected* if its components are points, and it is *totally disconnected* if its quasi-components are points.

THEOREM 3. *If C is a subset of a hereditarily locally connected continuum then the following conditions are equivalent:*

(a) C is zero-dimensional;

(b) C is totally disconnected;

(c) C is hereditarily disconnected.

PROOF. (a) $\Rightarrow$ (c) is obvious, and (c) $\Rightarrow$ (b) follows from Theorem 2. It remains to show that (b) $\Rightarrow$ (a).

Suppose that C is a totally disconnected subset of a hereditarily locally connected continuum X. Let $x \in C$ and U be a neighbourhood of x in X. Let V be a neighbourhood of x in X such that $\mathrm{cl}(V) \subset U$ and $\mathrm{bd}(V)$ is countable (see [9, Theorem 4.1]). Then $\mathrm{bd}(V) \cap C$ is at most countable, say $\mathrm{bd}(V) \cap C = \{y_1, y_2, \ldots\}$. For each n let W_n be a neighbourhood of y_n such that $x \notin W_n$ and $\mathrm{bd}(W_n)$ is disjoint from C.

Let $V_1 = W_1$ and $V_{n+1} = W_{n+1} - \mathrm{cl}(W_1 \cup \cdots \cup W_n)$ for $n > 1$. Then $V_1, V_2, \ldots$ are pairwise disjoint open subsets of X and $\mathrm{bd}(V) \cap C \subset V_1 \cup V_2 \cup \cdots$. For $n = 1, 2, \ldots$ let Y_n be the union of the components of V_n which meet $\mathrm{bd}(V) \cap C$. Let $Y = Y_1 \cup Y_2 \cup \cdots$. Since X is locally connected, Y is open in X. Clearly, the components of Y are components of the sets $V_1, V_2, \ldots$. By Theorem A, the components of Y form a null-family. It follows that $\mathrm{bd}(Y) \subset \big(\mathrm{bd}(V) - C\big) \cup \mathrm{bd}(V_1) \cup \mathrm{bd}(V_2) \cup \cdots$. Therefore, $\mathrm{bd}(Y)$ is disjoint from C. Hence, $V - \mathrm{cl}(Y)$ is a neighbourhood of x in U whose intersection with C is both open and closed in C. This completes the proof.

4. **Arcwise accessibility in hereditarily locally connected continua.**

The following Theorem 4 solves a problem posed in [15]. Some particular forms of the theorem were obtained in [15] and [7, Lemma 4.4].

THEOREM 4. *Let X be a hereditarily locally connected continuum, U a connected open subset of X and $x \in \mathrm{bd}(U)$. Then there is an arc I in X such that $x \in I \subset U \cup \{x\}$.*

PROOF. It suffices to prove Theorem 3 under the additional assumption that X is cyclic.

Let $y, z \in U$ with $y \neq z$. By Theorems B and E, there is a sequence $(A_n)_{n=1}^{\infty}$ of T-sets in X which T-approximates X and such that $x, y, z \in A_1$.

We start with the following auxiliary construction. Suppose that either $n = 0$, $C = X$ and $x_C = y$, or that $M \in \mathbf{K}(X - A_n)$ for some $n > 0$, and C is a non-degenerate cyclic element of $\mathrm{cl}(M)$ such that $\mathrm{bd}(C) = \{x, x_C\}$ for some $x_C \in C$ (recall that x is the our selected point in $\mathrm{bd}(U)$) and, moreover, $C \cap U$ is connected, $x_C \in U$ and $x \in \mathrm{cl}(C \cap U)$. Let $A = A_{n+1} \cap C$. Then A is a metrizable T-set in C, $|A| \geq 3$ and $x, x_C \in A$. Let C_A and $f: C \longrightarrow C_A$ be as in Theorem D. Then C_A is a metrizable hereditarily locally connected continuum and $f(A)$ is a T-set in C_A such that $f(x), f(x_C) \in C_A$. To simplify the notation, we let $\mathbf{K} = \mathbf{K}\big(C_A - f(A)\big)$.

Let $Z \in \mathbf{K}$. We let $P_Z \in \mathbf{K}(C - A)$ be such that $f(P_Z) \subset \mathrm{cl}(Z)$. Recall that then P_Z is unique; clearly, $P_Z \in \mathbf{K}(X - A_{n+1})$. Let $a_Z \in Z$.

Now, let

$$V = f(A\cap U\cap C)\cup\bigcup\{Z : Z\in \mathbf{K}, \mathrm{cl}(P_Z)\cap U \text{ is connected and } \mathrm{bd}(P_Z)\subset U\}$$
$$\cup\bigcup\big\{Z-\{a_Z\} : Z\in \mathbf{K}, \mathrm{cl}(P_Z)\cap U \text{ is not connected and } \mathrm{bd}(P_Z)\subset U\big\}$$
$$\cup\bigcup\{Z : Z\in \mathbf{K}, \mathrm{cl}(P_Z)\cap U\neq\emptyset \text{ and } |\,\mathrm{bd}(P_Z)\cap U|\leq 1\}.$$

By Theorem C, V is an open subset of C_A. Clearly, $f(x)\in \mathrm{bd}(V)$. Observe that V is connected because $C\cap U$ is connected. Since C_A is metrizable, there exists an arc J in C_A such that $f(x)\in J\subset V\cup\{f(x)\}$ and $f(x)$ and $f(x_C)$ are the end-points of J.

