J. Austral. Math. Soc. (*Series A*) **29** (1980), 61–70

NEAR-RINGS OF POLYNOMIALS AND POLYNOMIAL FUNCTIONS

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(Received 6 November 1978; revised 27 March 1979)

Communicated by R. Lidl

Abstract

In this paper we investigate near-rings of polynomials and polynomial functions. After some results which belong to universal algebra we turn our attention to the familiar case of polynomials and polynomial functions over a commutative ring with identity. We study the relation between ring- and near-ring homomorphisms, and the behaviour of polynomial near-rings when the ring splits into a direct sum. A discussion of the structure of these polynomial near-rings (radical, semisimplicity) finishes this paper. These investigations are motivated by Clay and Doi (1973).

1980 Mathematics subject classification (*Amer. Math. Soc.*): primary 16 A 76; secondary 08 A 40.

1. Some general concepts and results

1.1 DEFINITION. Let $A = (A, \Omega)$ be a universal algebra.

(a) $M(A) = (A^A, \Omega \cup \{\circ\})$, where $\circ$ means the composition of functions; the operations $\omega \in \Omega$ are defined pointwise in A^A.

(b) $C(A) = \{f \in M(A)$/for all congruence relations $\equiv$ on A we have that
$$a \equiv b \Rightarrow f(a) \equiv f(b) \text{ for all } a, b \in A\}.$$
The functions in $C(A)$ are said to be *compatible*.

(c) Let $P(A)$ be the subalgebra of $M(A)$ generated by id_A and the constant functions. The elements in there are called *polynomial functions*. Let $P_c(A)$ be the set of all constant maps in $P(A)$.

1.2 REMARK. We always have that $P(A) \subseteq C(A) \subseteq M(A)$. If A is an Ω-group (written additively with zero element 0) then these three algebras are near-rings with respect to addition and composition. For all notations and results concerning near-rings see Pilz (1977).

Let $A[x]$ be the algebra of polynomials in one indeterminate x over A as defined in Lausch and Nöbauer (1973) (if we want to specify the variety V of which A is taken then we write more precisely $(A[x], V)$. For all on polynomials see Lausch and Nöbauer (1973). Again, if A happens to be an Ω-group then $A[x]$ is a near-ring with respect to addition and substitution. $A[x]$ can be viewed as the free union of A and the free algebra over $\{x\}$ in V.

For later applications we compare polynomials in A and in (sub)direct components. Generalizing results of Lausch and Nöbauer (1973) and Nöbauer (1976) we get

1.3 Theorem. *Let A be a subdirect product of algebras A_i $(i \in I)$. For all $f \in C(A)$ there are uniquely determined $f_i \in C(A_i)$ with $f((\ldots, a_i, \ldots)) = (\ldots, f_i(a_i), \ldots)$ for all $(\ldots, a_i, \ldots) \in A$. If $f \in P(A)$ then all f_i are in $P(A_i)$.*

We remark that if π_i is the usual projection $A \to A_i$ then the f_i makes the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & A \\ {\scriptstyle \pi_i}\big\downarrow & & \big\downarrow{\scriptstyle \pi_i} \\ A_i & \xrightarrow{\ f_i\ } & A_i \end{array}$$

commutative.

Also, the map $f \to (\ldots, f_i, \ldots)$ is a monomorphism (see Lausch and Nöbauer (1973), Ch. 3, 3.53). After this general discussion we concentrate our attention to the much more familiar case of commutative rings with identity.

