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Modular operads

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Abstract. We develop a 'higher genus' analogue of operads, which we call modular operads, in which graphs replace trees in the definition. We study a functor F on the category of modular operads, the Feynman transform, which generalizes Kontsevich's graph complexes and also the bar construction for operads. We calculate the Euler characteristic of the Feynman transform, using the theory of symmetric functions: our formula is modelled on Wick's theorem. We give applications to the theory of moduli spaces of pointed algebraic curves.

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0. Introduction

Recently, there has been increased interest in applications of operads outside homotopy theory, much of it due to the relation between operads and moduli spaces of algebraic curves.

The formalism of operads is closely related to the combinatorics of trees [11], [13]. However, in dealing with moduli spaces of curves, one encounters general graphs, the case of trees corresponding to curves of genus 0.

This suggests considering a 'higher genus' analogue of the theory of operads, in which graphs replace trees. We call the resulting objects modular operads: their systematic study is the purpose of this paper.

The cobar functor B [13] is an involution on the category of differential graded (dg) operads. We will construct an analogous functor F on the category of dg-modular operads, the Feynman transform. This functor generalizes Kontsevich's graph complexes [24].

The behaviour of F is more mysterious than that of the cobar construction. For example, for such a simple operad as $\mathcal{C}om$, describing commutative algebras, $\mathsf{B}\mathcal{C}om$ is a resolution of the Lie operad. On the other hand, knowledge of the homology of $\mathsf{F}\mathcal{C}om$ implies complete information on the dimensions of the spaces of Vassiliev invariants of knots (by a theorem of Kontsevich and Bar-Natan [2]; see (5.10)).

Our main result about the Feynman transform is the calculation of its Euler characteristic; to do this, we use the theory of symmetric functions. As a model for this calculation, take the formula for the enumeration of graphs known in mathematical physics as Wick's Theorem [3]. Consider the asymptotic expansion of the integral

$$W(\xi,\hbar) = \log \int \operatorname{Exp} \frac{1}{\hbar}\left(x\xi - \frac{x^2}{2} + \sum_{2(g-1)+n>0} \frac{a_{g,n}\hbar^g x^n}{n!} \right) \frac{\mathrm{d}x}{\sqrt{2\pi\hbar}}, \quad (0.1)$$

considered as a power series in ξ and $\hbar$. (The asymptotic expansion is independent of the domain of integration, provided it contains 0.) Let $\Gamma(\!(g,n)\!)$ be the set of isomorphism classes of connected graphs G, with a map $v \mapsto g(v)$ from the vertices $\operatorname{Vert}(G)$ of G to $\{0,1,2,\ldots\}$ and having exactly n legs numbered from 1 to n, such that

$$g = \sum_{v\in\operatorname{Vert}(G)} g(v) + b_1(G),$$

where $b_1(G)$ is the first Betti number of the graph. If v is a vertex of G, denote by $n(v)$ its valence, and let $|\operatorname{Aut}(G)|$ be the cardinality of the automorphism group of G. Wick's Theorem states that

$$W \sim \frac{1}{\hbar}\left(\frac{\xi^2}{2} + \sum_{2(g-1)+n>0} \frac{\hbar^g\xi^n}{n!} \sum_{G\in\Gamma(\!(g,n)\!)} \frac{1}{|\operatorname{Aut}(G)|} \prod_{v\in\operatorname{Vert}(G)} a_{g(v),n(v)} \right). \quad (0.2)$$

The calculation of the Euler characteristic of F is a natural generalization of this, in which the coefficients $a_{g,n}$ are replaced by representations $\mathcal{V}(\!(g,n)\!)$ of the symmetric groups $\mathbb{S}_n$, sums and products are replaced by the operations $\oplus$ and $\otimes$, and the weight $|\operatorname{Aut}(G)|^{-1}$ is replaced by taking the coinvariants with respect to a natural action of $\operatorname{Aut}(G)$. Up to isomorphism, a sequence $\mathcal{V} = \{\mathcal{V}(\!(g,n)\!) \mid 2(g-1)+n>0\}$ of $\mathbb{S}_n$-modules is determined by its Frobenius characteristic, which is a symmetric function $f(x_1,x_2,\ldots)$ in infinitely many variables. In Sections 7 and 8, we define analogues of the Legendre and Fourier transforms for symmetric functions, which give formulas for the characteristics of $\mathsf{B}\mathcal{A}$ and $\mathsf{F}\mathcal{A}$, where $\mathcal{A}$ is a cyclic, respectively modular, operad.

There is an Euler characteristic associated to orbifolds, called the virtual Euler characteristic. This is the invariant obtained by giving each cell σ a coefficient $(-1)^{\dim(\sigma)}/|\operatorname{Aut}(\sigma)|$, very much like in (0.2). Harer and Zagier [16] calculated the virtual Euler characteristics of the moduli spaces $\mathcal{M}_{\gamma,\nu}$ and $\mathcal{M}_{\gamma,\nu}/\mathbb{S}_\nu$ (see also Penner [31]), and more recently, Kontsevich has given a very simple proof of their result using Wick's Theorem [23].

Let $|\mathcal{M}_{\gamma,\nu}|$ be the coarse moduli space of smooth algebraic curves of genus γ with ν labelled marked points, and $|\mathcal{M}_{\gamma,\nu}|/\mathbb{S}_\nu$ that of smooth algebraic curves

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of genus γ with ν unlabelled marked points. Let $e(|\mathcal{M}_{\gamma,\nu}|)$ and $e(|\mathcal{M}_{\gamma,\nu}|/\mathbb{S}_\nu)$ be their Euler characteristics; clearly, these coincide for $\nu = 1$. Harer and Zagier were able to calculate $e(|\mathcal{M}_{\gamma,1}|)$; however, for higher values of ν, little is known about Euler characteristics $e(|\mathcal{M}_{\gamma,\nu}|)$ and $e(|\mathcal{M}_{\gamma,\nu}|/\mathbb{S}_\nu)$. By applying our formulas to the modular operad $\mathcal{A}ss$ corresponding to associative algebras, we obtain in Section 9 a closed formula for the sum

$$\sum_{2(1-\gamma)-\nu=\chi} e(|\mathcal{M}_{\gamma,\nu}|/\mathbb{S}_\nu),$$

where χ is a fixed integer, representing the Euler characteristic of the punctured Riemann surfaces contributing to the sum.

The use of symmetric functions in enumeration of graphs goes back to Pólya [32]. Our approach is slightly different: while he associates symmetric functions to permutations of vertices of the graph, we associate them to permutations of flags of the graph (pairs consisting of a vertex and an incident edge). The idea of attaching arbitrary representations of symmetric groups to vertices of a tree appears (under the name 'lumps') in Hanlon–Robinson [15]; they obtain formulas resembling our formula for the characteristic of $\mathsf{B}\mathcal{A}$ (in Pólya's setting). The introduction of the Legendre transform in this problem leads to a new perspective on this class of problems by bringing out a hidden involutive symmetry, which is very natural from the point of view of operads.

Our analogue of Wick's Theorem may be viewed as a synthesis of the methods of graphical enumeration of quantum field theory with Pólya's ideas. Our formula for the character of $\mathsf{F}\mathcal{A}$ has another link to quantum field theory, since the space of symmetric functions is the Hilbert space for the basic representation of $\mathrm{GL}_{\mathrm{res}}(\infty)$ (Kac–Raina [21]); in this direction, we present a formal representation of the characteristic of the free modular operad $\mathbb{M}\mathcal{V}$ as a functional integral (8.18).

We now describe the contents of this paper. In Section 1, we recall the definition of cyclic operads from [12]; roughly speaking, these are operads in which the inputs and output may be permuted. In Section 2, we define modular operads as algebras over a certain triple, constructed by summing over graphs. Modular operads are actually a special sort of cyclic operad, in which there is an additional operation (which we call contraction) which reduces the number of inputs by two. In Section 3, we explain the axioms which must be imposed on such a contraction in order that it determines a modular operad structure; this may be viewed as a coherence theorem for modular operads.

In Section 4, we introduce a generalization of modular operads, in which certain signs are introduced into the structure maps, which we call cocycles. A cocycle is a certain functor from graphs to the Picard category of invertible graded vector spaces. The most important cocycle for us will be the determinant of the first cohomology of the graph, which is obviously trivial when restricted to trees; this explains why this twist is not needed in the theory of operads. In Section 5, we construct the Feynman transform F, which maps from the category of dg-modular operads to the category of dg-modular operads for this cocycle.

Another example of a Feynman transform is given in Section 6: roughly speaking, the complexes of currents on the moduli spaces of stable curves $\overline{\mathcal{M}}_{g,n}$ form a modular operad, and the Feynman transform of this modular operad may be identified with the differential forms on the open strata $\mathcal{M}_{g,n}$; these form a twisted modular operad, in the sense of Section 4, by a construction involving residues taken around divisors at infinity.

In Sections 7 and 8, we define analogues of the Legendre and Fourier transforms for symmetric functions. In this way, we obtain formulas for the characteristics of $\mathsf{B}\mathcal{A}$ and $\mathsf{F}\mathcal{A}$, where $\mathcal{A}$ is a cyclic, respectively modular, operad.

One of the pioneers of the use of Wick's Theorem as a tool in topology was Claude Itzykson, and he was an influence on us and many of our colleagues, in innumerable ways. We humbly offer this article in his memory.

1. Cyclic operads

In this section, we recall the definition of a cyclic operad – this will be useful later, since one way of looking at modular operads is as a special kind of cyclic operad.

Our presentation of the theory of cyclic operads is a little different from our previous account [12]; we need a non-unital version of the theory, due to Markl [27]. One advantage of this formulation is that the basic operations in an operad are bilinear. In any case, if one simply took the original definition of an operad (May [28]), and omitted the axioms involving the unit, one would not obtain the same notion.

(1.1) $\mathbb{S}$**-modules**. Throughout this paper, we work over a fixed field **k** of characteristic 0.

A chain complex (dg-vector space) is a graded vector space $V_\bullet$ together with a differential $\delta\colon V_i \to V_{i-1}$, such that $\delta^2 = 0$. A map of chain complexes is called a weak equivalence if it induces isomorphisms in homology. We denote by $V^\sharp$ the graded vector space underlying V, with vanishing differential.

The suspension $\Sigma V_\bullet$ of a chain complex $V_\bullet$ has components $(\Sigma V)_n = V_{n-1}$, and differential equal to minus that of $V_\bullet$. By $\Sigma^n V_\bullet$, $n \in \mathbb{Z}$, we denote the n-fold iterated suspension of $V_\bullet$.

If V is a chain complex and G is a finite group, we denote by V_G the chain complex of G-coinvariants of V

$$V_G = \frac{V}{\{gv - v \mid v \in V, g \in G\}}.$$

By $V_\bullet^*$, we denote the linear dual of $V_\bullet$, with $V_i^* = (V_{-i})^*$ and with differential $\delta^*\colon V_i^* \to V_{i-1}^*$ the adjoint of $\delta\colon V_{-i+1} \to V_{-i}$. All chain complexes which we

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consider in this article have finite dimensional homology.

Denote by $\mathbb{S}_n$ the group $\mathrm{Aut}\{1,\dots,n\}$ and by $\mathbb{S}_{n+}$ the group $\mathrm{Aut}\{0,1,\dots,n\}$. (This was denoted by $\mathbb{S}_{n+1}$ in [12]; we have changed the notation in order to distinguish between the (isomorphic) groups $\mathbb{S}_{n+} = \mathrm{Aut}\{0,\dots,n\}$ and $\mathbb{S}_{n+1} = \mathrm{Aut}\{1,\dots,n+1\}$.) An $\mathbb{S}$-module is a sequence of chain complexes $\mathcal{V} = \{\mathcal{V}(n) \mid n \geqslant 0\}$, together with an action of $\mathbb{S}_n$ on $\mathcal{V}(n)$ for each n.

A map of $\mathbb{S}$-modules is called a weak equivalence if it is a weak equivalence for each n.

(1.2) **Operads**. An operad is an $\mathbb{S}$-module $\mathcal{P}$ together with bilinear operations

$$\circ_i : \mathcal{P}(m) \otimes \mathcal{P}(n) \to \mathcal{P}(m+n-1), \quad 1 \leqslant i \leqslant m,$$

satisfying the following axioms.

(1.2.1). If $\pi \in \mathbb{S}_m$, $\rho \in \mathbb{S}_n$, $a \in \mathcal{P}(m)$ and $b \in \mathcal{P}(n)$, then

$$(\pi a) \circ_{\pi(i)} (\rho b) = (\pi \circ_i \rho)(a \circ_i b),$$

where $\pi \circ_i \rho \in \mathbb{S}_{m+n-1}$ is defined as follows: it permutes the interval $\{i,\dots, i+n-1\}$ according to the permutation ρ, and then reorders the m intervals

$$\{1\},\dots,\{i-1\},\{i,\dots,i+n-1\},\{i+n\},\dots,\{m+n-1\},$$

which partition $\{1,\dots,m+n-1\}$, according to the permutation π. Explicitly,

$$(\pi \circ_i \rho)(j) = \begin{cases} \pi(j), & j < i \text{ and } \pi(j) < \pi(i), \\ \pi(j)+n-1, & j < i \text{ and } \pi(j) > \pi(i), \\ \pi(j-n+1), & j \geqslant i+n \text{ and } \pi(j) < \pi(i), \\ \pi(j-n+1)+n-1, & j \geqslant i+n \text{ and } \pi(j) > \pi(i), \\ \pi(i)+\rho(j-i+1)-1, & i \leqslant j < i+n. \end{cases}$$

(1.2.2). For $a \in \mathcal{P}(k)$, $b \in \mathcal{P}(l)$ and $c \in \mathcal{P}(m)$, and $1 \leqslant i < j \leqslant k$,

$$(a \circ_i b) \circ_{j+l-1} c = (a \circ_j c) \circ_i b.$$

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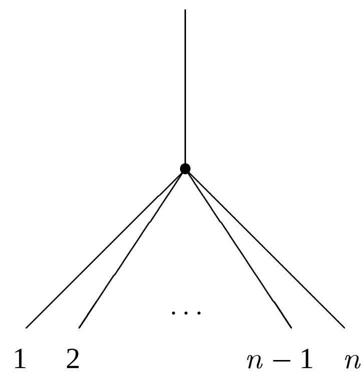
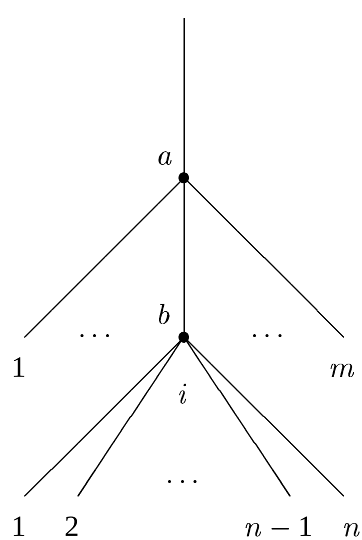
(1.2.3). For $a \in \mathcal{P}(k)$, $b \in \mathcal{P}(l)$ and $c \in \mathcal{P}(m)$, and $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l$,

$$(a \circ_i b) \circ_{i+j-1} c = a \circ_i (b \circ_j c).$$

(1.3) **Operads and trees**. We think of an element of $\mathcal{P}(n)$ as corresponding to a rooted tree with one vertex, n inputs numbered from 1 up to n (and one output).



The compositions correspond to grafting two such trees together along the input of the first tree numbered i. Axiom (1.2.1) expresses the equivariance of this construction.