Let $Z\in\mathbf{K}$ be such that $Z\cap J\neq\emptyset$. Then $\mathrm{cl}(Z)\subset J$ and $Z\subset V\cap J$.

If $f(x)\notin \mathrm{bd}(Z)$, then $U\cap \mathrm{cl}(P_Z)$ is a connected open subset of P_Z (by the definition of V) and $\mathrm{bd}(P_Z)\subset U$. Since connected open subsets of an hereditarily locally connected continuum are arc-connected, there is an arc K_Z in $\mathrm{cl}(P_Z)\cap U$ such that the end-points of K_Z coincide with $\mathrm{bd}(P_Z)$.

Suppose next that $f(x)\in \mathrm{bd}(Z)$. Then $x\in \mathrm{bd}(P_Z)$. Moreover, $x\in \mathrm{cl}(P_Z\cap U)$. Let y_Z denote the other point of $\mathrm{bd}(P_Z)$. There are two cases to consider.

First, suppose that there is a non-degenerate cyclic element D of $\mathrm{cl}(P_Z)$ such that $x\in D$. Then $\mathrm{bd}(D)=\{x,x_D\}$ for some point x_D such that either $x_D=y_Z$ or x_D is a separating point of P_Z. If $x_D=y_Z$, let $K_Z=D=\mathrm{cl}(P_Z)$. If x_D is a separating point of P_Z, let E be the component of $\mathrm{cl}(P_Z)-\{x_D\}$ such that $y_Z\in E$ and observe that $\mathrm{cl}(E)\cap U$ is a connected open subset of $\mathrm{cl}(E)=E\cup\{x_D\}$ which contains y_Z and x_D, whence there is an arc K_Z' in $\mathrm{cl}(E)\cap U$ from y_Z to x_D. In this case, let $K_Z=K_Z'\cup D$.

Now, suppose that there is no non-degenerate cyclic element of $\mathrm{cl}(P_Z)$ which contains x. Then x is the limit point of the set of separating points of P_Z. Since $\mathrm{cl}(P_Z)$ is a cyclic chain from x to y_Z and $U\cap(P_Z\cup\{y_Z\})$ is connected with $x\in \mathrm{cl}(P_Z\cap U)$, it follows that all separating points of P_Z belong to U. By hereditary local connectedness of $\mathrm{cl}(P_Z)$, there is an arc K_Z in $\mathrm{cl}(P_Z)$ such that $K_Z\subset U\cup\{x\}$ and x and y_Z are the end-points of K_Z.

Note that there is at most one $Z\in\mathbf{K}$ such that $Z\cap J\neq\emptyset$ and $f(x)\in\mathrm{bd}(Z)$, because $f(x)$ is an end-point of J. Let

$$I_C=\big(A\cap f^{-1}(J)\big)\cup\bigcup\{K_Z : Z\in\mathbf{K} \text{ and } Z\cap J\neq\emptyset\}.$$

Then either I_C is an arc from x to x_C such that $I_C\subset U\cup\{x\}$ or $I_C=I_C'\cup D$, where I_C' is an arc from x_D to x_C such that $I_C'\subset U$. In the latter case D is a non-degenerate cyclic element of the closure of a component of $X-A_{n+1}$ (*i.e*, of $X-A_1$ if we started with $C=X$) and, moreover, $D\cap U$ is connected, $x_D\in U$ and $x\in \mathrm{cl}(D\cap U)$. This concludes the auxiliary construction.

Let I_0 be the continuum I_{C_0} given in the case when $C_0=X$. If I_0 is an arc, we may stop the intended induction and let $I=I_0$. Suppose that I_0 is not an arc. Then $I_0=I_{C_0}'\cup C_1$, where C_1 is a non-degenerate cyclic element of $\mathrm{cl}(M_1)$ for some $M_1\in\mathbf{K}(X-A_1)$ with $\mathrm{bd}(C_1)=\{x,x_{C_1}\}$, and I_{C_0}' is an arc in U from x_{C_1} to y. Moreover, $x_{C_1}\in U, x\in\mathrm{cl}(C_1\cap U)$ and $C_1\cap U$ is connected.

The auxiliary construction in the case $C = C_1$ gives us a continuum $I_{C_1} = I'_{C_1} \cup C_2$, where I'_{C_1} is an arc and C_2 is either void or it is a non-degenerate cyclic element of the closure of a component M_2 of $X - A_2$. Let $I_1 = I'_{C_0} \cup I_{C_1}$ (note that $I'_{C_0} \cup I'_{C_1}$ is an arc). Proceed by induction.