2. Homomorphisms between (near-) rings of polynomials and polynomial functions

In all that follows (unless otherwise specified) let $R, R_1, R_2, \ldots$ be commutative rings with identity 1. In this case we get from Lausch and Nöbauer (1973) (Results no. 1, 4.5; 3, 3.11; 3, 3.21; 3, 3.61) and Nöbauer (1976) or from So (1977) or Werner (1971):

2.1 Theorem.

(a) *Let Φ: $R_1[x] \to R_2[x]$ be a (near-ring)homomorphism. Then Φ / R_1 is a ring-homomorphism $R_1 \to R_2$.*

(b) *Conversely, if Φ: $R_1 \to R_2$ is a ring-homomorphism then*

$$\Phi\colon R_1[x] \to R_2[x]\colon a_0 + a_1 x + \ldots + a_n x^n \to \Phi(a_0) + \Phi(a_1)x + \ldots + \Phi(a_n)x^n$$

is a ring and near-ring homomorphism (hence a composition-ring homomorphism from $R_1[x]$ to $R_2[x]$). All composition-ring epimorphisms arise in this way.

https://doi.org/10.1017/S1446788700020930 Published online by Cambridge University Press

(c) *Let* $R = \bigoplus_{i\in I} R_i$. *Then the correspondence* $f \to (\dots, f_i, \dots)$ *of* 1.3 *is an isomorphism, hence* $C(R) = \bigoplus_{i\in I} C(R_i)$ *and* $P(R) = \bigoplus_{i\in I} P(R_i)$.

(d) *If* $R_1 \cong R_2$ *then* $R_1[x] \cong R_2[x]$, $P(R_1) \cong P(R_2)$ *and* $C(R_1) \cong C(R_2)$ (*as composition rings*).

Part (b) of this theorem settles the question about all composition-ring epimorphisms between polynomial composition-rings. It is harder to determine all near-ring homomorphisms from $R_1[x]$ to $R_2[x]$.

2.2 Lemma (So (1977)). *Let* Φ *be a near-ring homomorphism from* $R_1[x]$ *to* $R_2[x]$.

(a) *If* R_2 *has no non-zero nilpotent elements then* $\Phi(x) = ax$ *with idempotent* a.

(b) *If* R_2 *is an integral domain and* Φ *non-zero or if* Φ *is onto then* $\Phi(x) = x$.

Proof. (a) Let $\Phi(x) = a_0 + a_1x + \dots + a_nx^n$ with $a_n \neq 0$ (if $\Phi(x) = 0$, (a) is trivially true). Now $a_0 = \Phi(x)\circ 0 = \Phi(x)\circ\Phi(0) = \Phi(x\circ 0) = \Phi(0) = 0$, and

$$\begin{aligned} a_nx^n + \dots + a_0 = \Phi(x) = \Phi(x\circ x) = \Phi(x)\circ\Phi(x) &= a_0 + a_1\Phi(x) + \dots + a_n(\Phi(x))^n \\ &= a_n^{n+1}x^{n^2} + \dots. \end{aligned}$$

Hence $n = 1$ and $a_1 = a_1^2$.

(b) If R_2 is even an integral domain, then either $a = 0$ (in this case we get $\Phi = \mathfrak{o}$, the zero map) or $a = 1$. If Φ is an epimorphism, $\Phi(x)$ is (as the image of the identity of $R_1[x]$) the identity in $R_2[x]$, that is $\Phi(x) = x$.

So we know something about the image of $x \in R_1[x]$. Every ring homomorphism from $R_1[x]$ to $R_2[x]$ is already determined by the images of 1 and x. However, this is not true for the near-ring homomorphisms:

2.3 Example (So (1977)). Let p be an odd prime.

(a) Define Φ: $\mathbf{Z}_{2p}[x] \to \mathbf{Z}_{2p}[x]$ by $\Phi(a_0 + \dots + a_nx^n) := p(a_0 + \dots + a_nx^n)$. Then Φ is a near-ring homomorphism (but no ring homomorphism) with $\Phi(x) = px$ and $\Phi(1) = p$.

(b) Define Ψ: $\mathbf{Z}_{2p}[x] \to \mathbf{Z}_{2p}[x]$ by $\Psi(a_0 + \dots + a_nx^n) := p(a_n + \dots + a_1)x + pa_0$. Then Ψ is again a near-ring homomorphism with $\Psi(x) = px$ and $\Psi(1) = p$, but $\Phi \neq \Psi$.