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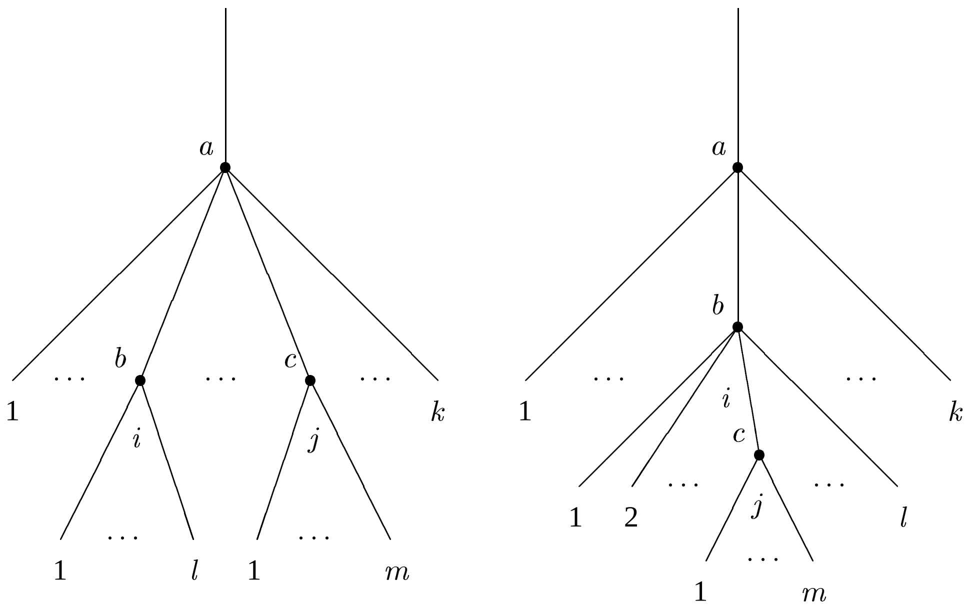
Axioms (1.2.2) and (1.2.3) mean that we can construct unambiguous compositions corresponding to the following two trees respectively



In fact, the axioms imply that the products $\circ_i$ give rise to an unambiguous definition of composition for any rooted tree [11], [13]. This point of view will be explained in greater detail, in the context of modular operads, in Section 2.

(1.4) **Cyclic $\mathbb{S}$-modules**. A cyclic $\mathbb{S}$-module $\mathcal{V}$ is a sequence of vector space $\mathcal{V}(n)$, with action of $\mathbb{S}_{n+}$ on $\mathcal{V}(n)$. In particular, each vector space $\mathcal{V}(n)$ is a module over the symmetric group $\mathbb{S}_n$, and over the cyclic group $\mathbb{Z}_{n+}$ generated by $\tau_n = (01\dots n)$.

If $\mathcal{V}$ is a cyclic $\mathbb{S}$-module, and I is a $(k+1)$-element set, define

$$\mathcal{V}(\!(\mathrm{I})\!) = \left(\bigoplus_{\substack{\text{bijections} \\ f:\{0,\dots,k\}\to \mathrm{I}}} \mathcal{V}(k) \right)_{\mathbb{S}_{k+}} .$$

This makes $\mathcal{V}$ into a functor from the category of nonempty finite sets and their bijections into the category of vector spaces. In the case when $k = n-1$ and $\mathrm{I} = \{1, \dots, n\}$, we write $\mathcal{V}(\!(n)\!)$ instead of $\mathcal{V}(\!(\mathrm{I})\!)$. Note that $\mathcal{V}(\!(n)\!) = \mathcal{V}(n-1)$.

(1.5) **Cyclic operads**. If $\mathcal{P}$ is a cyclic $\mathbb{S}$-module and $a \in \mathcal{P}(n)$, let $a^* \in \mathcal{P}(n)$ be the result of applying the cycle $(01 \dots n) \in \mathbb{S}_{n+}$ to a. (Thus, if $n = 1$, this operation exchanges the input and output of a; for $n > 1$, it generalizes this.) A cyclic operad

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[12] is a cyclic $\mathbb{S}$-module $\mathcal{P}$ whose underlying $\mathbb{S}$-module has the structure of an operad, such that

$$(a \circ_m b)^* = b^* \circ_1 a^*. \tag{1.6}$$

for any $a \in \mathcal{P}(m)$, $b \in \mathcal{P}(n)$. This formula shows that cyclic operads are a generalization of associative $*$-algebras, which are the special case in which $\mathcal{P}(n) = 0$ for $n \neq 1$.

(Cyclic) $\mathbb{S}$-modules may be defined, in exactly the same way, in any symmetric monoidal category. The most important case for us will be the category of chain complexes, whose operads will be called differential graded operads (abbreviated to dg-operad). Other examples are the category of topological spaces, giving rise to topological $\mathbb{S}$-modules and operads, and the opposite category to the category of chain complexes, whose operads are called dg-cooperads.

In the remainder of this paper, unless otherwise specified, by an $\mathbb{S}$-module, operad or cooperad, we mean a dg $\mathbb{S}$-module, operad or cooperad. A map of operads is called a weak equivalence if it is a weak equivalence of the underlying $\mathbb{S}$-module.

(1.7) **Endomorphism operads and cyclic algebras**. Let V be a chain complex such that its homogeneous subspaces V_i are finite-dimensional for all k. An inner product on V is a non-degenerate bilinear form $B(x, y)$ such that $B(\delta x, y) + (-1)^{|x|} B(x, \delta y) = 0$, where δ is the differential of V. Such a bilinear form is symmetric (respectively antisymmetric) if $B(y, x) = (-1)^{|x|\,|y|} B(x, y)$ (resp. $B(y, x) = -(-1)^{|x|\,|y|} B(x, y)$), and has degree k if $B(x, y) = 0$ unless $|x| + |y| = k$.

Let V be a chain complex with symmetric inner product $B(x, y)$ of degree 0. We define a cyclic $\mathbb{S}$-module $\mathcal{E}[V]$ by putting $\mathcal{E}[V](n) = V^{\otimes(n+)}$, with the natural action of $\mathbb{S}_{n+}$. This may be given the structure of a cyclic operad: if $a \in V^{\otimes(m+)}$ and $b \in V^{\otimes(n+)}$, the product $a \circ_i b \in V^{\otimes(m+n)}$ is defined by contracting $a \otimes b$ with the bilinear form B, applied to the ith factor of a and the 0th factor of b. Using the isomorphism $V^{\otimes(n+1)} \cong \mathrm{Hom}(V^{\otimes n}, V)$, the operad underlying this cyclic operad may be identified with the endomorphism operad of [28] and [13].

A cyclic algebra over a cyclic operad $\mathcal{P}$ is a chain complex A with inner product B, together with a morphism of cyclic operads $\mathcal{P} \to \mathcal{E}[A]$.

(1.8) EXAMPLES

(1.8.1) *Stable curves of genus* 0. Define a topological cyclic operad $\mathcal{M}_0$ by letting $\mathcal{M}_0(n)$ be the moduli space $\mathcal{M}_{0,n+}$ of stable curves of genus 0 with embedding of $\{0, \dots, n\}$ [22] (see also [13]). By definition, a point of $\mathcal{M}_{g,n+}$ is a system $(C, x_0, \dots, x_n)$, where C is a projective curve of arithmetic genus 0, with possibly nodal singularities, x_i are distinct smooth points, and C has no infinitesimal automorphisms preserving the points x_i (this amounts to saying that each component

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of C minus its singularities and marked points has negative Euler characteristic). The $\mathbb{S}_n$-action on $\mathcal{M}_{0,n}$ is given by renumbering the punctures. The composition $\circ_i$ takes two pointed curves $(C, x_0, \dots, x_m)$ and $(D, y_0, \dots, y_n)$ into

$$\left(\frac{C \coprod D}{(x_i \sim y_0)}, x_0, \dots, x_{i-1}, y_1, \dots, y_n, x_{i+1}, \dots, x_m\right).$$

(1.8.2) *Spheres with holes*. Define a topological cyclic operad $\widehat{\mathcal{M}}_0$ by letting $\widehat{\mathcal{M}}_0(n)$ be the moduli space of data $(C, f_0, \dots, f_n)$, where C is a complex manifold isomorphic to $\mathbb{CP}^1$, and f_i are biholomorphic maps of the unit disk

$$\Delta = \{z \in \mathbb{C} \mid |z| \leqslant 1\},$$

into C with disjoint images. The composition $\circ_i$ takes $(C, f_0, \dots, f_m)$ and $(D, g_0, \dots, g_n)$ into

$$\left(\left(C \setminus f_i[\mathring{\Delta}]\right) \coprod_{f_i(t) \sim g_0(t^{-1}), t \in \partial\Delta} \left(D \setminus g_0[\mathring{\Delta}]\right), \right.$$
$$\left. f_0, \dots, f_{i-1}, g_1, \dots, g_n, f_{i+1}, \dots, f_m\right).$$

Note that by applying the total homology functor $H_\bullet(-, \mathbf{k})$ to the topological operads $\overline{\mathcal{M}}_0$ and $\widehat{\mathcal{M}}_0$, we obtain cyclic operads in the category of graded vector spaces.

(1.8.3) *Commutative operad*. This operad, denoted $\mathcal{C}om$, has $\mathcal{C}om(\!(n)\!) = \mathbf{k}$ (the trivial representation of $\mathbb{S}_n$) for all $n \geqslant 3$, with the obvious composition maps. Cyclic algebras over $\mathcal{C}om$ are commutative algebras (possibly without unit) with an invariant scalar product in the ordinary sense: $B(xy, z) = B(x, yz)$.

(1.8.4) *Associative operad*. This operad, denoted $\mathcal{A}ss$, has $\mathcal{A}ss(\!(n)\!) = \mathrm{Ind}_{\mathbb{Z}_n}^{\mathbb{S}_n}(\mathbf{k})$, where $\mathbf{k}$ is the trivial representation of the cyclic group $\mathbb{Z}_n$. Thus, there is a natural basis for $\mathcal{A}ss(\!(n)\!)$ labelled by the *cyclic orders* of the set $\{1, \dots, n\}$, that is, the free $\mathbb{Z}_n$-actions on this set. Note that cyclic orders on $\{0, 1, \dots, n\}$ are in bijection with permutations of $\{1, \dots, n\}$; thus $\mathcal{A}ss(\!(n+)\!) = \mathcal{A}ss(n)$ is free as an $\mathbb{S}_n$-module, with generating vector e_n and basis $\{\sigma e_n \mid \sigma \in \mathbb{S}_n\}$. The composition is determined by the formulas $e_m \circ_1 e_n = e_{m+n-1}$ together with (1.2.1).

A cyclic $\mathcal{A}ss$-algebra is the same as an associative algebra A with an invariant scalar product. The basis element $\sigma e_n \in \mathcal{A}ss(n)$ acts on A as an n-ary operation $(x_1, \dots, x_n) \mapsto x_{\sigma^{-1}(1)} \dots x_{\sigma^{-1}(n)}$. See [13], [12] for more details.

(1.8.5) *Lie operad*. This operad, denoted $\mathcal{L}ie$, is determined by the requirement that its cyclic algebras are Lie algebras with invariant scalar product. Thus $\mathcal{L}ie(n) =$

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$\mathcal{L}ie(\!(n+1)\!)$ can be identified, as a module over $\mathbb{S}_n \subset \mathbb{S}_{n+}$, with the subspace in the free Lie algebra on generators $x_1, \dots, x_n$ spanned by all Lie monomials containing each x_i exactly once. The $\mathbb{S}_n$-module $\mathcal{L}ie(n)$ is isomorphic to the induced representation $\mathrm{Ind}_{\mathbb{Z}_n}^{\mathbb{S}_n}(\chi)$, where χ is a primitive character of $\mathbb{Z}_n$ (one which takes the generator of $\mathbb{Z}_n$ into a primitive root of 1).

2. Modular operads

(2.1) **Stable $\mathbb{S}$-modules**. A stable $\mathbb{S}$-module is a collection of chain complexes

$$\{\mathcal{V}(\!(g,n)\!) \mid n, g \geqslant 0\}$$

with an action of $\mathbb{S}_n$ on $\mathcal{V}(\!(g,n)\!)$, such that $\mathcal{V}(\!(g,n)\!) = 0$ if $2g + n - 2 \leqslant 0$.

A morphism $\mathcal{V} \to \mathcal{W}$ of stable $\mathbb{S}$-modules is a collection of equivariant maps of chain complexes $\mathcal{V}(\!(g,n)\!) \to \mathcal{W}(\!(g,n)\!)$.

We have borrowed the term ‘stable’ from the theory of moduli spaces of curves, since the condition of stability is the same in the two settings.

Any cyclic $\mathbb{S}$-module $\mathcal{V}$ such that $\mathcal{V}(\!(n)\!) = 0$ for $n \geqslant 2$ may be regarded as a stable $\mathbb{S}$-module by setting

$$\mathcal{V}(\!(g,n)\!) = \begin{cases} \mathcal{V}(\!(n)\!), & g = 0, \\ 0, & g > 0. \end{cases} \tag{2.2}$$

In the other direction, we have the forgetful functor, which we denote by Cyc. If $\mathcal{V}$ is a stable $\mathbb{S}$-module, then $\mathrm{Cyc}(\mathcal{V})$ is a cyclic $\mathbb{S}$-module, and

$$\mathrm{Cyc}(\mathcal{V})(\!(n)\!) = \mathcal{V}(\!(0,n)\!). \tag{2.3}$$

A stable $\mathbb{S}$-module $\mathcal{V}$ has a natural extension to all finite sets I

$$\mathcal{V}(\!(g,I)\!) = \Bigg(\bigoplus_{\substack{\text{bijections} \\ f:\{1,\dots,n\}\to \mathrm{I}}} \mathcal{V}(\!(g,n)\!) \Bigg)_{\mathbb{S}_n}. \tag{2.4}$$

(2.5) **Graphs**. (See [25].) A graph G is a finite set $\mathrm{Flag}(G)$ (whose elements are called flags) together with an involution σ and a partition λ. (By a partition of a set, we mean a disjoint decomposition into several unordered, possibly empty, subsets, called blocks.) We say that two flags $a, b \in \mathrm{Flag}(G)$ meet if they are equivalent under the partition λ.

The vertices of G are the blocks of the partition λ, and the set of them is denoted $\mathrm{Vert}(G)$. The subset of $\mathrm{Flag}(G)$ corresponding to a vertex v is denoted $\mathrm{Leg}(v)$. Its cardinality is called the valence of v, and denoted $n(v)$.

The edges of G are the pairs of flags forming a two-cycle of σ, and the set of them is denoted $\mathrm{Edge}(G)$. The legs of G are the fixed-points of σ, and the set of them is denoted $\mathrm{Leg}(G)$. Since a flag forms either a leg or half an edge, we see that

$$\sum_{v\in \mathrm{Vert}(G)} n(v) = 2|\mathrm{Edge}(G)| + n. \tag{2.6}$$

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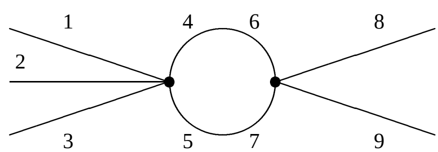
(2.7) **The geometric realization of a graph**. We may associate to a graph the finite one-dimensional cell complex $|G|$, obtained by taking one copy of $[0, \frac{1}{2}]$ for each flag, and imposing the following equivalence relation: the points $0 \in [0, \frac{1}{2}]$ are identified for all flags in a block of the partition λ, and the points $\frac{1}{2} \in [0, \frac{1}{2}]$ are identified for pairs of flags exchanged by the involution σ. Thus, two flags in G meet if and only if their corresponding loci in $|G|$ intersect in a point.

