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THE ABBENA-THURSTON MANIFOLD AS A CRITICAL POINT

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ABSTRACT. The Abbena-Thurston manifold (M, g) is a critical point of the functional $I(g) = \int_M(\frac{4}{3}\operatorname{tr}Q^3 - R)\,dV_g$, where Q is the Ricci operator and R is the scalar curvature, and then the index of $I(g)$ and also the index of $-I(g)$ are positive at (M, g).

1. **Introduction.** Let M be a compact symplectic manifold with a symplectic form Ω. From $\Omega(X, Y) = g(X, JY)$, g and J are created simultaneously by polarization. A metric g created in this way is called an associated metric and the set of these metrics will be denoted by $\mathcal{A}$. In particular $\mathcal{A}$ is the set of all almost Kähler metrics on M which have Ω as their fundamental 2-form. Let $\mathcal{M}$ be the set of all Riemannian metrics of volume 1 on M. The *$*$-Ricci tensor* and the *$*$-scalar curvature* of an almost Hermitian manifold are defined by

$$R^*_{ij} := R_{iklt}J^{kl}J^t_j, \quad R^* := R^{*i}_i,$$

where R_{iklt} is the component of the curvature tensor.

Blair and Ianus [3] showed that $g \in \mathcal{A}$ with $QJ = JQ$ is a critical point of $K(g) := \int_M (R - R^*)\,dV$ and $H(g) := \int_M R\,dV$ on $\mathcal{A}$. Here, R is the scalar curvature of (M, g). Since $R - R^* = -\frac{1}{2}|\nabla J|^2$ [7], Kähler metrics are maxima of functional $K(g)$. Moreover, Y. Muto [4] has studied whether a given Einstein metric gives a minimum of $H(g)$ or not.

It is natural to ask for some concrete functional I on $\mathcal{A}$ (or $\mathcal{M}$) such that a given metric g_o is a critical point of functional I. And then, it is interesting to compute the second derivative of $I(g)$ at the critical point g_o.

In this paper, we show that the *Abbena-Thurston manifold*, which is a compact symplectic and not Käehlerian and not Einstein manifold, is a critical point of some function $I(g)$, and investigate the index of $I(g)$ and the index of $-I(g)$.

2. **A.-T. manifold as a critical point.** Let G be the closed connected subgroup of $GL(4, C)$ defined by

$$\left\{ \begin{pmatrix} 1 & a_{12} & a_{13} & 0 \\ 0 & 1 & a_{23} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{2\pi i a} \end{pmatrix} \middle|\, a_{12}, a_{13}, a_{23}, \alpha \in R \right\},$$

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i.e., $G = H \times S^1$ is the product of the Heisenberg group H and S^1. Let Γ be the discrete subgroup of G with integer entries and $M = G/\Gamma$. Denote by x, y, z, t coordinates on G, say for $A \in G, x(A) = a_{12}, y(A) = a_{23}, z(A) = a_{13}, t(A) = a$. If L_B is left translation by $B \in G, L_B^* dx = dx, L_B^* dy = dy, L_B^*(dz - x\,dy) = dz - x\,dy, L_B^* dt = dt$. In particular, these forms are invariant under the action of Γ; let $\pi: G \longrightarrow M$, then there exist 1-forms $\alpha_1, \alpha_2, \alpha_3$ and α_4 on M such that $dx = \pi^*\alpha_1$, $dy = \pi^*\alpha_2, dz - x\,dy = \pi^*\alpha_3$ and $dt = \pi^*\alpha_4$. Setting $\Omega = \alpha_4 \wedge \alpha_1 + \alpha_2 \wedge \alpha_3$, we see that $\Omega \wedge \Omega \neq 0$ and $d\Omega = 0$ on M giving M a symplectic structure. The vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial t}$$

are dual to $dx, dy, dz - x\,dy, dt$ and are left invariant. Moreover, $\{e_i\}$ is orthonormal with respect to the left invariant metric on G given by

$$ds^2 = dx^2 + dy^2 + (dz - x\,dy)^2 + dt^2.$$

On M, the corresponding metric is $g = \sum \alpha_i \otimes \alpha_i$. The Riemannian manifold (M, g) is referred to as the *Abbena-Thurston manifold*. Moreover, M carries an almost complex structure J defined by

$$Je_1 = e_4, \quad Je_2 = -e_3, \quad Je_3 = e_2, \quad Je_4 = -e_1.$$

Then noting that $\Omega(X, Y) = g(X, JY)$, we see that g is an associated metric.