Nevertheless, we can prove

2.4 Proposition. *Let* Φ: $R_1[x] \to R_2[x]$ *be a composition ring homomorphism and* R_2 *be an integral domain. Then, as in* 2.1(b),

$$\Phi(a_0 + \dots + a_nx^n) = \Phi(a_0) + \dots + \Phi(a_n)x^n.$$

PROOF. By 2.2(b), we have $\Phi(x)=x$. Since Φ is also a ring homomorphism, $\Phi(x^n)=(\Phi(x))^n$, and from that we get our result.

2.5 REMARK. One can improve 2.1(c) by

$$R_1 \cong R_2 \Leftrightarrow R_1[x] \cong R_2[x] \Leftrightarrow P(R_1) \cong P(R_2)$$

(as near-rings). See So (1977).

3. $P(\mathbf{Z}_n)$

Now we study the polynomial functions from the residue-class rings $\mathbf{Z}_n$ into itself. If $n=p_1^{k(1)}\cdots p_r^{k(r)}$ then $\mathbf{Z}_n \cong \mathbf{Z}_{p_1{}^{k(1)}} \oplus \cdots \oplus \mathbf{Z}_{p_r{}^{k(r)}}$ and hence by 2.1(c)

$$P(\mathbf{Z}_n) \cong P(\mathbf{Z}_{p_1{}^{k(1)}}) \oplus \cdots \oplus P(\mathbf{Z}_{p_r{}^{k(r)}}).$$

This reduces our attention to near-rings of the type $P(\mathbf{Z}_{p^k})$.

If $k=1$ we know that $\mathbf{Z}_p$ is a finite field and all functions $\mathbf{Z}_p \to \mathbf{Z}_p$ are polynomial functions by Lausch and Nöbauer (1973); so $P(\mathbf{Z}_p)=M(\mathbf{Z}_p)$. If $k>1$, the situation becomes much more complicated. Of course, $P(\mathbf{Z}_n)$ is in

$$C(\mathbf{Z}_n)=\{f\in M(\mathbf{Z}_n)/\text{for all } I\trianglelefteq \mathbf{Z}_n,\ i\in I \text{ and } t\in \mathbf{Z}_n \text{ we have } f(t+i)-f(t)\in I\}.$$

By the way, functions which fulfil $f(r+i)-f(r)\in I$ for all $r\in R$ and $i\in I$ (for some ideal $I\trianglelefteq R$) are called *I-loyal*. The set of all I-loyal functions forms a composition ring $C_I(R)$ between $C(R)$ and $M(R)$ and $C(R)=\bigcap_{I\trianglelefteq R} C_I(R)$. I-loyal functions are studied in So (1977).

Returning to $P(\mathbf{Z}_{p^k})$: its cardinal number is studied, for example, in Kempner (1921), Keller and Olson (1968), Müller and Eigenthaler (1979) and Nöbauer (1974); the first explicit descriptions can be found in Kempner (1921).

We give the following descriptions, taken from So (1977). The basic idea is the following: Find in $\mathbf{Z}_n[x]$ the polynomials of lowest degree (normed and not normed) which induce the zero function in $P(\mathbf{Z}_n)$. The remaining polynomials of smaller degree can be shown to yield exactly all different elements of $P(\mathbf{Z}_n)$.

This gives a possibility to describe their number as well as to characterize $P(\mathbf{Z}_n)$ in several cases. One more remark: if n is the product of distinct primes then $P(\mathbf{Z}_n)=C(\mathbf{Z}_n)$ by 2.1 (for instance).