For example, the following corresponds to the set of flags $\{1, \ldots, 9\}$, the involution $\sigma = (46)(57)$ and the partition $\{1,2,3,4,5\} \cup \{6,7,8,9\}$.



Let $H_i(G)$ be the ith homology group of the cell complex $|G|$ with coefficients in **k**, and let $b_i(G)$ be the dimension of $H_i(G)$. Then $b_0(G)$ is the number of components of G, and $b_1(G)$ is the number of circuits of G. The graph G is connected if $b_0(G) = 1$.

(2.8) **Stable graphs**. A labelled graph is a connected graph G together with a map g from Vert(G) into the non-negative integers. The value $g(v)$ of this map at a given vertex v is called the genus of v. The genus $g(G)$ of a labelled graph G is defined by the formula

$$g(G) = \sum_{v \in \mathrm{Vert}(G)} g(v) + b_1(G) = \sum_{v \in \mathrm{Vert}(G)} (g(v) - 1) + |\mathrm{Edge}(G)| + 1. \tag{2.9}$$

Adding twice (2.9) to (2.6), we see that

$$2(g-1) + n = \sum_{v \in \mathrm{Vert}(G)} \big(2(g(v) - 1) + n(v)\big) . \tag{2.10}$$

Likewise, adding three times (2.9) to (2.6), we see that

$$3(g-1) + n = |\mathrm{Edge}(G)| + \sum_{v \in \mathrm{Vert}(G)} \big(3(g(v) - 1) + n(v)\big) . \tag{2.11}$$

Both of these formulas will be needed later.

A forest is a (labelled) graph of genus 0; a tree is a connected forest. This definition is slightly different from the definition of trees in [12]: here, we do not admit the tree with two legs and no vertices.

A connected labelled graph is called stable if $2(g(v) - 1) + n(v) > 0$ for each vertex v.

If $\mathcal{V}$ is a stable $\mathbb{S}$-module and G be a stable graph, let $\mathcal{V}(\!(G)\!)$ be the tensor product

$$\mathcal{V}(\!(G)\!) = \bigotimes_{v \in \mathrm{Vert}(G)} \mathcal{V}(\!(g(v), v)\!). \tag{2.12}$$

(2.13) **Morphisms of graphs**. Let G_0 and G_1 be two graphs. A morphism $f\colon G_0 \to G_1$ is an injection $f^*\colon \mathrm{Flag}(G_1) \to \mathrm{Flag}(G_0)$ such that

(1) $\sigma_0 \circ f^* = f^* \circ \sigma_1$, where σ_i, $i = 0, 1$, are the involutions of $\mathrm{Flag}(G_i)$;
(2) σ_0 acts freely on the complement of the image of f^* in $\mathrm{Flag}(G_0)$ (i.e. G_1 is obtained from G_0 by contracting a subset of its edges);
(3) two flags a and b in G_1 meet if and only if there is a chain $(x_0, \ldots, x_k)$ of flags in G_0 such that $f^*a = x_0$, $\sigma_0 x_{i-1}$ and x_i meet for all $1 \leqslant i \leqslant k$, and $f^*b = \sigma_0 x_k$.

This definition is equivalent to that of Kontsevich–Manin [25].

A morphism $f\colon G_0 \to G_1$ defines a surjective cellular map $|f|\colon |G_0| \to |G_1|$ which is bijective on the legs.

The preimage of a vertex $v \in \mathrm{Vert}(G_1)$ under a morphism f, denoted $f^{-1}(v)$, is the graph consisting of those flags in G_0 which are connected to a flag in $\mathrm{Leg}(v)$ by a chain of edges in G_0 contracted by the morphism. Note that $\mathrm{Leg}(f^{-1}(v)) = \mathrm{Leg}(v)$.

A morphism $f\colon G_0 \to G_1$ of labelled graphs is a morphism of the underlying graphs such that the genus of a vertex v of G_1 is equal to the genus of its inverse image $f^{-1}(v)$ in G_0.

Let Γ be the category of all stable graphs and their morphisms.

(2.14) **Contractions of graphs**. Let G be a stable graph and let $\mathrm{I} \subset \mathrm{Edge}(G)$ is a subset of its edges. Then there is a unique stable graph G/I with the following properties.

(1) $\mathrm{Flag}(G/\mathrm{I})$ is obtained from $\mathrm{Flag}(G)$ by deleting the flags constituting the edges in I;
(2) the inclusion $\mathrm{Flag}(G/\mathrm{I}) \hookrightarrow \mathrm{Flag}(G)$ is a morphism of graphs $\pi_{G,\mathrm{I}}\colon G \to G/\mathrm{I}$.

The graph G/I is called the contraction of G along the set of edges I. Any morphism $f\colon G \to G'$ of stable graphs is isomorphic to a morphism of this form. Note that the realization $|G/\mathrm{I}|$ is obtained from $|G|$ by contracting each edge of I to a point.

If I is a set with just one edge e, we will abbreviate $G/\{e\}$ and $\pi_{G,\{e\}}$ to G/e and $\pi_{G,e}$.

(2.15) **The category** $\Gamma(\!(g,n)\!)$. Let $\Gamma(\!(g,n)\!)$ be the category whose objects are pairs (G, ρ) where G is a stable graph of genus g and ρ is a bijection between $\mathrm{Leg}(G)$ and the set $\{1, \ldots, n\}$, and whose morphisms are morphisms of stable graphs preserving the labelling ρ of the legs. This category has a terminal object $*_{g,n}$, the graph with no edges and one vertex v of genus g and valence n.

(2.16) LEMMA. *The category* $\Gamma(\!(g,n)\!)$ *has a finite number of isomorphism classes of objects, the set of which we will denote by* $[\Gamma(\!(g,n)\!)]$.

Proof. Since the graph G is stable, the integer $3(g(v)-1)+n(v)$ is non-negative at each vertex. By (2.11), this gives a bound of $3(g-1)+n$ for the number of edges of G, and hence a bound of $6(g-1)+2n$ for the number of flags. Since there are a finite number of stable graphs of genus g with a given number of flags, the lemma follows. $\square$

Denote by $\mathrm{Aut}(G)$ the automorphism group of a graph G in $\Gamma(\!(g,n)\!)$. Observe that trees have no non-trivial automorphisms, since each vertex is uniquely determined by its distance from each of the legs (or even any two).

(2.17) **The triple of stable graphs**. If $\mathcal{C}$ is a category, let $\mathrm{Iso}\,\mathcal{C}$ be the subcategory of isomorphisms of $\mathcal{C}$. We define an endofunctor $\mathbb{M}$ of the category of stable $\mathbb{S}$-modules by the formula

$$\mathbb{M}\mathcal{V}(\!(g,n)\!) = \operatorname*{colim}_{G\in \mathrm{Iso}\Gamma(\!(g,n)\!)} \mathcal{V}(\!(G)\!) \cong \bigoplus_{G\in[\Gamma(\!(g,n)\!)]} \mathcal{V}(\!(G)\!)_{\mathrm{Aut}(G)}. \tag{2.18}$$

The functor $\mathbb{M}$ is a triple; that is, there are natural transformations $\mu : \mathbb{M}\mathbb{M}\mathcal{V} \to \mathbb{M}\mathcal{V}$ and $\eta : \mathcal{V} \to \mathbb{M}\mathcal{V}$ making it into a monoid in the monoidal category of endofunctors of the category of stable $\mathbb{S}$-modules. We will now construct these and verify the axioms of a triple.

We may associate to any category $\mathcal{C}$ a simplicial category $\mathrm{Iso}_\bullet\mathcal{C}$; the objects of $\mathrm{Iso}_k\mathcal{C}$ are diagrams

$$(f_1,\dots,f_k) = [G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} \dots \xrightarrow{f_{k-1}} G_{k-1} \xrightarrow{f_k} G_k]$$

in $\mathcal{C}$, while the morphisms are isomorphisms of such diagrams. The face maps are given by the usual formulas

$$\partial_i(f_1,\dots,f_k) = \begin{cases} (f_2,\dots,f_k), & i=0,\\ (f_1,\dots,f_{i+1}f_i,\dots,f_k), & 1\leqslant i\leqslant k-1,\\ (f_1,\dots,f_{k-1}), & i=k,\end{cases}$$

as are the degeneracies

$$\sigma_i(f_1,\dots,f_k) = (f_1,\dots,f_i,\mathrm{Id}_{G_i},f_{i+1},\dots,f_k), \quad 0\leqslant i\leqslant k.$$

In particular, $\mathrm{Iso}_0\,\mathcal{C} = \mathrm{Iso}\,\mathcal{C}$.

Every object of $\mathrm{Iso}_k\,\Gamma(\!(g,n)\!)$ is isomorphic to an object made up of a sequence of contractions $[G \to G/\mathrm{I}_1 \to \cdots \to G/\mathrm{I}_k]$, where $G \in \mathrm{Ob}\,\Gamma(\!(g,n)\!)$ is a stable graph and $\mathrm{I}_1 \subset \ldots \subset \mathrm{I}_k \subset \mathrm{Edge}(G)$ is a chain of subsets of $\mathrm{Edge}(G)$.

The proof that $\mathbb{M}$ is a triple rests on the identity

$$(\mathbb{M}^{k+1}\mathcal{V})(\!(g,n)\!) = \operatorname*{colim}_{\substack{[G_0 \xrightarrow{f_1} \cdots \xrightarrow{f_k} G_k] \\ \in \mathrm{Iso}_k\Gamma(\!(g,n)\!)}} \mathcal{V}(\!(G_0)\!).$$

Equivalently,

$$(\mathbb{M}^{k+1}\mathcal{V})(\!(g,n)\!) \cong \operatorname*{colim}_{G\in\mathrm{Iso}\Gamma(\!(g,n)\!)} \bigoplus_{\mathrm{I}_1\subset\ldots\subset\mathrm{I}_k\subset\mathrm{Edge}(G)} \mathcal{V}(\!(G)\!). \tag{2.19}$$

The multiplication $\mu : \mathbb{M}\mathbb{M}\mathcal{V} \to \mathbb{M}\mathcal{V}$ of $\mathbb{M}$ is induced by $\partial_1 : \mathrm{Iso}_1\Gamma(\!(g,n)\!) \to \mathrm{Iso}_0\Gamma(\!(g,n)\!)$, which maps the contraction $G \to G/\mathrm{I}$ to G. The unit $\eta : \mathcal{V} \to \mathbb{M}\mathcal{V}$ of $\mathbb{M}$ is the inclusion of the summand $\mathcal{V}(\!(*_{g,n})\!) \cong \mathcal{V}(\!(g,n)\!)$ of $\mathbb{M}\mathcal{V}(\!(g,n)\!)$ associated to the graph $*_{g,n}$ with no edges.

The natural transformations $\mu\mathbb{M}$ and $\mathbb{M}\mu : (\mathbb{M}^3\mathcal{V})(\!(g,n)\!) \to (\mathbb{M}^2\mathcal{V})(\!(g,n)\!)$ are induced by the functors $\partial_1, \partial_2 : \mathrm{Iso}_1\Gamma(\!(g,n)\!) \to \mathrm{Iso}_0\Gamma(\!(g,n)\!)$, which send the sequence of contractions $G \to G/\mathrm{I}_1 \to G/\mathrm{I}_2$ respectively to the contractions $G \to G/\mathrm{I}_1$ and $G \to G/\mathrm{I}_2$. Since $\partial_1\partial_1 = \partial_1\partial_2$, composing either of these with μ gives the same natural transformation, proving associativity of multiplication in the triple $\mathbb{M}$. It is easy to see that η is a unit.

(2.20) **Modular operads**. A modular operad $\mathcal{A}$ is an algebra over the triple $\mathbb{M}$ in the category of stable $\mathbb{S}$-modules. This means that there is a structure map $\mu : \mathbb{M}\mathcal{A} \to \mathcal{A}$ such that $\mu(\mu\mathcal{A}) = \mu(\mathbb{M}\mu) : \mathbb{M}\mathcal{A} \to \mathcal{A}$, and $\mu(\eta\mathcal{A}) = \mathrm{Id}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$. For example, for any stable $\mathbb{S}$-module $\mathcal{V}$, $\mathbb{M}\mathcal{V}$ is a modular operad, called the free modular operad generated by $\mathcal{V}$. Modular operads may be considered in any symmetric monoidal categories.

(2.21) **Coherence for modular operads**. By a modular pre-operad, we mean a stable $\mathbb{S}$-module $\mathcal{A}$ together with a structure map $\mu : \mathbb{M}\mathcal{A} \to \mathcal{A}$. We now give a criterion which determines when a modular pre-operad is a modular operad.

If $\mathcal{A}$ is a modular pre-operad and $G \in \mathrm{Ob}\Gamma(\!(g,n)\!)$, denote by $\mu_G : \mathcal{A}(\!(G)\!) \to \mathcal{A}(\!(g,n)\!)$ the $\mathbb{S}_n$-equivariant map obtained by composing the universal map

$$\mathcal{A}(\!(G)\!) \to \mathbb{M}\mathcal{A}(\!(g,n)\!) = \operatorname*{colim}_{H\in\mathrm{Iso}\Gamma(\!(g,n)\!)} \mathcal{A}(\!(H)\!),$$

with the structure map $\mu : \mathbb{M}\mathcal{A}(\!(g,n)\!) \to \mathcal{A}(\!(g,n)\!)$. We call this map composition along the graph G. We may use the technique of (2.4) to define maps $\mu_G : \mathcal{A}(\!(G)\!) \to \mathcal{A}(\!(g(G), \mathrm{Leg}(G))\!)$ for any stable graph G (no longer requiring that the legs of G be numbered).

Given a morphism $f: G_0 \to G_1$ of stable graphs, define a morphism $\mathcal{A}(\!(f)\!)$: $\mathcal{A}(\!(G_0)\!) \to \mathcal{A}(\!(G_1)\!)$ to be the composition

$$\mathcal{A}(\!(G_0)\!) = \bigotimes_{u\in \mathrm{Vert}(G_0)} \mathcal{A}(\!(g(u), \mathrm{Leg}(u))\!) \cong \bigotimes_{v\in \mathrm{Vert}(G_1)} \mathcal{A}(\!(f^{-1}(v))\!)$$

$$\xrightarrow{\bigotimes_v \mu_{f^{-1}(v)}} \bigotimes_{v\in \mathrm{Vert}(G_1)} \mathcal{A}(\!(g(v), \mathrm{Leg}(v))\!) = \mathcal{A}(\!(G_1)\!). \tag{2.22}$$

(2.23) PROPOSITION. *A modular pre-operad $\mathcal{A}$ is a modular operad if and only if the morphisms $\mathcal{A}(\!(f)\!)$ define a functor on the category of stable graphs, that is, if*

$$\mathcal{A}(\!(f_1 f_0)\!) = \mathcal{A}(\!(f_1)\!)\mathcal{A}(\!(f_0)\!),$$

for any two composable morphisms.