The curvature of g was computed by E. Abbena in [1]. With respect to the basis $\{e_i\}$, the non-zero components of the curvature tensor are

$$K_{1221} = -\frac{3}{4}, \quad K_{2332} = \frac{1}{4}, \quad K_{1331} = \frac{1}{4}.$$

Thus the Ricci operator Q is given by the matrix

$$\begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and we note that Q^2 is parallel with respect to the Levi-Civita connection of g but that Q is not parallel.

REMARK. From the expression for Q it is clear that (M, g) is not Einstein nor is $QJ = JQ$. Thus the metric is not a critical point of $H(g) := \int_M R\,dV_g$ considered as a functional on $\mathcal{M}$ or on $\mathcal{A}$ or for $K(g) := \int_M (R - R^*)\,dV_g$ on $\mathcal{A}(\subset \mathcal{M})$. Here, $\mathcal{M}$ is the set of all Riemannian metrics of volume 1 on $G/\Gamma = M$, and $\mathcal{A}$ is the set of all associated metrics on (M, Ω).

In the following we use local coordinates, and tensors are expressed in their components with respect to the natural frame. When we take a C^∞ curve $g(t)$ in $\mathcal{M}$, we get several tensor fields defined by

$$\begin{gathered} D_{ji} = \frac{\partial}{\partial t} g_{ji}, \quad D_i^h = D_{ik} g^{kh}, \quad D^{ih} = D_{kj} g^{ki} g^{jh}, \\ D_{ji}^h = \frac{1}{2}(\nabla_j D_i^h + \nabla_i D_j^h - \nabla^h D_{ji}), \\ D_{kji}^h = \nabla_k D_{ji}^h - \nabla_j D_{ki}^h, \end{gathered} \tag{1}$$

where ∇ means the covariant differentiation with respect to the metric $g(t)$. Then,

$$\frac{\partial}{\partial t}\left\{ {h \atop ji} \right\} = D_{ji}^h, \quad \frac{\partial}{\partial t} K_{kji}^h = D_{kji}^h, \quad \frac{\partial}{\partial t} R_{ji} = \nabla_s D_{ji}^s - \nabla_j D_{si}^s, \tag{2}$$

where $\left\{ {h \atop ji} \right\}$, K_{kji}^h and R_{ji} denote the Christoffel symbol of the metric g, the components of the curvature and Ricci tensors respectively.

Then we get

PROPOSITION 1. *The Abbena-Thurston manifold is a critical point of the functional*

$$I(g) = \int_M \left(\frac{4}{3} \operatorname{tr} Q^3 - R\right) dV_g$$

$\mathcal{M}$, *where R is the scalar curvature.*

PROOF. By straightforward computation, we get in general

$$\begin{aligned} \frac{d}{dt} I\big(g(t)\big) = \int_M [2(&\nabla_m \nabla_i R_{jk} R^{km} + \nabla_m \nabla_j R_{ik} R^{km} - \nabla^m \nabla_m R_i^k R_{kj} \\ &- g_{ij} \nabla_m \nabla_l R^{lk} R_k^m - 2 R_j^k R_k^m R_{mi} + \frac{1}{2} R_{ij}) \\ &+ \frac{1}{2}\left(\frac{4}{3} \operatorname{tr} Q^3 - R\right) g_{ij}] D^{ij} \, dV_g. \end{aligned} \tag{3}$$

Since Q^2 is parallel and $Q^3 = \frac{1}{4} Q$ on the *Abbena-Thurston manifold*, we see [3, Lemma of p. 25] that this metric on the underlying manifold $M = G/\Gamma$ is a critical point of $I(g)$.

REMARK. This Proposition stems from conversations between D. E. Blair and the second author.

Now, differentiating $\int_M dV = 1$, we get

$$\begin{gathered} \int_M g^{ji} \frac{\partial g_{ji}}{\partial t} \, dV = 0, \\ \int_M g^{ji} \frac{\partial^2 g_{ji}}{\partial t^2} \, dV = \int_M [D^{ji} D_{ji} - \frac{1}{2} (D_i^i)^2] \, dV. \end{gathered} \tag{4}$$