3.1 EXAMPLES. Let $p>2$ be a prime.

$$P(\mathbf{Z}_{p^2})=\{f\in C(\mathbf{Z}_{p^2})/f(kp+c)=kf(p+c)-(k-1)f(c) \text{ for } k\in\{2,\ldots,p-1\} \text{ and } c\in\{0,\ldots,p-1\}\}.$$

$P(\mathbf{Z}_{p^3}) = \{f \in C(\mathbf{Z}_{p^3}) / \text{for all } k \in \{3,4,\ldots,p^2-1\} \text{ and all } c \in \{0,\ldots,p-1\} \text{ there are } a,b,d \in \mathbf{Z}_{p^3} \text{ with } p^2 f(p+c) = p^2 f(c) \wedge pf(2p+c) = p(2f(p+c)-f(c)) \wedge f(kp+c) = af((k-1)p+c)+bf((k-2)p+c)+df(c)\}.$

3.2 Corollary (So (1977) and Nöbauer (1976)). For $p>2$,

$$|P(\mathbf{Z}_p)| = p^p, \quad |P(\mathbf{Z}_{p^2})| = p^{3p} < p^{p+p^2} = |C(\mathbf{Z}_{p^2})|,$$
$$|P(\mathbf{Z}_{p^3})| = p^{6p} < p^{p+p^2+p^3} = |C(\mathbf{Z}_{p^3})|.$$

As usual, $p=2$ causes some trouble.

3.3 Examples.

$$P(\mathbf{Z}_4) = C(\mathbf{Z}_4) = \{f\colon x \to a_0+a_1x+a_2x^2+a_3x^3 / a_0, a_1 \in \mathbf{Z}_4 \wedge a_2, a_3 \in \{0,1\}\}$$
$$\begin{aligned} P(\mathbf{Z}_8) &= \{f\colon x \to a_0+a_1x+a_2x^2+a_3x^3 / a_0, a_1 \in \mathbf{Z}_8 \wedge a_2, a_3 \in \{0,1,2,3\}\} \\ &= \{f \in C(\mathbf{Z}_8) / f(4) = 2f(2)-f(0) \wedge f(5) = 2f(3)-f(1) \\ &\qquad \wedge f(6) = f(3)+f(4)-f(0) \wedge f(7) = 6f(1)+3f(3)\} \\ &\neq C(\mathbf{Z}_8) \end{aligned}$$

3.4 Remark. The recursion formula given in Keller and Olson (1968) is $|P(\mathbf{Z}_{p^k})| = p^{\beta(k)}|P(\mathbf{Z}_{p^{k-1}})|$ for $k \geqslant 2$, where $\beta(k)$ is the smallest $t \in \mathbf{N}$ with $p^k/t!$

4. R-subgroups

4.1 Definition. A subgroup S of $(R[x],+)$ or of $(P(R),+)$ is called an *R-subgroup* if $r.s \in S$ for all $r \in R$ and $s \in S$.

The importance of R-subgroups stems from:

4.2 Remark. If N is the near-ring $R[x]$ or $P(R)$ then every left ideal, ideal, N- or N_0-subgroup of N is an R-subgroup. This is true because of $rx \circ s = r.s$ for all $r \in R$ and s in a (left) ideal or N_0-subgroup S.

Hence R-subgroups are common generalizations of left ideals and N_0-subgroups in polynomial near-rings.

4.3 Examples.

(a) For $I \trianglelefteq R$, let $\bar{I} = \{a_0+a_1x+\ldots+a_nx^n \in R[x] / a_k \in I \text{ for all } k \geqslant 1\}$. Then $\bar{I}$ is an R-subgroup of $R[x]$, but no left ideal.

(b) Let $I \trianglelefteq R$ be such that $|R/I| = 2$, and $a \in R \setminus I$. Then
$\{p \in R[x] / p \circ a - p \circ 0 \in I\}$
is a maximal left ideal, maximal right ideal, maximal R-subgroup and a maximal ideal of $R[x]$, but in general not a ring ideal.

https://doi.org/10.1017/S1446788700020930 Published online by Cambridge University Press

4.4 THEOREM. *Let $R = \bigoplus_{i \in I} R_i$. Then every R-subgroup S of $R[x]$ or $P(R)$ is (in the group-theoretical sense) the direct sum of R_i-subgroups of $R_i[x]$ ($P(R_i)$), respectively).*

The proof is accomplished by using the fact that both $R[x]$ and $P(R)$ are near-rings with identity.