Proof. The associativity of the functor $f \mapsto \mathcal{A}(\!(f)\!)$ implies that $\mathcal{A}$ is a modular operad; $\mathcal{A}$ is an $\mathbb{M}$-algebra precisely when $\mathcal{A}(\!(f_1 f_0)\!) = \mathcal{A}(\!(f_1)\!)\mathcal{A}(\!(f_0)\!)$ for all diagrams of the form

$$G_0 \xrightarrow{f_0} G_1 \xrightarrow{f_1} *_{g,n}.$$

On the other hand, if $\mathcal{A}$ is a modular operad, then given a composable pair of morphisms $G_0 \xrightarrow{f_0} G_1 \xrightarrow{f_1} G_2$, we see that $\mathcal{A}(\!(f_1)\!)\mathcal{A}(\!(f_0)\!)$ is the composition

$$\mathcal{A}(\!(G_0)\!) = \bigotimes_{u\in \mathrm{Vert}(G_0)} \mathcal{A}(\!(g(u), \mathrm{Leg}(u))\!) \cong \bigotimes_{v\in \mathrm{Vert}(G_1)} \mathcal{A}(\!(f_1^{-1}(v))\!)$$

$$\xrightarrow{\bigotimes_v \mu_{f_1^{-1}(v)}} \bigotimes_{v\in \mathrm{Vert}(G_1)} \mathcal{A}(\!(g(v), \mathrm{Leg}(v))\!) \cong \bigotimes_{w\in \mathrm{Vert}(G_2)} \mathcal{A}(\!(f_2^{-1}(w))\!)$$

$$\xrightarrow{\bigotimes_w \mu_{f_2^{-1}(w)}} \bigotimes_{w\in \mathrm{Vert}(G_2)} \mathcal{A}(\!(g(w), \mathrm{Leg}(w))\!) = \mathcal{A}(\!(G_2)\!). \tag{2.24}$$

But $\mathcal{A}$ is a modular operad, so that associativity holds for $\mathcal{A}$ applied to the diagram

$$(f_1 f_0)^{-1}(w) \to f_1^{-1}(w) \to *_{g,n},$$

for all $w \in \mathrm{Vert}(G_2)$. This allows us to rewrite (2.24) as

$$\mathcal{A}(\!(G_0)\!) = \bigotimes_{u\in \mathrm{Vert}(G_0)} \mathcal{A}(\!(g(u), \mathrm{Leg}(u))\!) \cong \bigotimes_{w\in \mathrm{Vert}(G_2)} \mathcal{A}(\!((f_1 f_0)^{-1}(w))\!)$$

$$\xrightarrow{\bigotimes_w \mu_{(f_1 f_0)^{-1}(w)}} \bigotimes_{w\in \mathrm{Vert}(G_2)} \mathcal{A}(\!(g(w), \mathrm{Leg}(w))\!) = \mathcal{A}(\!(G_2)\!),$$

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which is $\mathcal{A}(\!(f_1 f_0)\!)$.

(2.25) **Endomorphism operads and modular algebras**. Let V be a chain complex with symmetric inner product $B(x,y)$ of degree 0. The endomorphism modular operad $\mathcal{E}[V]$ of V has as its underlying stable $\mathbb{S}$-module

$$\mathcal{E}[V](\!(g,n)\!) = V^{\otimes n}.$$

The composition maps of $\mathcal{E}[V]$ are defined as follows: if G is a graph, the vector space $\mathcal{E}[V](\!(G)\!)$ may be identified with $V^{\otimes \mathrm{Flag}(G)}$, and the composition map is obtained by contracting elements of this chain complex with the multlinear form $B^{\otimes \mathrm{Edge}(G)}$, which contracts with the factors of $V^{\otimes \mathrm{Flag}(V)}$ corresponding to the flags which are paired up as edges of the graph G.

It is easily seen that the cyclic operad underlying $\mathcal{E}[V]$ is the endomorphism cyclic operad introduced in (1.7).

An algebra over a modular operad $\mathcal{A}$ is a chain complex V with inner product B, together with a morphism of modular operads $\mathcal{A} \to \mathcal{E}[V]$.

3. The structure of modular operads

In this section, we show that a modular operad is a cyclic operad with additional structure (a grading by genus and contractions on pairs of legs) satisfying certain conditions.

(3.1) **Cyclic operads and the triple of trees**. For a cyclic $\mathbb{S}$-module $\mathcal{V}$ we define a cyclic $\mathbb{S}$-module $\mathbb{T}\mathcal{V}$ by summing over trees

$$\mathbb{T}\mathcal{V}(\!(n)\!) = \bigoplus_{T \in \Gamma(\!(0,n)\!)} \mathcal{V}(\!(T)\!).$$

(In [12], this triple was denoted $\mathbb{T}_+$. Since we have no need for $\mathbb{T}_-$ in this paper, we omit the subscript from our notation.) The following result is Theorem (2.2) of [12].

(3.2) THEOREM. *The functor $\mathbb{T}$ is a triple and a cyclic operad $\mathcal{P}$ with $\mathcal{P}(0) = \mathcal{P}(1) = 0$ is the same as an algebra over $\mathbb{T}$.*

In particular, we have free cyclic operads $\mathbb{T}\mathcal{V}$, where $\mathcal{V}$ is a cyclic $\mathbb{S}$-module. Note that we have the following commutative diagram of triples

$$\begin{array}{ccc} \text{stable } \mathbb{S}\text{-modules} & \xrightarrow{\mathbb{M}} & \text{stable } \mathbb{S}\text{-modules} \\ \Big\downarrow{\scriptstyle \mathrm{Cyc}} & & \Big\downarrow{\scriptstyle \mathrm{Cyc}} \\ \text{cyclic } \mathbb{S}\text{-modules} & \xrightarrow{\mathbb{T}} & \text{cyclic } \mathbb{S}\text{-modules}. \end{array}$$

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(3.3) **Graded cyclic operads**. A graded cyclic operad is a cyclic operad $\mathcal{P}$ such that $\mathcal{P}(\!(n)\!)$ has an $\mathbb{S}_n$-invariant decomposition

$$\mathcal{P}(\!(n)\!) = \bigoplus_{g=0}^{\infty} \mathcal{P}(\!(g,n)\!)$$

and if $a \in \mathcal{P}(\!(g,m)\!)$ and $b \in \mathcal{P}(\!(h,m)\!)$, then $a \circ_i b \in \mathcal{P}(\!(g+h, n+m-2)\!)$. We say that $\mathcal{P}$ is a stable graded cyclic operad if $\mathcal{P}(\!(g,n)\!) = 0$ for $2(g-1)+n \leqslant 0$.

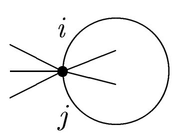
(3.4) LEMMA. *If $\mathcal{A}$ is a modular operad, then the cyclic $\mathbb{S}$-module*

$$\mathcal{A}^\flat(\!(n)\!) = \bigoplus_{g} \mathcal{A}(\!(g,n)\!)$$

is a stable graded cyclic operad.

Proof. If $\mathcal{V}$ is a stable $\mathbb{S}$-module, the sub-triple of $\mathbb{M}\mathcal{V}$ induced by summing over simply connected graphs alone is isomorphic to the triple $\mathbb{T}\mathcal{V}^\flat$. It follows that if $\mathcal{V}$ is an $\mathbb{M}$-algebra, then $\mathcal{V}^\flat$ is a $\mathbb{T}$-algebra, that is, a cyclic operad. It is clear that it is stable and graded. □

(3.5) **The contraction maps**. Given a finite set I and distinct elements $i, k \in \mathrm{I}$, let $G^{ij}_{g,\mathrm{I}}$ be the stable graph with $\mathrm{Flag}(G^{ij}_{g,\mathrm{I}}) = \mathrm{I}$, a single vertex with genus g, and a single edge (a loop) joining the flags i and j.

$G^{ij}_{g,\mathrm{I}} =$ 

If $\mathcal{A}$ is a modular operad, denote by ξ_{ij} the composition map

$$\mu_{G^{ij}_{g,\mathrm{I}}} : \mathcal{A}(\!(G^{ij}_{g,\mathrm{I}})\!) \cong \mathcal{A}(\!(g,\mathrm{I})\!) \to \mathcal{A}(\!(g+1, \mathrm{I} \setminus \{i,j\})\!).$$

(Here we make use of the notation (2.4).) We call ξ_{ij} the contraction map. These maps are equivariant, in the sense that for any bijection $\sigma : \mathrm{I} \to J$ of finite sets and $i, j \in \mathrm{I}$,

$$\xi_{\sigma(i)\sigma(j)}\sigma = \sigma\xi_{ij}, \quad \text{on} \in \mathcal{A}(\!(g,\mathrm{I})\!). \tag{3.6}$$

We now determine the coherence relations that the contractions ξ_{ij} on a stable graded cyclic operad must satisfy in order for them to define a modular operad structure.

(3.7) THEOREM. *Let $\mathcal{A}$ be a stable graded cyclic operad with contraction maps*

$$\xi_{ij} : \mathcal{A}(\!(g,\mathrm{I})\!) \to \mathcal{A}(\!(g+1, \mathrm{I} \setminus \{i,j\})\!),$$

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equivariant in the sense of (3.6). *These data determine a modular operad if and only if the following coherence conditions are satisfied:*

(1) *For any finite set* I *and distinct elements* $i, j, k, l \in \mathrm{I}$,

$$\xi_{ij} \circ \xi_{kl} = \xi_{kl} \circ \xi_{ij}.$$

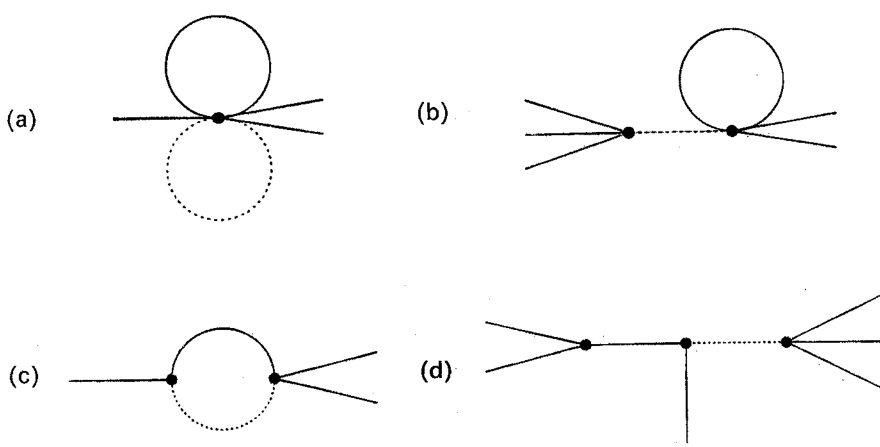
The remaining conditions concern the composition $a \circ_m b$ *of* $a \in \mathcal{A}(m)$ *and* $b \in \mathcal{A}(n)$*:*

(2) $\xi_{12}(a \circ_m b) = (\xi_{12}a) \circ_m b$;
(3) $\xi_{m,m+1}(a \circ_m b) = a \circ_m (\xi_{12}b)$;
(4) $\xi_{m-1,m}(a \circ_m b) = \xi_{m+n-2,m+n-1}(a \circ_{m-1} b^*)$.

Proof. 'Only if': Let $\mathcal{A}$ be a modular operad. By (2.21), we obtain a functor $f \mapsto \mathcal{A}(\!(f)\!)$ from the category Γ of stable graphs to $\mathcal{C}$. If e, e' are two edges in a stable graph G, the contractions of e and e' commute in Γ, in the sense that

$$\pi_{G/e,e'}\pi_{G,e} = \pi_{G/e',e}\pi_{G,e'} = \pi_{G,\{e,e'\}}: G \to G/\{e,e'\}.$$

We now obtain relations (91–4) in the statement of the theorem by evaluating the functor $\mathcal{A}(\!(f)\!)$ on these identities, for all stable graphs with two edges. Indeed, a graph with two edges has one of the following forms



Graphs of type (a) give rise to the relations of type (1) in the statement of the theorem, graphs of type (b) to the relations of type (2) and (3) and graphs of type (c) to relations of type (3). (Graphs of type (d) of course imply that $\mathcal{A}^\flat$ is a (graded) cyclic operad.)

'If': Let $\mathcal{A}$ be a stable graded cyclic operad equipped with contraction maps ξ_{ij} as in the statement of the theorem. We will construct a functor, which we still denote by $\mathcal{A}$, on the category Γ of stable graphs. On objects G of Γ (stable graphs), we define $\mathcal{A}(\!(G)\!)$ as in (2.12). If $\phi\colon G_0 \to G_1$ is an isomorphism of stable graphs, we define $\phi\colon \mathcal{A}(\!(G_0)\!) \to \mathcal{A}(\!(G_1)\!)$ in the evident way: this definition is functorial on the subcategory $\operatorname{Iso}\Gamma$.

Next, we turn to the case where $f = \pi_{G,e}\colon G \to G/e$ is a contraction. There are two sub-cases:

(1) The edge e has two distinct ends, the vertices v and v': Let H be the graph whose flags are the legs of v and v'. Then H is a tree with one edge e and two vertices v and v', and H/e has a single vertex $\overline{v}$. We have

$$\mathcal{A}(\!(G)\!) \cong \mathcal{A}(\!(H)\!) \otimes \bigotimes_{w \neq v,v'} \mathcal{A}(\!(g(w), \operatorname{Leg}(w))\!),$$

$$\mathcal{A}(\!(G/e)\!) \cong \mathcal{A}(\!(g(\overline{v}), \operatorname{Leg}(\overline{v}))\!) \otimes \bigotimes_{w \neq v,v'} \mathcal{A}(\!(g(w), \operatorname{Leg}(w))\!)$$

and we define

$$\mathcal{A}(\!(\pi_{G,e})\!) = \mu_H \otimes \bigotimes_{w \neq v,v'} \operatorname{Id}_{\mathcal{A}(\!(g(w),\operatorname{Leg}(w))\!)},$$

where μ_H is the composition along H in the graded cyclic operad $\mathcal{A}^\flat$.

(2) The edge e has one end, the vertex v: Let $i, j \in \operatorname{Leg}(v)$ be the two flags of e, and let $\overline{v}$ be the image of the vertex v in G/e. We have

$$\mathcal{A}(\!(G)\!) \cong \mathcal{A}(\!(g(v), \operatorname{Leg}(v))\!) \otimes \bigotimes_{w \neq v} \mathcal{A}(\!(g(w), \operatorname{Leg}(w))\!),$$

$$\mathcal{A}(\!(G/e)\!) \cong \mathcal{A}(\!(g(\overline{v}) + 1, \operatorname{Leg}(\overline{v}) \setminus \{i, j\})\!) \otimes \bigotimes_{w \neq v} \mathcal{A}(\!(g(w), \operatorname{Leg}(w))\!)$$

and we define $\mathcal{A}(\!(\pi_{G,e})\!) = \xi_{ij} \otimes \bigotimes_{w \neq v} \operatorname{Id}_{\mathcal{A}(\!(g(w),\operatorname{Leg}(w))\!)}$.