Using general facts (2),(4) and Green's Theorem, and the facts $\operatorname{tr} Q = -\frac{1}{2}$, $Q^3 = \frac{1}{4}Q$ and $\nabla Q^2 = 0$ on (M, g), we get by computing

$$
\begin{aligned}
(5)\qquad \left(\frac{d^2 I(g)}{dt^2}\right)_0 = \int_M [&(\nabla^i D^j_j)(\nabla^h D_{hi}) + \frac{1}{2}(\nabla^h D^{ji})(\nabla_h D_{ji}) \\
&- (\nabla^j D^{ih})(\nabla_h D_{ji}) - \frac{1}{2}(\nabla^l D^s_s)(\nabla_l D^i_i) + 2R^{is}R^k_s(\nabla_i D^l_l)(\nabla_k D^j_j) \\
&+ 2R^{sj}(\nabla_l \nabla_j R^l_i)(\nabla_k \nabla_s R^{ik}) + 2R^{is}(\nabla_l \nabla_j D^l_i)(\nabla_k \nabla^j D^k_s) \\
&+ 4R^{sj}(\nabla_k \nabla_i D^k_s)(\nabla_l \nabla_j D^{il}) - 2R^{is}(\nabla_l \nabla^l D_{ji})(\nabla_k \nabla^k D^j_s) \\
&- 8R^{is}(\nabla_k \nabla^k D^j_s)(\nabla_l D^l_{ji}) + 8R^{sj}(\nabla_j D^l_{li})(\nabla^i D^k_{ks}) \\
&- 16R^j_s(\nabla_j D^l_{li})(\nabla_k D^{isk}) - 8R_{sb}D^{bj}R_{ji}(\nabla_l D^{sil}) \\
&+ 16R^{js}R_{sk}D^{ki}(\nabla_j D^l_{li}) + 8R^{ik}D_{ks}R^{sj}(\nabla_j D^l_{li}) \\
&+ 4R^{si}R_{ij}D^j_{bl}(\nabla_s D^{bl}) + 8R^{si}R_{ij}D^l_{bs}(\nabla_l D^{jb}) \\
&+ 8D^{jb}R_{bs}R^s_l D^{li}R_{ij}]\, dV_g.
\end{aligned}
$$

The right hand side of (4) is a functional of the tensor field D_{ji}. Denote this integral by $J(D)$.

DEFINITION 2. Let $\mathcal{D}$ be the set of all symmetric tensor fields D

$$
(6)\qquad \int_M \operatorname{tr} D\, dV = 0.
$$

Let us say that the dimension of the vector space $\{D \in \mathcal{D} \mid J(D) < 0\}$ (resp. $\{D \in \mathcal{D} \mid -J(D) < 0\}$) is the *index* of the functional $I(g)$ (resp. $-I(g)$) at the critical point (M, g) of $I(g)$.

Then we obtain

THEOREM 3. *Let $I(g)$ be the integral as defined in Proposition* 1. *Then the index of $I(g)$ and also the index of $-I(g)$ are positive at the Abbena-Thurston metric on $\mathcal{M}$.*

PROOF. If we put $D_{ji} = fg_{ji}$ where f is a C^∞ function such that $\int_M f\, dV = 0$, then we have from (4)

$$
\begin{aligned}
(7)\qquad \left(\frac{d^2 I(g)}{dt^2}\right)_{t=0} = \int_M [&8R^{kj}(\nabla_k \nabla^l f)(\nabla_l \nabla_j f) + 8R^{ij}(\nabla_j \nabla_i f)(\nabla_k \nabla^k f) \\
&- 4R^{li}R_i{}^b(\nabla_l f)(\nabla_b f) - f^2 - 9(\nabla^i f)(\nabla_i f) \\
&- (\nabla_l \nabla^l f)(\nabla_k \nabla^k f)]\, dV.
\end{aligned}
$$

All the local calculations on M will be done on G and on its Lie algebra $\mathfrak{g}$ because G is locally isomorphic to M. Let x^1, x^2, x^3, x^4 be local coordinates of M and $G = H \times S^1$ such that x^1, x^2, x^3 are local coordinates of H and x^4 is a local coordinates of S^1. The local components R_4^4 and R^{44} with respect to local coordinates x^1, x^2, x^3, x^4 of (M, g) are zero. We can choose functions f on M which make (7) negative. This proves that the index of $I(g)$ is positive.

Now, let's prove that the index of $-I(g)$ is positive.

Let U be a coordinate neighbourhood of M, and let $N \subset U$ be a neighbourhood of a point $p_0 \in U$, where the local coordinates are such that

$$g_{ji} = \delta_{ji}, \quad \{{}^{h}_{ji}\} = 0$$

at p_0. We assume that N is sufficiently small so that there exists a positive number ε such that g satisfies in N

$$|g_{ij} - \delta_{ij}| < \epsilon, \quad |g^{ij} - \delta_{ij}| < \epsilon, \quad |\{{}^{h}_{ji}\}| < \epsilon.$$