4.5 COROLLARY. *In the situation as in* 4.4, *S is maximal if and only if S is of the form $S = S_i \oplus \bigoplus_{j \neq i} R_j[x]$ (or $S = S_i \oplus \bigoplus_{j \neq i} P(R_j)$), where S_i is a maximal R_i-subgroup in $R_i[x]$ ($P(R_i)$, respectively).*

4.6 REMARK. The last two results remain true if "R-subgroup" is changed into "$R_0[x]$-subgroup" ($P_0(R)$-subgroup, respectively) or into "left ideal". Here, $R_0[x]$ and $P_0(R)$ denote the zero-symmetric parts of $R[x]$ and $P(R)$, respectively.

5. Radicals of polynomial near-rings

Again, we adopt the notations and results of Pilz (1977). Since $R[x]$ has an identity, $\mathfrak{J}_1(R[x]) = \mathfrak{J}_2(R[x])$. We get an upper bound for these radicals:

5.1 THEOREM. $\mathfrak{J}_2(R[x]) \subseteq (\mathfrak{J}(R) : R)$, *where $\mathfrak{J}(R)$ is the Jacobson radical of R.*

PROOF. Let M be a maximal ideal of R. It is shown in So (1977) that for each $a \in R$, $(M : a)$ is a maximal left ideal and a maximal $R_0[x]$-subgroup of $R[x]$. Since $(R[x])$ is the intersection of all maximal left ideals of $R[x]$ which are at the same time maximal $R_0[x]$-subgroups, we get that

$$\mathfrak{J}_2(R[x] \subseteq \bigcap_{M \text{ max}} \bigcap_{a \in R} (M : a) = \bigcap_{a \in R} \bigcap_{M \text{ max}} (M : a) = (\mathfrak{J}(R) : R).$$

5.2 COROLLARY. *If R is a semisimple ring of characteristic* 0 *then* $\mathfrak{J}_2(R[x]) = \{0\}$. *This holds since in this case* $(\mathfrak{J}(R) : R) = (\{0\} : R) = \{0\}$.

5.3 COROLLARY. *If R is an infinite field then $R[x]$ is 2-semisimple.*

5.4 PROPOSITION. *Let R be a finite field of order >2. Then* $\mathfrak{J}_2(R[x]) = (\{0\} : R)$.

PROOF. Since $\mathfrak{J}(R) = \{0\}$ we get $\mathfrak{J}_2(R[x]) \subseteq (\{0\} : R)$ by 5.1. Conversely, the radical $\mathfrak{J}(R[x])$, as defined in Clay and Doi (1973), is there shown to be $= (\{0\} : R)$; one easily sees that $\mathfrak{J}(R[x]) \subseteq \mathfrak{J}_2(R[x])$. Hence

$$(\{0\}) : R) = \mathfrak{J}_2(R[x]).$$

5.5 THEOREM. *Let R be a field with char $R \neq 2$. Then* $\mathfrak{J}_{\frac{1}{2}}(R[x]) = \{0\}$.

https://doi.org/10.1017/S1446788700020930 Published online by Cambridge University Press

PROOF. By 7.94 of Pilz (1977), the left ideals of $R[x]$ are under these assumptions just the ring-ideals; the maximal ones are those (ring-)ideals which are generated by irreducible polynomials. But their intersection is zero.

To handle the case of characteristic 2 to some extent, we have to consider when $R_0[x]$ happens to be a ring.