Now let $f\colon G_0 \to G_1$ be a general morphism of stable graphs, and let I be the set of edges of G contracted by f. The morphism f decomposes as a composition

$$G_0 \xrightarrow{\pi_{G_0,\mathrm{I}}} G_0/\mathrm{I} \xrightarrow{\phi} G_1,$$

where $\phi\colon G_0/\mathrm{I} \to G_1$ is an isomorphism. Choosing an ordering $\{e_1, \dots, e_k\}$ of the edges in I, we obtain a factorization

$$G_0 \xrightarrow{\pi_{G_0,e_1}} G_0/e_1 \xrightarrow{\pi_{G_0\setminus\{e_1\},e_2}} G_0/\{e_1, e_2\} \xrightarrow{\pi_{G_0\setminus\{e_1,e_2\},e_3}}$$

$$\dots \xrightarrow{\pi_{G_0\setminus\{e_1,\dots,e_{k-1}\},e_k}} G_0/\mathrm{I} \xrightarrow{\phi} G_1,$$

where each morphism is a contraction along one edge except the last, which is an isomorphism. We define

$$\mathcal{A}(\!(f)\!) = \mathcal{A}(\!(\phi)\!)\mathcal{A}(\!(\pi_{G_0\setminus\{e_1,\dots,e_{k-1}\},e_k})\!) \dots \mathcal{A}(\!(\pi_{G_0,e_1})\!).$$

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We must prove that this definition of $\mathcal{A}((f))$ is independent of the ordering of the elements of I. It suffices to prove the product does not change if we interchange two consecutive edges e_i and e_{i+1}. If the two edges do not meet, this is evident. If they do meet, then they form a stable graph with two edges, whose topology is one of the four types (a–d) catalogued above. From conditions (1–4) and equivariance (3.6), it follows that composition is well-defined along every graph with two edges, regardless of the numbering of its legs.

Thus, under the hypotheses of the theorem, we have defined a functor $f \mapsto \mathcal{A}((f))$ on the category of stable graphs, using the graded cyclic operad structure $\mathcal{A}^\flat$ and the contractions ξ_{ij}. It remains to show that these functors are related by (2.22) to the underlying modular pre-operad structure $\mu\colon \mathbb{M}\mathcal{A} \to \mathcal{A}$ associated to the special morphisms of graphs $G \to *_{g,n}$. This is evident if we order the vertices of G_1, and then order I in a compatible fashion; the decomposition of $\mathcal{A}((f))$ obtained from this ordering clearly corresponds to (2.22). □

4. Twisted modular operads

In this section, we introduce twisted triples $\mathbb{M}_{\mathfrak{D}}$, which will be used in the next section in the construction of the Feynman transform .

(4.1) **Hyperoperads**. A hyperoperad $\mathfrak{D}$ in a symmetric monoidal category $\mathcal{C}$ is a collection of functors from the categories $\mathrm{Iso}\Gamma((g,n))$, $g,n \geqslant 0$, of stable graphs and their isomorphisms to $\mathcal{C}$, together with the following data.

(4.1.1). To each morphism $f\colon G_0 \to G_1$ of $\Gamma((g,n))$ is assigned a morphism in $\mathcal{C}$

$$\nu_f\colon \mathfrak{D}(G_1) \otimes \bigotimes_{v\in\mathrm{Vert}(G_1)} \mathfrak{D}(f^{-1}(v)) \to \mathfrak{D}(G_0),$$

natural with respect to isomorphisms.

(4.1.2). If $*_{g,n}$ is the graph in $\Gamma((g,n))$ with no edges, $\mathfrak{D}(*_{g,n}) \cong \mathbb{1}$, the unit object of $\mathcal{C}$.

These data are required to satisfy the following conditions.

(4.1.3). Given a sequence of morphisms $G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2$ in $\Gamma((g,n))$, the following diagram commutes

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$$\begin{array}{ccc}
\mathfrak{D}(G_2) \otimes \bigotimes_{v \in \mathrm{Vert}(G_2)} \mathfrak{D}(f_2^{-1}(v)) \otimes \bigotimes_{v \in \mathrm{Vert}(G_1)} \mathfrak{D}(f_1^{-1}(v)) & \xrightarrow{\nu_{f_2}} & \mathfrak{D}(G_1) \otimes \bigotimes_{v \in \mathrm{Vert}(G_1)} \mathfrak{D}(f_1^{-1}(v)) \\
\cong \Big\| & & \Big\downarrow \nu_{f_1} \\
\mathfrak{D}(G_2) \otimes \bigotimes_{v \in \mathrm{Vert}(G_2)} (\mathfrak{D}(f_2^{-1}(v)) \otimes \bigotimes_{w \in f_2^{-1}(v)} \mathfrak{D}(f_1^{-1}(w))) & & \\
\Big\downarrow \mathfrak{D}(G_2) \otimes \bigotimes_{v \in \mathrm{Vert}(G_2)} \nu_{f_1|f_2^{-1}(v)} & & \\
\mathfrak{D}(G_2) \otimes \bigotimes_{v \in \mathrm{Vert}(G_2)} \mathfrak{D}((f_2 \cdot f_1)^{-1}(v)) & \xrightarrow{\nu_{f_2 \cdot f_1}} & \mathfrak{D}(G_0).
\end{array}$$

Here, $f_1|f_2^{-1}(v)$ denotes the restriction of f_1 to the subgraph $f_2^{-1}(v)$ of G_1.
(4.1.4). If $f\colon G_0 \to G_1$ is an isomorphism, the following diagram commutes

$$\begin{array}{ccc}
\mathfrak{D}(G_1) \otimes \otimes_{v \in \mathrm{Vert}(G_1)} \mathfrak{D}(f^{-1}(v)) & \xrightarrow{\nu_f} & \mathfrak{D}(G_0) \\
& \searrow\!\!\!\!= & \Big\downarrow \mathfrak{D}(f) \\
& & \mathfrak{D}(G_1).
\end{array}$$

(4.2) **Modular $\mathfrak{D}$-operads**. If $\mathfrak{D}$ is a hyperoperad, define an endofunctor $\mathbb{M}_{\mathfrak{D}}$ on the category of stable $\mathbb{S}$-modules by the formula

$$\mathbb{M}_{\mathfrak{D}} \mathcal{V}((g,n)) = \bigoplus_{G \in \mathrm{Iso}\Gamma((g,n))} \mathfrak{D}(G) \otimes \mathcal{V}((G)).$$

We show that $\mathbb{M}_{\mathfrak{D}}$ is a triple by imitating the proof that $\mathbb{M}$ is. The unit of the triple is again defined by the inclusion of the summand associated to the graph $*_{g,n}$ with no edges in $\Gamma((g,n))$: we may identify $\mathcal{V}(g,n))$ with $\mathfrak{D}(*_{g,n}) \otimes \mathcal{V}((*_{g,n}))$, and $\mathfrak{D}(*_{g,n}) \cong \mathbb{1}$ by (4.1.2).

We have the identity

$$(\mathbb{M}^2_{\mathfrak{D}} \mathcal{V})((g,n)) = \operatorname*{colim}_{\substack{[G_0 \xrightarrow{f_1} G_1] \\ \in \mathrm{Iso}_1 \Gamma((g,n))}} \left\{ \mathfrak{D}(G_1) \otimes \bigotimes_{v \in \mathrm{Vert}(G_1)} \mathfrak{D}(f_1^{-1}(v)) \otimes \mathcal{V}((G_0)) \right\}.$$

Using the hyperoperad structure maps ν_f, it is easy to define a natural transformation μ from $(\mathbb{M}^2_{\mathfrak{D}} \mathcal{V})((g,n))$ to $(\mathbb{M}_{\mathfrak{D}} \mathcal{V})((g,n))$. By (4.1.4), we see that η is a unit for this multiplication.

We also have the identity

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$$(\mathbb{M}^3_{\mathfrak{D}}\mathcal{V})(\!(g,n)\!)$$

$$= \operatorname*{colim}_{\substack{[G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2] \\ \in \mathrm{Iso}_2\Gamma(\!(g,n)\!)}} \Bigg\{ \mathfrak{D}(G_2) \otimes \bigotimes_{v\in \mathrm{Vert}(G_2)} \mathfrak{D}(f_2^{-1}(v))$$

$$\otimes \bigotimes_{v\in \mathrm{Vert}(G_1)} \mathfrak{D}(f_1^{-1}(v)) \otimes \mathcal{V}(\!(G_0)\!) \Bigg\}.$$

By (4.1.3), we see that the multiplication μ is associative.

A modular $\mathfrak{D}$-operad is an algebra over the triple $\mathbb{M}_{\mathfrak{D}}$.

(4.3) **Cocycles**. If $\mathcal{C}$ is a symmetric monoidal category, an object L is said to be invertible if there is an object L^{-1} and an isomorphism $\mathbb{1} \cong L \otimes L^{-1}$.

A cocycle is a hyperoperad $\mathfrak{D}$ such that $\mathfrak{D}(G)$ is invertible for all stable graphs G, and such that the morphisms $\mathfrak{D}_f$ associated to morphisms of stable graphs $f\colon G_0 \to G_1$ are isomorphisms. The inverse of a cocycle $\mathfrak{D}$ is again a cocycle, which we denote by $\mathfrak{D}^{-1}$.

(4.4) **Coboundaries**. Let $\mathfrak{l}$ be an $\mathbb{S}$-module such that each object $\mathfrak{l}(\!(g,n)\!)$ is invertible. Tensoring with $\mathfrak{l}$ defines a functor on $\mathbb{S}$-modules, which we denote by $\mathcal{V} \mapsto \mathfrak{l}\mathcal{V}$. There is a natural structure of a cocycle on

$$\mathfrak{D}_{\mathfrak{l}}(G) = \mathfrak{l}(\!(g,n)\!) \otimes \bigotimes_{v\in \mathrm{Vert}(G)} \mathfrak{l}(\!(g(v),n(v))\!)^{-1}$$

and a natural isomorphism of triples $\mathbb{M}_{\mathfrak{D}\otimes\mathfrak{D}_{\mathfrak{l}}} \cong \mathfrak{l} \circ \mathbb{M}_{\mathfrak{D}} \circ \mathfrak{l}^{-1}$. We call this cocycle the coboundary of $\mathfrak{l}$. It follows that if $\mathfrak{D}$ is a cocycle, the functor $\mathfrak{l}$ induces an equivalence between the category of modular $\mathfrak{D}$-operads and the category of modular $\mathfrak{D} \otimes \mathfrak{D}_{\mathfrak{l}}$-operads.

Let $\mathfrak{D}_{\mathfrak{s}}$ be the coboundary associated to the invertible stable $\mathbb{S}$-module $\mathfrak{s}$ given by

$$\mathfrak{s}(\!(g,n)\!) = \Sigma^{-2(g-1)-n}\varepsilon_n,$$

where ε_n is the alternating character of $\mathbb{S}_n$. Equation (2.10) shows that $\mathfrak{D}_{\mathfrak{s}}$ is concentrated in degree 0.

The functor $\mathcal{V} \mapsto \mathfrak{s}\mathcal{V}$ is called suspension. Since $\mathfrak{D}_{\mathfrak{s}}^2 \cong \mathbb{1}$, the double suspension of a modular operad is a modular operad. Note that the suspension of cyclic $\mathbb{S}$-modules, considered as stable $\mathbb{S}$-modules, coincides with the definition of suspension on cyclic $\mathbb{S}$-modules [12], given by the formula $\Lambda\mathcal{V}(n) = \Sigma^{1-n}\varepsilon_{n+1} \otimes \mathcal{V}(n)$.

Two further coboundaries which will be of interest are associated to the invertible stable $\mathbb{S}$-modules $\mathfrak{p}(\!(g,n)\!) = \Sigma^{-6(g-1)-2n}\mathbf{k}$ and $\Sigma(\!(g,n)\!) = \Sigma\mathbf{k}$.

(4.5) **Determinants**. For a finite-dimensional vector space V of dimension n, let $\mathrm{Det}(V)$ be the graded vector space

$$\mathrm{Det}(V) = \Sigma^{-n}\Lambda^n V;$$

this is the one-dimensional top exterior power of V, concentrated in degree $-n$. If S is a finite set, let $\mathrm{Det}(S) = \mathrm{Det}(\mathbf{k}^S)$. Observe that there is a natural isomorphism

$$\mathrm{Det}(S)^2 \cong \Sigma^{-2|S|}\mathbf{k}. \tag{4.6}$$

(4.7) LEMMA. *Given a collection of vector spaces $(V_i)_{i\in\mathrm{I}}$, there is natural identification*

$$\mathrm{Det}\,(\oplus_i V_i) \simeq \bigotimes_{i\in\mathrm{I}} \mathrm{Det}(V_i).$$

(4.8) **The dualizing cocycle**. By (4.7), we see that $\mathfrak{K}(G) = \mathrm{Det}(\mathrm{Edge}(G))$ is a cocycle, which we call the dualizing cocycle. Given a cocycle $\mathfrak{D}$, we denote the cocycle $\mathfrak{K}\otimes\mathfrak{D}^{-1}$ by $\mathfrak{D}^\vee$ and call it the dual of $\mathfrak{D}$. This duality will be important in the definition of the Feynman transform for modular operads.

(4.9) PROPOSITION. *There is a natural isomorphism of cocycles $\mathfrak{K}^2 \cong \mathfrak{D}_{\mathfrak{p}}$.*

Proof. There is a natural isomorphism $\mathfrak{D}_{\mathfrak{p}}(G) \cong \Sigma^{-2\ell}\mathbf{k}$, where

$$\ell = 3(g-1) + n - \sum_{v\in\mathrm{Vert}(G)} \big(3(g(v)-1) + n(v)\big)\,.$$

By (2.11), we see that $\ell = |\mathrm{Edge}(G)|$, from which the result follows. □

(4.10) **The orientation cocycle**. Let $\mathrm{Or}(e)$ be the orientation line of an edge e in a graph G, that is, the determinant $\Sigma^2\mathrm{Det}(\{s,t\})$, where s and t are the pair of flags making up the edge e. The orientation cocycle $\mathfrak{T}(G)$ of a graph G is the one-dimensional vector space

$$\mathfrak{T}(G) = \mathrm{Det}\left(\bigoplus_{e\in\mathrm{Edge}(G)} \mathrm{Or}(e)\right).$$

(4.11) PROPOSITION. *There is a natural isomorphism $\mathfrak{K} \cong \mathfrak{T}\otimes\mathfrak{D}_{\mathfrak{s}}$.*

Proof. If x and y are two independent elements of a vector space V, denote by $\Sigma^{-1}x\wedge\Sigma^{-1}y$ the corresponding element of $\mathrm{Det}(\mathrm{Span}\{x,y\})$. If s and t are the two flags making up an edge e, then $\mathrm{Or}(e)$ is spanned by $\Sigma^2(\Sigma^{-1}s\wedge\Sigma^{-1}t)$, and thus $(\mathrm{Or}(e))$ is spanned by a vector $\Sigma(\Sigma^{-1}s\wedge\Sigma^{-1}t)$. We may identify this with the element $\Sigma e\otimes(\Sigma^{-1}s\wedge\Sigma^{-1}t)$ of $\mathrm{Det}(\{e\})^{-1}\otimes\mathrm{Det}(\{s,t\})$. Tensoring over all edges of G, we obtain a natural isomorphism

$$\mathfrak{T}(G) \cong \mathrm{Det}(\mathrm{Edge}(G))^{-1}\otimes\mathrm{Det}(\mathrm{Flag}(G))\otimes\mathrm{Det}(\mathrm{Leg}(G))^{-1}.$$

(Here, we use the fact that $\operatorname{Det}(\operatorname{Flag}(G)) \otimes \operatorname{Det}(\operatorname{Leg}(G))^{-1}$ is the Det of the set of *internal* flags of G, those which are not legs.)

Thus, it remains to show that

$$\mathfrak{D}_{\mathfrak{s}}(G) \cong \operatorname{Det}(\operatorname{Edge}(G))^2 \otimes \operatorname{Det}(\operatorname{Flag}(G))^{-1} \otimes \operatorname{Det}(\operatorname{Leg}(G)).$$

Since $\mathfrak{D}_{\mathfrak{s}}$ is concentrated in degree 0, we see that

$$\mathfrak{D}_{\mathfrak{s}}(G) \cong \Sigma^{-2|\operatorname{Edge}(G)|} \operatorname{Det}(\operatorname{Flag}(G))^{-1} \otimes \operatorname{Det}(\operatorname{Leg}(G)).$$

The proposition now follows from (4.6), which shows that $\operatorname{Det}(\operatorname{Edge}(G))^2 \cong \Sigma^{-2|\operatorname{Edge}(G)|}$. □

Modular $\mathfrak{T}$-operads admit a notion of algebra parallel to that for modular operads, as is shown by the following proposition.