We want to take a suitable C^∞ tensor field D_{ji}. We know that for any given tensor field D_{ji} there exist $g(t)$ such that

$$\left(\frac{\partial g_{ji}}{\partial t}\right)_0 = D_{ji}.$$

First we assume $D_l^l = 0$ on M. Then we get from (1) and (5)

$$\left(-\frac{d^2 I(g)}{dt^2}\right)_0 = \int_M (F_1 + F_2 + F_3 + F_4)\, dV, \tag{8}$$

where

$$F_1 := (\nabla^j D^{ih})(\nabla_h D_{ji}) - \frac{1}{2}(\nabla^h D_{ji})(\nabla_h D^{ji}) - 8D^{jk} R_{ks} R_l^s D^{li} R_{ij}, \tag{9}$$

$$F_2 := 8R_{sk} D^{kj} R_{ji} (\nabla_l \nabla^s D^{il}) - 4R_{sk} D^{kj} R_{ji} (\nabla_l \nabla^l D^{si}), \tag{10}$$

$$\begin{aligned} F_3 := {} & 2R^{ks} R_s^j (\nabla_j D_i^l)(\nabla_k D_l^i) - 4R^{si} R_{ij} (\nabla_l D^{jk})(\nabla_k D_s^l) \\ & + 4R^{si} R_{ij} (\nabla_l D^{jk} (\nabla^l D_{ks}) - 8R^{si} R_{ij} (\nabla_l D^{jk})(\nabla_s D_k^l), \end{aligned} \tag{11}$$

$$\begin{aligned} F_4 := {} & -2R_s^j (\nabla_l \nabla_j D_i^l)(\nabla_k \nabla^s D^{ik} - 2R^{is} (\nabla_l \nabla_j D_i^l)(\nabla_k \nabla^j D_s^k) \\ & - 2R^{sj} (\nabla_l \nabla^l D_{ij})(\nabla_k \nabla^k D_s^i) - 4R^{sj} (\nabla_k \nabla_i D_s^k)(\nabla_l \nabla_j D^{il}) \\ & + 4R^{sj} (\nabla_l \nabla^l D_{ij})(\nabla_k \nabla_s D^{ik}) + 4R^{is} (\nabla_l \nabla_j D_i^l)(\nabla_k \nabla^k D_s^j). \end{aligned} \tag{12}$$

Define S_{ji} by

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$$g^{ji} = \delta_{ji} + \epsilon S_{ji}.$$

Then S_{ji} satisfy $|S_{ji}| < 1$ on N.

Assume D_{ji} vanishes everywhere except in the interior of N, and define M_1, M_2, M_3 and M_4 by

$$\begin{aligned} M_1 &:= \max\{|D_{ji}(p)|; p \in N; i,j = 1,2,3,4\}, \\ M_2 &:= \max\{|\partial_j D_{ih}(p)|; p \in N; j,i,h = 1,2,3,4\}, \\ M_3 &:= \sup\{|\partial_j \{{}^{l}_{ik}\}(p)|; p \in N; l,i,j,k = 1,2,3,4\}, \\ M_4 &:= \max\{|\partial_l(\partial_j D_{ik})(p)|; p \in N; l,i,j,k = 1,2,3,4\}. \end{aligned} \tag{13}$$

From (13), we obtain on N

$$\begin{aligned} |R_{ji}| &\le 8M_3 + o(\varepsilon^2), \\ |\nabla_j D_{il}| &\le M_2 + 8M_1\varepsilon, \\ |\partial_l(\nabla_j D_{ik})| &\le M_5, \end{aligned} \tag{14}$$

where $M_5 := M_4 + 8M_1M_3 + 8M_2\varepsilon$, and $j,i,l,k = 1,\ldots,4$. In the following we put $n = 4$. Using (13) and (14), we find

$$\begin{aligned} &(\text{The first term of } F_4) \\ &\quad = -2R_{cb}(\nabla_l\nabla_j D_{ie})(\nabla_k\nabla_s D_{da})g^{cs}g^{bj}g^{el}g^{di}g^{ak} \\ &\quad = \sum_{c,b}\sum_{l,j,i,e}\sum_{k,s,d,a} -2R_{cb}[\partial_l(\nabla_j D_{ie}) - \{{}^{r}_{lj}\}\nabla_r D_{ie} - \{{}^{r}_{li}\}\nabla_j D_{re} - \{{}^{r}_{le}\}\nabla_j D_{ir}] \\ &\qquad \cdot [\partial_k(\nabla_s D_{da}) - \{{}^{q}_{ks}\}\nabla_q D_{da} - \{{}^{q}_{kd}\}\nabla_s D_{qa} - \{{}^{q}_{ka}\}\nabla_s D_{dq}] \\ &\qquad \cdot (\delta_{cs} + \varepsilon S_{cs})(\delta_{bj} + \varepsilon S_{bj})(\delta_{el} + \varepsilon S_{el})(\delta_{ak} + \varepsilon S_{ak})(\delta_{di} + \varepsilon S_{di}) \\ &\quad \le 4n^6M_3(M_4)^2 + 16n^7M_1(M_3)^2M_4 + 16n^8(M_1)^2(M_3)^3 \\ &\qquad + \big(32n^8M_1M_2M_3 + 16n^7M_2M_4 + 24n^7M_2M_5 + 20n^8(M5)^2\big)M_3\varepsilon + o(\varepsilon^2). \end{aligned}$$