5.6 THEOREM. *Let R be a ring (not necessarily commutative with identity).*

(a) *If R has an identity then: $R_0[x]$ is a ring$\Rightarrow P_0(R)$ is a ring$\Leftrightarrow R$ is Boolean.*

(b) *If R is simple then: $R_0[x]$ is a ring$\Rightarrow P_0(R)$ is a ring$\Leftrightarrow R \cong \mathbf{Z}_2 \vee R$ is a zero ring.*

PROOF. Let R be arbitrary and $R_0[x]$ a ring; then $P_0(R)$ is a ring. Let $\mathrm{id}_R = i$, $i^2 = i.i$, and so on.

(1) $i^2 \circ (i+i) = i^2 \circ i + i^2 \circ i$ implies $i^2 + i^2 = 0$, hence for all $r \in R$,

$$0 = (i^2+i^2)(r) = r^2 + r^2.$$

(2) For all $r \in R$, $i^2 \circ (i+ri) = i^2 \circ i + i^2 \circ (ri)$; so $i^2 + iri + ri^2 + riri = i^2 + riri$. As in (1) we get $(sr+rs)s = 0$ for all s, $r \in R$.

(3) For all $t \in R$, $i^2 \circ (i+it) = i^2 \circ i + i^2 \circ it$ implies that $r(rt+tr) = 0$ for r, $t \in R$.

(4) By (3) and (2) we get $a^2 b = ba^2$ $(a, b \in R)$.

(5) From that and (2) one deduces that for all $a, b \in R$, $(a+b)^3 = a^3 + a^2 b + ab^2 + b^3$.

(6) Now

$$\begin{aligned}(a+b)^4 &= (a+b)^3 (a+b) \\ &= (a^3 + a^2 b + ab^2 + b^3)(a+b) \\ &= a^4 + a^2 ba + ab^2 a + b^3 a + a^3 b + a^2 b^2 + ab^3 + b^4 \\ &= a^4 + a^3 b + a^3 b + a^2 b^2 + a^2 b^2 + ab^3 + ab^3 + b^4 \\ &= a^4 + b^4\end{aligned}$$

by (1)–(3).

(7) Since $i^3 \circ (i+i^2) = i^3 \circ i + i^3 \circ i^2$ we get for $a \in R$: $(a+a^2)^3 = a^3 + (a^2)^3$. By using (5) we arrive at $a^4 = a^5 = a^8$ for all $a \in R$.

(8) Since $a^4 + b^4 = (a+b)^4 = (a+b)^5 = (a+b)^4 (a+b) = a^4 + b^4 a + a^4 b + b^4$ we get for all $a, b \in R$: $a^4 b = ab^4$.

(9) Now suppose that R contains an identity. Then by (8) with $b = 1$ we see that for all $a \in R$: $a^4 = a$, hence $a^2 = (a^4)^2 = a^8 = a$, and R is shown to be Boolean.

(10) Conversely, if R is Boolean, then it remains to show that $P_0(R)$ fulfils the left distributive law. Since a Boolean ring is a commutative ring with identity, we

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can use the usual normal form for polynomials. Consider

$$\Big(\sum_{k\geqslant 1} a_k i^k\Big)\circ\Big(\sum_{j\geqslant 0} b_j i^j+\sum_{j\geqslant 0} c_j i^j\Big)=\sum_k a_k\Big(\sum_j (b_j+c_j)\, i^j\Big)^k=f,$$

and

$$\Big(\sum_k a_k i^k\Big)\circ\Big(\sum_j b_j i^j\Big)+\Big(\sum_k a_k i^k\Big)\circ\Big(\sum_j c_j i^j\Big)$$
$$=\sum_k a_k\Big(\sum_j b_j i^j\Big)^k+\sum_k a_k\Big(\sum_j c_j i^j\Big)^k=g.$$

Now for all $r\in R$:

$$f(r)=\sum_k a_k\Big(\sum_j (b_j+c_j)\, r^j\Big)^k$$
$$=\sum_k a_k\Big(\sum_j (b_j+c_j)\, r\Big)=\ldots=g(r).$$

(11) Suppose now that R is simple. Take some $a\in R$. We show that Ra^2 is an ideal of R. Of course, it is a left ideal. Now take $b\in R$, and some $ra^2\in Ra^2$. Then $(ra^2)b=r(a^2 b)=rba^2\in Ra^2$. If $Ra^2\neq\{0\}$ then by the simplicity of R, $Ra^2=R$. Hence there is some

$$e\in R\colon ea^2=a^2.$$

But then for all

$$c=da^2\in R\colon ec=eda^2=ea^2 d=a^2 d=da^2=c$$

and similarly $ce=c$. Hence R has an identity and R must be Boolean by (9). But a simple Boolean ring is isomorphic to $\mathbf{Z}_2$.