(4.12) PROPOSITION. *Let V be a chain complex with antisymmetric inner product $B(x, y)$ of degree -1. Define the stable $\mathbb{S}$-module $\mathcal{E}[V]$ of endomorphisms*

$$\mathcal{E}[V]((g, n)) = V^{\otimes n}.$$

There is a modular $\mathfrak{T}$-operad structure on $\mathcal{E}[V]$.

Proof. For a graph $G \in \Gamma((g, n))$, the composition $\mathcal{E}[V]((G)) \otimes \mathfrak{T}(G) \to \mathcal{E}[V]((g, n))$ is defined in the same way as in the untwisted case: we identify $\mathcal{E}[V]((G))$ with $V^{\otimes \operatorname{Flag}(G)}$ and contract with $B^{\otimes \operatorname{Edge}(G)}$. The resulting map is well-defined, and invariant under the action of the groups $\operatorname{Aut}(G)$ and $\mathbb{S}_n$: the antisymmetry of B is needed since reversing an edge $e \in \operatorname{Edge}(G)$ changes the sign of $\mathfrak{T}(G)$, while the degree of B must be -1, since supressing an edge $e \in \operatorname{Edge}(G)$ changes the degree of $\operatorname{Det}(G)$ by 1. □

(4.13) **The determinant of a graph**. The determinant of a graph G is defined to be $\operatorname{Det}(G) = \operatorname{Det}(H_1(G))$.

(4.14) PROPOSITION. *There is a natural isomorphism*

$$\operatorname{Det} \cong \mathfrak{T} \otimes \mathfrak{D}_{\Sigma}^{-1} \cong \mathfrak{K} \otimes \mathfrak{D}_{\mathfrak{s}}^{-1} \otimes \mathfrak{D}_{\Sigma}^{-1}.$$

In particular, Det *is a cocycle.*

Proof. This follows from applying (4.7) to the exact sequence of vector spaces arising from the complex of cellular chains of the graph G

$$0 \to H_1(G) \to \bigoplus_{e \in \operatorname{Edge}(G)} \operatorname{Or}(e) \to \mathbf{k} \otimes \operatorname{Vert}(G) \to H_0(G) \cong \mathbf{k} \to 0. \qquad \square$$

Since $\operatorname{Det}(H_1(G))$ is trivial when G is a tree, we see that a cyclic operad may be considered as a modular $\mathfrak{D}$-operad for either the trivial cocycle $\mathfrak{D} = \mathbb{1}$ or the determinant cocycle $\mathfrak{D} = \operatorname{Det}$.

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5. The Feynman transform of a modular operad

In this section, we define a functor $\mathsf{F}_{\mathfrak{D}}$ from the category of dg modular $\mathfrak{D}$-operads to the category of dg modular $\mathfrak{D}^\vee$-operads, where we recall that $\mathfrak{D}^\vee = \mathfrak{K} \otimes \mathfrak{D}^{-1}$, and $\mathfrak{K}$ is the dualizing cocycle (4.8). We call it the Feynman transform, since $\mathsf{F}_{\mathfrak{D}}\mathcal{A}$ is a sum over graphs, as is Feynman's expansion for amplitudes in quantum field theory.

The most important properties of $\mathsf{F}_{\mathfrak{D}}$ are that it is a homotopy functor, in the sense that it maps weak equivalences to weak equivalences, and that it has homotopy inverse $\mathsf{F}_{\mathfrak{D}^\vee}$: that is, there is a natural transformation from $\mathsf{F}_{\mathfrak{D}^\vee}\mathsf{F}_{\mathfrak{D}}$ to the identity functor such that for any modular operad $\mathcal{A}$, $\mathsf{F}_{\mathfrak{D}^\vee}\mathsf{F}_{\mathfrak{D}}\mathcal{A} \to \mathcal{A}$ is a weak equivalence. In this way, we see that the homotopy categories of modular $\mathfrak{D}$-operads and modular $\mathfrak{D}^\vee$-operads are equivalent.

In the special case where $\mathfrak{D} = 1\!\!1$ is the trivial cocycle, we denote $\mathsf{F}_{1\!\!1}$ by F, and $\mathsf{F}_{1\!\!1^\vee}$ by F^{-1}.

(5.1) **Definition of the Feynman transform.** As a stable $\mathbb{S}$-module, but ignoring differentials, $\mathsf{F}_{\mathfrak{D}}\mathcal{A}$ equals $\mathbb{M}_{\mathfrak{D}^\vee}\mathcal{A}^*$, the underlying stable $\mathbb{S}$-module of the free modular $\mathfrak{D}^\vee$-operad generated by the linear dual $\mathcal{A}^*$ of $\mathcal{A}$. The differential $\delta_{\mathsf{F}_{\mathfrak{D}}\mathcal{A}}$ is the sum $\delta_{\mathsf{F}_{\mathfrak{D}}\mathcal{A}} = \delta_{\mathcal{A}^*} + \partial$, where $\delta_{\mathcal{A}^*}$ is the differential on $\mathbb{M}_{\mathfrak{D}^\vee}\mathcal{A}^*$ induced by the differential on $\mathcal{A}^*$, and ∂ is defined as follows.

If G is a stable graph and e is an edge of G, the adjoint of the structure map of the morphism $\pi_{G,e}: G \to G/e$ is a map

$$(\mu_{\pi_{G,e}})^*: \mathfrak{D}(G/e)^* \otimes \mathcal{A}(\!(G/e)\!)^* \to \mathfrak{D}(G)^* \otimes \mathcal{A}(\!(G)\!)^*,$$

of degree 0. There is natural map $\varepsilon_e: \mathfrak{K}(G/e) \to \mathfrak{K}(G)$, given by tensoring with the natural basis element e of $\mathfrak{K}(\{e\}) = \mathrm{Det}(\{e\})$. Tensoring these two maps together, we obtain a map

$$\begin{array}{ccc} \mathfrak{K}(G/e) \otimes \mathfrak{D}(G/e)^* \otimes \mathcal{A}(\!(G/e)\!)^* & \xrightarrow{\varepsilon_e \otimes (\mu_{\pi_{G,e}})^*} & \mathfrak{K}(G) \otimes \mathfrak{D}(G)^* \otimes \mathcal{A}(\!(G)\!)^* \\ \| & & \| \\ \mathfrak{D}^\vee(G/e) \otimes \mathcal{A}(\!(G/e)\!)^* & \xrightarrow{\delta_{G,e}} & \mathfrak{D}^\vee(G) \otimes \mathcal{A}(\!(G)\!)^*, \end{array}$$

of degree -1.

Recall that $\mathbb{M}_{\mathfrak{D}^\vee}\mathcal{A}(\!(g,n)\!)$ is the sum of complexes

$$\big(\mathfrak{K}(G) \otimes \mathfrak{D}(G)^* \otimes \mathcal{A}(\!(G)\!)^*\big)_{\mathrm{Aut}(G)}$$

over isomorphism classes of stable graphs $G \in \mathrm{Ob}\Gamma(\!(g,n)\!)$. Given two stable graphs G and H, define the matrix element

$$(\mathfrak{K}(G) \otimes \mathcal{A}(\!(G)\!)^*)_{\mathrm{Aut}(G)} \xrightarrow{\partial_{G,H}} (\mathfrak{K}(H) \otimes \mathcal{A}(\!(H)\!)^*)_{\mathrm{Aut}(H)},$$

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to be the sum of the maps $\delta_{H,e}$ over all edges e of H such that $G \cong H/e$; in particular, $\partial_{G,H}$ vanishes if $|\mathrm{Edge}(G)| \neq |\mathrm{Edge}(H)| - 1$. The term $\delta_{H,e}$ does not depend on the isomorphism of H/e with G which is used, since any two such isomorphisms differ by an automorphism of G, and we have taken coinvariants with respect to $\mathrm{Aut}(G)$.

(5.2) THEOREM.

(1) *The map $\delta_{\mathsf{F}_{\mathfrak{D}}\mathcal{A}}$ has square zero.*
(2) *The pair $(\mathsf{F}_{\mathfrak{D}}\mathcal{A} = \mathbb{M}_{\mathfrak{D}^\vee}\mathcal{A}^*, \delta_{\mathsf{F}_{\mathfrak{D}}\mathcal{A}})$ is a modular $\mathfrak{D}^\vee$-operad of chain complexes.*
(3) *The Feynman transform $\mathsf{F}_{\mathfrak{D}}$ is a homotopy functor: if $f : \mathcal{A} \to \mathcal{B}$ is a weak equivalence of modular $\mathfrak{D}$-operads, then so is $\mathsf{F}_{\mathfrak{D}}f : \mathsf{F}_{\mathfrak{D}}\mathcal{B} \to \mathsf{F}_{\mathfrak{D}}\mathcal{A}$.*

Proof. The matrix element

$$(\mathfrak{K}(G) \otimes \mathcal{A}(\!(G)\!)^*)_{\mathrm{Aut}(G)} \xrightarrow{(\partial^2)_{G,K}} (\mathfrak{K}(K) \otimes \mathcal{A}(\!(K)\!)^*)_{\mathrm{Aut}(K)},$$

of ∂^2 is a sum over pairs (e_1, e_2) of distinct edges of K such that $G \cong K/\{e_1, e_2\}$. The exchange $(e_1, e_2) \mapsto (e_2, e_1)$ is a fixed-point free involution on the set of such pairs. The respective contributions $\delta_{K/e_1,e_2}\delta_{K,e_1}$ and $\delta_{K/e_2,e_1\}}\delta_{K,e_2}$ to ∂^2 cancel, since the two isomorphisms

$$\mathfrak{K}(K) \cong \mathfrak{K}(G) \otimes \mathfrak{K}(\{e_1, e_2\}),$$

in their definition are negatives of each other, showing that $\partial^2 = 0$.

It is clear that $\partial \circ \delta_{\mathcal{A}^*} + \delta_{\mathcal{A}^*} \circ \partial = 0$, since the differential in $\mathcal{A}$ which induces $\delta_{\mathcal{A}^*}$ is compatible with the structure maps of the modular $\mathfrak{D}$-operad $\mathcal{A}$. Together, these results show that $\delta_{\mathsf{F}_{\mathfrak{D}}\mathcal{A}}$ has square zero, proving part 1.

It is obvious that the internal differential $\delta_{\mathcal{A}^*}$ is compatible with the modular $\mathfrak{D}^\vee$-operad structure. To prove part (2), it remains to show that the differential ∂ is compatible with the structure maps of the modular $\mathfrak{D}$-operad $\mathcal{A}$

$$\begin{aligned}
&\mathfrak{D}^\vee(G) \otimes \mathbb{M}_{\mathfrak{D}^\vee}\mathcal{A}^*(\!(G)\!) \\
&\quad\cong \operatorname*{colim}_{\substack{[f:G'\to G] \\ \in \mathrm{Iso}\Gamma(\!(g,n)\!)/G}} \mathfrak{D}^\vee(G) \otimes \bigotimes_{v \in \mathrm{Vert}(G)} (\mathfrak{D}^\vee(f^{-1}(v)) \otimes \mathcal{A}^*(\!(f^{-1}(v))\!)) \\
&\quad\xrightarrow{\mu_G} \mathcal{A}(\!(g,n)\!);
\end{aligned}$$

recall that $\Gamma(\!(g,n)\!)/G$ is the comma category, whose objects are morphisms $f : G' \to G$ in $\Gamma(\!(g,n)\!)$ with target G. The differential induced by ∂ on $\mathsf{F}_{\mathfrak{D}}\mathcal{A}(\!(G)\!)$ is a sum index by the vertices v of G, of terms which are themselves a sum, over the vertices u of the graph $f^{-1}(v)$, of all ways of inserting an edge at u. This is clearly the same as summing over all ways of inserting an edge at all the vertices of G'. On application of the structure map μ_G, this goes into the differential ∂ of

$\mathsf{F}_{\mathfrak{D}}\mathcal{A}(\!(g,n)\!)$, showing that ∂ is compatible with the modular $\mathfrak{D}^\vee$-operad structure on $\mathsf{F}_{\mathfrak{D}}\mathcal{A}$.

Part (3) is proved by considering a spectral sequence associated to the cone of the map $\mathsf{F}_{\mathfrak{D}}f$: we filter by number of edges, in such a way that the E^1-term of the spectral sequence equals the cone of the map $\mathsf{F}_{\mathfrak{D}}H_*(f)$, which is zero by hypothesis. The convergence of the spectral sequence follows from the fact that $\mathsf{F}_{\mathfrak{D}}\mathcal{A}(\!(g,n)\!)$ and $\mathsf{F}_{\mathfrak{D}}\mathcal{B}(\!(g,n)\!)$ have contributions from a finite number of graphs, so that the spectral sequence is uniformly bounded in one direction. □

(5.3) **The homotopy inverse of the Feynman transform**. Under our hypothesis that $\mathcal{A}(\!(g,n)\!)$ is finite dimensional in each degree, there are isomorphisms of stable $\mathbb{S}$-modules

$$\mathsf{F}_{\mathfrak{D}^\vee}\mathsf{F}_{\mathfrak{D}}\mathcal{A} \cong \mathbb{M}_{\mathfrak{D}}\left(\mathbb{M}_{\mathfrak{D}^\vee}\mathcal{A}^*\right)^* \cong \mathbb{M}_{\mathfrak{D}}\mathbb{M}_{\mathfrak{K}^{-1}\otimes\mathfrak{D}}\mathcal{A},$$

which shows that $(\mathsf{F}_{\mathfrak{D}^\vee}\mathsf{F}_{\mathfrak{D}}\mathcal{A})(\!(g,n)\!)$ is a colimit over $[G_0 \xrightarrow{f} G_1] \in \mathrm{Iso}\Gamma_1(\!(g,n)\!)$ of the functor

$$[G_0 \xrightarrow{f} G_1] \mapsto \mathfrak{D}(G_1) \otimes \bigotimes_{v\in\mathrm{Vert}(G_1)} \Big(\mathfrak{K}(f^{-1}(v))^{-1} \otimes \mathfrak{D}(f^{-1}(v))\Big) \otimes \mathcal{A}(\!(G_0)\!).$$

The summand associated to the object $[*_{g,n} \xrightarrow{\mathrm{Id}} *_{g,n}]$ is isomorphic to $\mathcal{A}(\!(g,n)\!)$. Let $\tau\colon \mathsf{F}_{\mathfrak{D}^\vee}\mathsf{F}_{\mathfrak{D}}\mathcal{A} \to \mathcal{A}$ be the map induced by projection onto this summand. We now arrive at the main result of this section.

(5.4) THEOREM. *If $\mathcal{A}$ is a modular $\mathfrak{D}$-operad, the canonical map $\tau\colon \mathsf{F}_{\mathfrak{D}^\vee}\mathsf{F}_{\mathfrak{D}}\mathcal{A} \to \mathcal{A}$ is a weak equivalence, that is, induces an isomorphism on homology*.

Proof. Fix g and n. We start by analyzing the complex $S = (\mathsf{F}_{\mathfrak{D}^\vee}\mathsf{F}_{\mathfrak{D}}\mathcal{A})(\!(g,n)\!)$. The following lemma identifies the underlying graded $\mathbb{S}_n$-module $S^\sharp$.