If the induction stops after $n+1$ steps, then $I_n = I'_{C_0} \cup I'_{C_1} \cup \cdots \cup I'_{C_{n-1}} \cup I_{C_n}$ is an arc from x to y such that $I_n \subset U \cup \{x\}$. Hence, it suffices to consider the case when the induction does not stop. Let $I = \bigcap_{n=0}^{\infty} I_n$. Note that $x, y \in I_{n+1} \subset I_n$ for each n. Therefore, I is a continuum and $x, y \in I$.

Since C_n is a cyclic element of $\mathrm{cl}(M_n)$ for some $M_n \in \mathbf{K}(X - A_n)$ and $C_{n+1} \subset C_n$, it follows that $M_1 \supset M_2 \supset \cdots$. Recall that, by [7, Lemma 3.4], $\bigcup_{n=1}^{\infty} A_n$ is dense in X. By [16, Theorem 8], $\bigcap_{n=1}^{\infty} \mathrm{cl}(M_n)$ consists of a single point. Since $x \in C_n$ for each n, we have $\bigcap_{n=1}^{\infty} \mathrm{cl}(M_n) = \bigcap_{n=1}^{\infty} \mathrm{cl}(C_n) = \{x\}$. Therefore, $I = \{x\} \cup \bigcup_{n=0}^{\infty} I'_{C_n}$. Obviously, $\bigcup_{n=0}^{\infty} I'_{C_n} \subset U$. Since I is a continuum and $\bigcup_{n=0}^{m} I'_{C_n}$ is an arc for each m, it follows that I is an arc. This completes the proof.

REFERENCES

1. J. L. Cornette, *Image of a Hausdorff arc is cyclically extensible and reducible*, Trans. Amer. Math. Soc. **199**(1974), 255–267.

2. R. Engelking, *General Topology*, Polish Scientific Publishers, Warsaw, 1977.

3. K. Kuratowski, *Topology, vol. II*, Academic Press, New York, 1968.

4. S. Mardešić, *On the Hahn-Mazurkiewicz problem in non-metric spaces*. In: General Topology and its Relations to Modern Analysis and Algebra II, Prague, 1966, 248–255.

5. ———, *Images of ordered compacta are locally peripherally metric*, Pacific J. Math. **23**(1967), 557–568.

6. L. Mohler, *A note on hereditarily locally connected continua*, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys. **12**(1969), 699–701.

7. J. Nikiel, *Images of arcs—a nonseparable version of the Hahn-Mazurkiewicz theorem*, Fund. Math. **129** (1988), 91–120.

8. ———, *A continuous partial ordering for images of arcs*. In: General Topology and its Relations to Modern Analysis and Algebra VI, Proc. Sixth Prague Topological Symp. 1986, (ed. Z. Frolik), Heldermann Verlag, Berlin, 1988, 361–370.

9. ———, *The Hahn-Mazurkiewicz theorem for hereditarily locally connected continua*, Topology Appl. **32**(1989), 307–323.

10. ———, *On continuous images of arcs and compact orderable spaces*, Topology Proc. **14**(1989), 163–193 and 279–280.

11. J. Nikiel, H. M. Tuncali and E. D. Tymchatyn, *On the rim-structure of continuous images of ordered compacta*, Pacific J. Math. **149**(1991), 145–155.

12. ———, *A locally connected rim-countable continuum which is the continuous image of no arc*, Topology Appl. **42**(1991), 83–93.

13. ———, *Continuous images of arcs and inverse limit methods*, Mem. Amer. Math. Soc., to appear.

14. J. N. Simone, *Concerning hereditarily locally connected continua*, Colloq. Math. **39**(1978), 243–251.

15. ———, *A property of hereditarily locally connected continua related to arcwise accessibility*, J. Austral. Math. Soc. (A) **25**(1978), 35–40.

16. L. B. Treybig, *Arcwise connectivity in continuous images of ordered compacta*, Glasnik Mat. **21**(1986), 201–211.

17. L. B. Treybig and L. E. Ward, Jr., *The Hahn-Mazurkiewicz problem*. In: Topology and order structures, I Math. Centre Tracts, **142**, Amsterdam, 1981, 95–105.

18. E. D. Tymchatyn, *Compactification of hereditarily locally connected spaces*, Canadian J. Math. **29**(1977), 1223–1229.

19. G. T. Whyburn, *Concerning hereditarily locally connected continua*, Amer. J. Math. 53 (1931), 374-394.
20. ———, *Analytic topology*, Amer. Math. Soc., Providence, R.I., 1942.

Technical University of Crete
Chania, Crete
Greece

Department of Mathematics
Texas A & M University
College Station, Texas 77843
U.S.A.

Department of Mathematics
University of Saskatchewan
Saskatoon, Saskatchewan
S7N 0W0

8901 W74th Street #25
Kansas City, Missouri 66204
U.S.A.