Similarly, we get by computing

$$\begin{aligned} F_1 \le{}& \Big|\sum_{h,i,j}(\partial_j D_{ih} + \partial_i D_{jh})\partial_h D_{ji}\Big| - \sum_{h,i,j}(\partial_j D_{ih})^2 + 64n^8M_1^2M_3^3 \\ &+ 12n^4M_1M_2\varepsilon + 9n^4(M_2)^2\varepsilon + 320n^8M_1^2M_3^3 + o(\varepsilon^2), \end{aligned} \tag{15}$$

$$\begin{aligned} F_2 \le{}& 48n^7M_1(M_3)^2M_4 + 96n^8(M_1)^2(M_3)^3 \\ &+ 240n^7M_1M_3(n\,M_2M_3 + M_3M_4 + 2n\,M_1(M_3)^2) + o(\varepsilon^2), \end{aligned} \tag{16}$$

$$F_3 \le 72n^7(M_2)^2(M_3)^2 + (360M_2 + 288n\,M_1)n^7M_2(M_3)^2\varepsilon + o(\varepsilon^2), \tag{17}$$

(18)
$$F_4 \leq 36n^6\big(M_4^2 + 4nM_1M_3M_4 + 4n^2(M_1)^2(M_3)^2\big)M_3 + \big(288nM_1M_2M_3 + 144M_2M_4 + 216M_2M_5 + 180n(M_5)^2\big)n^7M_3\varepsilon + o(\varepsilon^2).$$

Now let us consider a tensor field T_{ji}, which vanishes everywhere except in the interior of N, such that all components are identically zero except

$$T_{12} = T_{21} = f,$$

where f is a C^∞ function. By putting

$$D_{ji} := T_{ji} - \frac{1}{n}T_{lk}g^{lk}g_{ji} = T_{ji} - \frac{1}{2}f\,g^{12}g_{ji},$$

we get $D_i^i = 0$ and

$$|\partial_j D_{ih} - \partial_j T_{ih}| \leq (|f| + \frac{1}{2}|\partial_j f|)\delta_{ih}\varepsilon + o(\varepsilon^2).$$

Hence,
$$M_1 = (\max|f|)\big(1 + o(\varepsilon)\big), \quad M_2 \leq \max(|\partial_j f| + |f|\varepsilon)\big(1 + o(\varepsilon^2)\big)$$

and $M_4 = \max|\partial_l\partial_j f|$. Moreover, M_3 is constant which is the geometric quantity of (M, g) and $M_5 = M_4 + 8M_1M_3 + 8M_2\varepsilon$. Therefore, we can neglect all minor terms in $F_1 + F_2 + F_3 + F_4$. Now we replace $\partial_j D_{ih}$ by $\partial_j T_{ih}$ to obtain

$$-\left(\frac{d^2I}{dt^2}\right)_0 \leq \int_M [\sum_{h,i,j}(\partial_j T_{ih})(\partial_h T_{ji}) - \frac{1}{2}\sum_{h,i,j}(\partial_j T_{ih})^2 + C(M_1, M_2, M_3, M_4)]\,dV$$
$$= \int_M [-(\partial_3 f)^2 - (\partial_4 f)^2 + C(M_1, M_2, M_3, M_4)]\,dV,$$

where $C(M_1, M_2, M_3, M_4) := 304n^8(M_1)^2(M_3)^3 + 192n^7M_1(M_3)^2M_4 + 72n^7(M_2)^2(M_3)^2 + 36n^6M_3(M_4)^2$.

As there exist functions f on (M, g) for which the last integral is negative, the index of $-I(g)$ is positive.

Thus we have proved this theorem.

REMARK. (M, g) is also a critical point for K in a different context; C. M. Wood [6] showed that the Abbena metric on the Thurston manifold is a critical point of K defined with respect to variations through almost complex structures J which preserve g. For this problem the critical point condition is $[J, \nabla^*\nabla J] = 0$, where $\nabla^*\nabla J$ is the rough Laplacian of the metric in question.

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