(5.5) LEMMA. (a) *There is an isomorphism*

$$S = \operatorname*{colim}_{G\in\mathrm{Iso}\Gamma(\!(g,n)\!)} S(G) \cong \oplus_{G\in[\Gamma(\!(g,n)\!)]} S(G)_{\mathrm{Aut}(G)},$$

where

$$S(G) = \bigoplus_{I\subset\mathrm{Edge}(G)} S(G,\mathrm{I})$$

and

$$S(G,\mathrm{I}) = \mathfrak{D}(G/\mathrm{I}) \otimes \bigotimes_{v\in\mathrm{Vert}(G/\mathrm{I})} \Big(\mathfrak{K}(\pi_{G,\mathrm{I}}^{-1}(v))^{-1} \otimes \mathfrak{D}(\pi_{G,\mathrm{I}}^{-1}(v))\Big) \otimes \mathcal{A}(\!(G)\!).$$

(b) *The hyperoperad structure of $\mathfrak{D}$ induces a natural isomorphism* $S(G,\mathrm{I}) \cong \mathrm{Det}(\mathrm{I})^{-1} \otimes \mathfrak{D}(G) \otimes \mathcal{A}(\!(G)\!)$.

Proof. The proof of (a) is similar to the formula of (2.19) for $\mathbb{M}^2\mathcal{V}$, except that now, the cocycle factors associated to the two Feynman transforms must be inserted at the appropriate points.

The proof of (b) is as follows. The structure maps associated to the morphism $\pi_{G,\mathrm{I}}$ for the hyperoperads $\mathfrak{K}$ and $\mathfrak{D}$ induce isomorphisms

$$\mathfrak{D}(G/\mathrm{I}) \otimes \bigotimes_{v\in \mathrm{Vert}(G/\mathrm{I})} \mathfrak{D}(\pi_{G,\mathrm{I}}^{-1}(v)) \cong \mathfrak{D}(G),$$

$$\mathfrak{K}(G/\mathrm{I}) \otimes \bigotimes_{v\in \mathrm{Vert}(G/\mathrm{I})} \mathfrak{K}(\pi_{G,\mathrm{I}}^{-1}(v)) \cong \mathfrak{K}(G),$$

and the ratio of these formulas gives

$$\mathfrak{K}(G/\mathrm{I})^{-1} \otimes \mathfrak{D}(G/\mathrm{I}) \otimes \bigotimes_{v\in \mathrm{Vert}(G/\mathrm{I})} \Big(\mathfrak{K}(\pi_{G,\mathrm{I}}^{-1}(v))^{-1} \otimes \mathfrak{D}(\pi_{G,\mathrm{I}}^{-1}(v))\Big)$$
$$\cong \mathfrak{K}(G)^{-1} \otimes \mathfrak{D}(G).$$

Multiplying both sides by $\mathfrak{K}(G/\mathrm{I})$ and observing that $\mathfrak{K}(G/\mathrm{I})\otimes\mathfrak{K}(G)^{-1} \cong \mathrm{Det}(\mathrm{I})^{-1}$, part b) of the lemma follows. □

Recall that for any cocycle $\mathfrak{E}$ and any $\mathfrak{E}$-operad $\mathcal{B}$, the differential in $\mathsf{F}_{\mathfrak{E}}\mathcal{B}$ is a sum of two terms $\delta_{\mathcal{B}^*} + \partial$, where $\delta_{\mathcal{B}^*}$ is induced by the differential of $\mathcal{B}$, and ∂ is induced by the modular $\mathfrak{E}$-operad structure of $\mathcal{B}$, as explained in (5.1).

Applying this with $\mathfrak{E} = \mathfrak{D}^\vee$ and $\mathcal{B} = \mathsf{F}_{\mathfrak{D}}\mathcal{A}$, we find that the differential in $\mathsf{F}_{\mathfrak{D}^\vee}\mathsf{F}_{\mathfrak{D}}\mathcal{A}$ is a sum of three terms $\delta_0 + \delta_1 + \delta_2$:

(1) $\delta_0 = (\partial_0)_{\mathrm{Aut}(G)}: S(G)_{\mathrm{Aut}(G)} \to S(G)_{\mathrm{Aut}(G)}$ is the map induced on $\mathrm{Aut}(G)$-coinvariants by $\partial_0: S(G,\mathrm{I}) \to S(G,\mathrm{I})$, where ∂_0 is the differential induced on the summand $S(G,\mathrm{I})$ by the differential of $\mathcal{A}$;

(2) the differential

$$\delta_1: S(G)_{\mathrm{Aut}(G)} \to \bigoplus_{\substack{H\in[\Gamma((g,n))]\\ |\mathrm{Edge}(H)|=|\mathrm{Edge}(G)|+1}} S(H)_{\mathrm{Aut}(H)}$$

is induced by the differential ∂ in $\mathsf{F}_{\mathfrak{D}}\mathcal{A}$, which itself is induced by the modular $\mathfrak{D}$-operad structure of $\mathcal{A}$;

(3) $\delta_2 = (\partial_2)_{\mathrm{Aut}(G)}: S(G)_{\mathrm{Aut}(G)} \to S(G)_{\mathrm{Aut}(G)}$ is the map induced on $\mathrm{Aut}(G)$-coinvariants by

$$\partial_2: S(G,\mathrm{I}) \to \bigoplus_{e\in \mathrm{I}} S(G,\mathrm{I}\setminus\{e\}),$$

where ∂_2 comes from the differential ∂ on $\mathsf{F}_{\mathfrak{D}^\vee}\mathcal{B}$, and $\mathcal{B} = \mathsf{F}_{\mathfrak{D}}\mathcal{A}$.

Thus, δ_0 depends only on the internal differential of $\mathcal{A}$, δ_1 depends on the modular $\mathfrak{D}$-operad structure of $\mathcal{A}$, while δ_2 is purely combinatorial and only depends on the graded stable $\mathbb{S}$-module structure underlying $\mathcal{A}$.

(5.6) LEMMA. *The map* $\tau\colon (S, \delta_1 + \delta_2) \to \mathcal{A}(\!(g, n)\!)$ *is a weak equivalence of complexes*.

Let us first show how this lemma implies Theorem (5.4). Observe that δ_1 has the effect of increasing both $|\mathrm{I}|$ and $|\mathrm{Edge}(G)|$ by 1, while δ_2 leaves $|\mathrm{Edge}(G)|$ unchanged, and decreases $|\mathrm{I}|$ by 1. Therefore, S is the total complex of a double complex $(S_{\bullet\bullet}, \delta_0, \delta_1 + \delta_2)$, where

$$S_{pq} = \bigoplus_{G\in[\Gamma(\!(g,n)\!)]} \left(\bigoplus_{\substack{\mathrm{I}\subset \mathrm{Edge}(G) \\ |\mathrm{I}|=q+2|\mathrm{Edge}(G)|}} S_{p+q}(G, \mathrm{I}) \right)_{\mathrm{Aut}(G)},$$

where $S_{p+q}(G, \mathrm{I})$ is the degree $p+q$ subspace of the graded $\mathbb{S}_n$-module $S(G, \mathrm{I})$. Since there are a finite number of terms, indexed by G and I, contributing to this direct sum, this double complex has $p + q \geqslant 0$ and q bounded below, and thus its associated spectral sequence is convergent, yielding the desired implication.

We now turn to the proof of Lemma (5.6). Introduce the decreasing filtration Φ of S given by

$$\Phi^q(S) = \bigoplus_{\substack{G\in[\Gamma(\!(g,n)\!)] \\ |\mathrm{Edge}(G)|\geqslant q}} S(G)_{\mathrm{Aut}(G)}.$$

From the properties of δ_1 and δ_2 discussed above, we see that

$$\mathrm{gr}^{\Phi}(S) \cong (S, \delta_2).$$

This reduces the proof to that of the following lemma.

(5.7) LEMMA. *For each stable graph* G *with* $|\mathrm{Edge}(G)| > 0$, *the complex* $(S(G), \partial_2)$ *is acyclic*.

Indeed, on taking $\mathrm{Aut}(G)$-coinvariants, this lemma implies that the differential $\delta_2 = (\partial_2)_{\mathrm{Aut}(G)}$ on $S(G)_{\mathrm{Aut}(G)}$ is acyclic, since the group $\mathrm{Aut}(G)$ is finite and we work over a field of characteristic zero.

The proof of Lemma (5.7) is based on the identification

$$(S(G), \partial_2) \cong C_\bullet(\mathrm{Edge}(G)) \otimes \mathfrak{D}(G)^\sharp \otimes \mathcal{A}(\!(G)\!)^\sharp, \tag{5.8}$$

where, for any finite set X

$$C_\bullet(X) = \bigoplus_{\mathrm{I}\subset X} \mathrm{Det}(\mathrm{I})^{-1}$$

is the augmented chain complex of the simplex with vertices X. Of course, this chain complex is acyclic for X non-empty. The identification (5.8) follows from Lemma (5.5) (b).

This concluded the proof of Theorem (5.4). □

(5.9) **The Feynman transform and the cobar operad of a cyclic operad**. Let $\mathsf{B}\mathcal{A}$ be the cobar operad the cyclic operad $\mathcal{A}$, introduced in Section 3.2 of [13]. We may regard $\mathcal{A}$ as a modular operad (2.2); then $\mathrm{Cyc}(\mathsf{F}\mathcal{A})$ is related to $\mathsf{B}\mathcal{A}$ by the formula

$$\mathrm{Cyc}(\mathsf{F}\mathcal{A}) \cong \Sigma\mathfrak{s}\mathsf{B}\mathcal{A}.$$

(5.10) **The Feynman transform and Vassiliev invariants**. Vassiliev has introduced a filtered space $V = \bigcup_{k=0}^{\infty} V_k$ of knot invariants of finite order (see Theorems 8 and 9 of [2]). The associated graded space $W = \operatorname{gr} V$ is a commutative cocommutative Hopf algebra. Let $P = \bigoplus P_k$ be its space of primitives. One of the chief results of Kontsevich and Bar-Natan identifies P_k with the lowest homology groups of certain graph complex. In our language,

$$P_k \cong \bigoplus_{\substack{k=g-1+n \\ n>0}} H_{1-g}(\mathfrak{s}^{-1}\mathsf{F}\mathcal{C}om)(\!(g,n)\!)_{\mathbb{S}_n},$$

where $\mathcal{C}om$ is the commutative operad.

6. Modular operads and moduli spaces of curves

In this section, we give some basic examples of modular operads, coming from the theory of moduli spaces of stable algebraic curves. Throughout this section, the base field is taken to be the field of complex numbers $\mathbb{C}$.

(6.1) **Orbifolds**. Let $\mathcal{G}$ be a groupoid in the category of varieties over $\mathbb{C}$, with morphisms $\mathrm{Mor}(\mathcal{G})$, objects $\mathrm{Ob}(\mathcal{G})$, and source and target maps $s, t\colon \mathrm{Mor}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G})$. The groupoid $\mathcal{G}$ is called

(1) proper if the morphism $s \times t\colon \mathrm{Mor}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G}) \times \mathrm{Ob}(\mathcal{G})$ is proper;
(2) étale if s and t are étale;
(3) smooth if $\mathrm{Mor}(\mathcal{G})$ and $\mathrm{Ob}(\mathcal{G})$ are smooth.

An orbifold (smooth algebraic stack) is an equivalence class of smooth proper étale groupoids: two groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ are equivalent if there is an étale map $f\colon \mathrm{Ob}(\mathcal{G}_1) \to \mathrm{Ob}(\mathcal{G}_2)$ and an equivalence of categories $\mathcal{G}_1 \cong f^*\mathcal{G}_2$, or more generally, if they are joined by a chain of such equivalences. For more on orbifolds, see [6] and [7].

A sheaf $(\mathcal{S}, p)$ on an orbifold $\mathcal{G}$ is a sheaf $\mathcal{S}$ on $\mathrm{Ob}(\mathcal{G})$ together with an isomorphism $p: s^*\mathcal{S} \cong t^*\mathcal{S}$. A global section of such a sheaf is a global section f of $\mathcal{S}$ over $\mathrm{Ob}(\mathcal{G})$ such that s^*f and t^*f are identifed by p.

The coarse space $|\mathcal{G}|$ of an orbifold $\mathcal{G}$ is the quotient of $\mathrm{Ob}(\mathcal{G})$ by the action of $\mathcal{G}$, in other words, the space of isomorphism classes of objects of $\mathcal{G}$. Note that the coarse space $|\mathcal{G}|$ need not be smooth.

If G is a group acting on an orbifold $\mathcal{G}$, the quotient $\mathcal{G}/G$ is the orbifold with the same objects as $\mathcal{G}$, and whose morphisms are $G \times \mathrm{Mor}(\mathcal{G})$. The structure maps are defined as follows

$$s(g,x) = s(x), \qquad t(g,x) = g(t(x)), \qquad (g,x)(h,y) = (gh, h(x)y).$$

The coarse space of $\mathcal{G}/G$ is isomorphic to the quotient $|\mathcal{G}|/G$.

(6.2) **Deligne-Mumford moduli spaces**. If $2(g-1)+n > 0$, the (large) groupoid of smooth complex curves of genus g with n marked points, with isomorphisms as arrows, represents an orbifold $\mathcal{M}_{g,n}$, of dimension $3(g-1)+n$. As g and n are varied, we obtain an $\mathbb{S}$-orbifold, which we denote by $\mathcal{M}$.

Knudsen [22] proves that the (large) groupoid of stable complex curves of genus g with n marked points, again with isomorphisms as arrows, represents an orbifold $\overline{\mathcal{M}}_{g,n}$, of dimension $3(g-1)+n$. As g and n are varied, we obtain an $\mathbb{S}$-orbifold, which we denote by $\overline{\mathcal{M}}$, which contains $\mathcal{M}$ as a dense open subset.

The dual graph $G(C, x_1, \dots, x_n) \in \Gamma(\!(g,n)\!)$ of a stable curve $(C, x_1, \dots, x_n) \in \overline{\mathcal{M}}(\!(g,n)\!)$ is the labelled graph defined as follows. Its flags are pairs (K, y) where y is either a nodal point or a marked point x_i and K is a branch of the curve C at y. (Note that the curve has one branch at a marked point and two branches at a node.) Its vertices are the components of C, its edges are the nodes, and its legs are the points x_i. If $v \in G(C, x_1, \dots, x_n)$ is the vertex corresponding to the component $K \in C$, label v by the genus $g(v)$ of the desingularization of K.

Given $G \in \Gamma(\!(g,n)\!)$, denote by $\mathcal{M}_G \subset \overline{\mathcal{M}}(\!(g,n)\!)$ the orbifold of stable curves whose dual graph is G; note that $\mathcal{M}_G$ is isomorphic to the orbifold $\mathcal{M}(\!(G)\!)/\mathrm{Aut}(G)$. This gives a stratification of $\overline{\mathcal{M}}(\!(g,n)\!)$ whose strata correspond to elements of $\Gamma(\!(g,n)\!)$; the open stratum $\mathcal{M}(\!(g,n)\!)$ corresponds to the graph with no edges. The closure $\overline{\mathcal{M}}_G$ of $\mathcal{M}_G$ is isomorphic to the orbifold $\overline{\mathcal{M}}(\!(G)\!)/\mathrm{Aut}(G)$.