(12) Now suppose that R is simple, but $Ra^2=\{0\}$ for all $a\in R$. Since

$$A(R):=\{x\in R/Rx=\{0\}\}\trianglelefteq R,$$

either $A(R)=R$ (whence $R^2=\{0\}$ and R is a zero ring) or $A(R)=\{0\}$. Assume that $A(R)=\{0\}$. Since all $a^2\in A(R)$, $a^2=0$ for all $a\in R$. Also,

$$0=(a+b)^2=a^2+ab+ba+b^2=ab+ba,$$

so for all $a,b\in R$: $ab+ba=0$. But then all Ra are ideals of R as can be seen as in (11). Since $R^2\neq\{0\}$, there is some $a_0\in R$ with $Ra_0\neq\{0\}$, hence $Ra_0=R$. Again we can deduce the existence of an identity, a contradiction to $a^2=0$ for each $a\in R$.

(13) Conversely, if R is either a zero ring or if $R=\mathbf{Z}_2$ then $P_0(R)$ is easily shown to be a ring.

5.7 Remark. Even if $R=\mathbf{Z}_2$, $R_0[x]$ is not a ring:

$$x^3\circ(x+x^2)=x^3\circ x+x^3\circ x^2$$

would result in the impossible equation $x^4+x^5=0$.

5.8 COROLLARY. *Let R be a ring with identity. P(R) is an abstract affine near-ring if and only if R is a Boolean ring. This follows from a quick inspection of part* (10) *of the last proof: zero symmetric polynomial functions distribute even over all other polynomial functions.*

5.9 COROLLARY. *Let R be a Boolean ring. Then*

$$\mathfrak{J}_0(P(R)) = \dots = \mathfrak{J}_2(P(R)) = \mathfrak{J}(P_0(R)) + \mathfrak{J}({}_{P_0(R)}P_c(R)),$$

where $\mathfrak{J}(P_c(R))$ *is the "Jacobson-radical" of the* $P_0(R)$*-module* $P_c(R)$ *(which is isomorphic to R itself), namely the intersection of all maximal submodules.*

PROOF. This result follows immediately from 5.8 and Theorem 9.77 of Pilz (1977).

We close our considerations by a result which can be deduced from 4.5 and 4.6 as is done for $\mathfrak{J}_2$ in Pilz (1977).

5.10 THEOREM (So (1977). *Let* $R \cong \bigoplus_{i\in I} R_i$. *Then for all*

$$v \in \{0, \tfrac{1}{2}, 1, 2\} : \mathfrak{J}_v(R[x]) \cong \bigoplus_{i\in I} \mathfrak{J}_v(R_i[x])$$

and

$$\mathfrak{J}_v(P(R)) \cong \bigoplus_{i\in I} \mathfrak{J}_v(P(R_i)).$$

5.11 COROLLARY (So (1977)). *Let* $n \in \mathbf{N}$ *be a product of distinct primes. Then*

$$\mathfrak{J}_2(P(\mathbf{Z}_n)) = \{0\}.$$

This follows from 5.10 and the fact that for prime p

$$\mathfrak{J}_2(P(\mathbf{Z}_p)) = \mathfrak{J}_2(M(\mathbf{Z}_p)) = \{0\}$$

(Pilz (1977)).

The authors thank the "Österreichische Forschungsfonds" (Projekt Nr. 3479) for highly appreciated assistance.

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