The $\mathbb{S}$-orbifold $\overline{\mathcal{M}}$ is a modular operad $\overline{\mathcal{M}}$, with product defined as follows: if $G \in \Gamma(\!(g,n)\!)$ is a stable graph, the composition map

$$\mu_G: \overline{\mathcal{M}}(\!(G)\!) = \prod_{v \in \mathrm{Vert}(G)} \overline{\mathcal{M}}(\!(g(v), v)\!) \to \overline{\mathcal{M}}(\!(g,n)\!) \tag{6.3}$$

is defined by gluing the marked points of the curves from $\overline{\mathcal{M}}(\!(g(v), v)\!)$, $v \in \mathrm{Vert}(g)$, according to the graph G (see [13], 1.4.3). This map induces the embedding of $\overline{\mathcal{M}}_G$ as a closed stratum of $\overline{\mathcal{M}}$.

Taking homology, we obtain a modular operad $H_\bullet(\overline{\mathcal{M}})$ in the category of graded vector spaces. An algebra over this operad is the same as a cohomological field theory in the sense of Kontsevich-Manin [25].

(6.4) **Differential forms with logarithmic singularities and principal values**. Let X be a compact n-dimensional complex manifold and $D \subset X$ a divisor with normal crossings. Let $D^k \subset X$ be the locus of k-fold self-intersection of D (so that $D^0 = X$ and $D^1 = D$), with inclusion morphisms $i^k \colon D^k \hookrightarrow X$ and $j^k \colon D^k \setminus D^{k+1} \hookrightarrow X$. Let $\pi^k \colon \tilde{D}^k \to D^k \subset X$ be the normalization morphism of D^k. The variety $\tilde{D}^k$ is smooth, and the preimage

$$\widehat{D}^{k+1} = (\pi^k)^{-1}(D^{k+1})$$

is a divisor in $\tilde{D}^k$ with normal crossings.

Let $\mathcal{E}_X^\bullet$ be the complex of sheaves of C^∞ differential forms on X. The complex $\mathcal{E}_X^\bullet(\log D)$ of sheaves of C^∞ differential forms with logarithmic singularities is the sheaf of subalgebras of $j^1_* \mathcal{E}^\bullet_{X \setminus D}$ generated by $\mathcal{E}_X^\bullet$ and forms $\mathrm{d}f/f$ where f is a holomorphic equation of D.

Let $\mathcal{E}^\bullet(X)$ and $\mathcal{E}^\bullet(X, \log D)$ be the spaces of global sections of the sheaves $\mathcal{E}_X^\bullet$ and $\mathcal{E}_X^\bullet(\log D)$. Each of the spaces $\mathcal{E}^i(X, \log D)$ and $\mathcal{E}^i(X)$ are nuclear Fréchet spaces, since they are spaces of C^∞ global sections of smooth vector bundles.

Let $\mathcal{C}_{X,\bullet}$ be the sheaf of de Rham currents on X. The space of global sections $\mathcal{C}_i(X) = \Gamma(X, \mathcal{C}_{X,i})$ is the topological dual of $\mathcal{E}^i(X)$, and the differential δ on $\mathcal{C}_{X,\bullet}$ has degree -1 and is adjoint to the exterior differential d on $\mathcal{E}_X^\bullet$.

The principal value (Herrera–Lieberman [17]) is the continuous map of graded sheaves

$$\mathrm{pv} \colon \mathcal{E}_X^\bullet(\log D) \to \mathcal{C}_{X,2n-\bullet}$$

defined as follows: if $U \subset X$ is an open set, $\alpha \in \Gamma(U, \mathcal{E}_X^i(\log D))$ and $\omega \in \Gamma_c(U, \mathcal{E}_X^{2n-i})$

$$\langle \mathrm{pv}\, \alpha, \omega \rangle = \lim_{\varepsilon \to 0} \int_{|\phi| \geqslant \varepsilon} \alpha \wedge \omega,$$

where ϕ is a holomorphic defining equation of $D \cap U$ in U. (The limit is independent of ϕ.)

The Poincaré residue is the map of graded sheaves

$$\mathrm{Res} \colon \mathcal{E}_X^\bullet(\log D) \to \pi^1_* \mathcal{E}^{\bullet-1}_{\tilde{D}}(\widehat{D}^2),$$

which measures the deviation of pv from being a map of complexes (Prop. 5.3 of [17]):

$$\delta(\mathrm{pv}(\alpha)) - \mathrm{pv}(\mathrm{d}\alpha) = \mathrm{pv}\left(2\pi i \mathrm{Res}(\alpha)\right), \qquad \alpha \in \mathcal{E}_X^\bullet(\log D). \tag{6.5}$$

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(6.6) **Currents with logarithmic singularities**. Denote the image of the injective map

$$\pi^k_* \mathcal{E}^\bullet_{\tilde{D}^k}(\log \widehat{D}^{k+1}) \xrightarrow{\text{pv}} \mathcal{C}_{X,2(n-k)-\bullet},$$

by $\mathcal{C}_{X,\bullet}(D,k)$, and the sum of these spaces as k varies between 0 and k by $\mathcal{C}_{X,\bullet}(D)$. (Note that these spaces have zero intersection, so the sum is direct.) We call elements of $\mathcal{C}_{X,\bullet}(D)$ currents with logarithmic singularities. By 6.5, the differential δ maps $\mathcal{C}_{X,\bullet}(D,k)$ to $\mathcal{C}_{X,\bullet}(D,k)\oplus\mathcal{C}_{X,\bullet}(D,k+1)$; thus, $\mathcal{C}_{X,\bullet}(D)$ is a complex of sheaves. Note that the spaces of global sections $\mathcal{C}_i(X,D)$ are nuclear Fréchet spaces.

(6.7) PROPOSITION. *The inclusion $\mathcal{C}_{X,\bullet}(D) \hookrightarrow \mathcal{C}_{X,\bullet}$ is a weak equivalence of complexes of sheaves.*

Proof. We may assume that $X = \mathbb{C}^n$, with the divisor D is given by the equation $z_1 \dots z_m = 0$, $m \leqslant n$; we must prove the weak equivalence for the stalks of the two complexes of sheaves at $0 \in \mathbb{C}^n$. If $I \subset \{1,\dots,n\}$, let $\mathbb{C}^{\mathrm{I}} \subset \mathbb{C}^n$ be the corresponding coordinate subspace, let $D^{\mathrm{I}} \subset \mathbb{C}^{\mathrm{I}}$ be the divisor given by $\prod_{i\in \mathrm{I}} z_i = 0$, and let $(\mathbb{C}^\times)^{\mathrm{I}} = \mathbb{C}^{\mathrm{I}} \setminus D^{\mathrm{I}}$, with embedding $j^{\mathrm{I}}\colon (\mathbb{C}^\times)^{\mathrm{I}} \hookrightarrow \mathbb{C}^n$. Let $i(\mathrm{I})$ be the least element of the set I.

The graded sheaf $\mathcal{C}_{\mathbb{C}^n,\bullet}(D)$ decomposes as a direct sum

$$\mathcal{C}_{\mathbb{C}^n,\bullet}(D) \cong \bigoplus_{\mathrm{I}\subset\{1,\dots,m\}} \mathcal{E}^{2(n-|\mathrm{I}|)-\bullet}_{\mathbb{C}^{\mathrm{I}}}(\log D^{\mathrm{I}}).$$

We write the associated decomposition of a current T lying in the stalk $\mathcal{C}_{\mathbb{C}^n,i}(D)_0$ of $\mathcal{C}_{\mathbb{C}^n,i}(D)$ at 0 as

$$T = \sum_{\mathrm{I}\subset\{1,\dots,m\}} j^{\mathrm{I}}_* \mathrm{pv}(\alpha_{\mathrm{I}}),$$

where α_{I} is in the stalk at 0 of $\mathcal{E}^{2(n-|\mathrm{I}|)-i}_{\mathbb{C}^{\mathrm{I}}}(\log D^{\mathrm{I}})$. Define a map $h\colon \mathcal{C}_{\mathbb{C}^n,i}(D)_0 \to \mathcal{C}_{\mathbb{C}^n,i+1}(D)_0$ by

$$h(T) = \frac{1}{2\pi i} \sum_{\substack{\mathrm{I}\subset\{1,\dots,m\}\\ |\mathrm{I}|>0}} j^{\mathrm{I}\setminus\{i(\mathrm{I})\}}_* \mathrm{pv}\left(\frac{dz_{i(\mathrm{I})}}{z_{i(\mathrm{I})}} \wedge \alpha_{\mathrm{I}}\right).$$

It is easily seen that this is a contracting homotopy from $\mathcal{C}_{\mathbb{C}^n,\bullet}(D)_0$ to $(\mathcal{C}_{\mathbb{C}^n,\bullet})_0$, proving the proposition. □

The above constructions generalize to the situation of a divisor D with normal crossings in an orbifold X. The orbifold X may be represented by a groupoid $\mathcal{G}$, and the divisor D gives rise to divisor with normal crossings in $\mathrm{Ob}(\mathcal{G})$, invariant under the action of $\mathcal{G}$ (that is, $s^{-1}(D) = t^{-1}(D)$). The sheaves $\mathcal{E}^\bullet_{\mathcal{G}}(\log D)$ and $\mathcal{C}_{\mathcal{G},\bullet}(D)$ are defined to be the subsheaves of $\mathcal{E}^\bullet_{\mathrm{Ob}(\mathcal{G})}(\log D)$ and $\mathcal{C}_{\mathrm{Ob}(\mathcal{G}),\bullet}(D)$ invariant under the action of $\mathcal{G}$, that is, such that $s^*\omega = t^*\omega$. (Pullbacks of currents on $\mathrm{Ob}(\mathcal{G})$ by the maps s and t are well-defined, since these maps are étale.)

(6.8) **The log-complex of $\overline{\mathcal{M}}_{g,n}$.** The compactification divisor $D_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ is a divisor with normal crossings in the orbifold $\overline{\mathcal{M}}_{g,n}$, which decomposes into an intersection of smooth divisors, corresponding to the graphs in $\Gamma(\!(g,n)\!)$ with one edge. Let $\mathcal{C}_\bullet(\overline{\mathcal{M}}, D)$ and $\mathcal{C}_\bullet(\overline{\mathcal{M}})$ be the stable $\mathbb{S}$-modules

$$\mathcal{C}_\bullet(\overline{\mathcal{M}}, D)(\!(g,n)\!) = \mathcal{C}_\bullet(\overline{\mathcal{M}}_{g,n}, D_{g,n}) \quad \text{and} \quad \mathcal{C}_\bullet(\overline{\mathcal{M}})(\!(g,n)\!) = \mathcal{C}_\bullet(\overline{\mathcal{M}}_{g,n}).$$

The operation of pushing forward currents along the inclusion of strata makes these into modular operads of chain complexes, which are weakly equivalent, and whose homology is the operad $H_\bullet(\overline{\mathcal{M}})$ of graded vector spaces.

(6.9) **The topological Feynman transform**. Recall [14] that nuclear Fréchet spaces form a symmetric monoidal category $\mathcal{NF}$, with operation $\widehat{\otimes}$ (projective tensor product). Furthermore, the opposite symmetric monoidal category $\mathcal{NF}^{\mathrm{op}}$ is identified, via the operation $V \mapsto V'$ (strong dual) with the category $\mathcal{DF}$ of nuclear DF-spaces, also with the projective tensor product.

Let $C(\mathcal{NF})$ and $C(\mathcal{DF})$ be the symmetric monoidal categories of bounded chain complexes with finite-dimensional homology over $\mathcal{NF}$ and $\mathcal{DF}$. The strong dual identifies $C(\mathcal{NF})^{\mathrm{op}}$ with $C(\mathcal{DF})$, and the homology of the dual complex is naturally dual to the homology of the original complex.

Imitating the construction of the Feynman transform in the topological setting, substituting the strong dual and projective tensor product for their algebraic analogues, we obtain a functor $\mathsf{F}^{\mathrm{top}}$, the topological Feynman transform, from modular operads in $C(\mathcal{NF})$ to modular $\mathfrak{K}$-operads in $C(\mathcal{DF})$. This functor has a homotopy inverse $\mathsf{F}^{\mathrm{top}}_\omega$, constructed in the analogous way.

The stable $\mathbb{S}$-module $\mathcal{C}_\bullet(\overline{\mathcal{M}}, D)$ is an example of a modular operad in $C(\mathcal{NF})$. By (6.7), its homology may be identified with $H_\bullet(\overline{\mathcal{M}}, D)$, the homology operad of the topological modular operad $\overline{\mathcal{M}}$.

(6.10) **The gravity operad**. Consider the stable $\mathbb{S}$-module Grav, given by

$$\mathsf{Grav}(\!(g,n)\!) = \mathcal{E}^\bullet(\overline{\mathcal{M}}_{g,n}, \log D_{g,n})'.$$

For any graph $G \in \Gamma(\!(g,n)\!)$ with one edge, we have the adjoint of the residue map

$$\mathrm{Res}^*_G : \mathsf{Grav}(\!(G)\!) \otimes \mathfrak{K}(G)^{-1} \to \mathsf{Grav}(\!(g,n)\!).$$

Iterating these maps, we may define Res^*_G for any stable graph. The maps $(2\pi i)^{|\mathrm{Edge}(G)|}\mathrm{Res}^*_G$ are the composition maps making Grav into a modular $\mathfrak{K}^{-1}$-operad in $C(\mathcal{DF})$, which we call the gravity operad.

Note that the homology $\mathcal{G}rav(\!(g,n)\!)$ of $\mathsf{Grav}(\!(g,n)\!)$ form a modular $\mathfrak{K}^{-1}$-operad in the category of finite-dimensional graded vector spaces, such that

$$\mathcal{G}rav(\!(g,n)\!) \cong H_\bullet(\mathcal{M}_{g,n}).$$

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The results of [9] show that the S^1-equivariant cohomology of a topological conformal field theory in two dimensions is a modular algebra over the suspension $\mathfrak{s}\mathsf{Grav}$. This paper also gives an explicit presentation for the cyclic operad $\mathrm{Cyc}(\Sigma\mathfrak{s}\mathsf{Grav})$. (See also [10].) This cyclic operad is formal, in the sense that there is a weak equivalence between it and and its homology. (See [13] and [10].) It seems unlikely that this is true for Grav and its homology $\mathcal{G}rav$.

Recall (4.4) the invertible stable $\mathbb{S}$-module $\mathfrak{p}(\!(g,n)\!) = \Sigma^{-6(g-1)-2n}$, whose coboundary satisfies $\mathfrak{D}_{\mathfrak{p}} \cong \mathfrak{K}^2$. We see that $\mathfrak{p}\mathsf{Grav}$ is a modular $\mathfrak{K}$-operad in $C(\mathcal{DF})$.

(6.11) PROPOSITION. *We have an isomorphism* $\mathsf{F}^{\mathrm{top}}_{\mathfrak{K}}\mathfrak{p}\mathsf{Grav} \cong \mathcal{C}_\bullet(\overline{\mathcal{M}}, D)$.

Proof. We have identifications

$$\begin{aligned}\mathcal{C}_\bullet(\overline{\mathcal{M}}_{g,n}, D_{g,n}) &\cong \bigoplus_k \mathcal{C}_\bullet(\overline{\mathcal{M}}_{g,n}, D_{g,n})\\ &\cong \bigoplus_k \mathcal{E}^{6(g-1)+2n-2k-\bullet}(D^k_{g,n}, \log D^{k+1}_{g,n})\\ &\cong \bigoplus_{G\in\Gamma(\!(g,n)\!)} \mathcal{E}^{6(g-1)+2n-2|\mathrm{Edge}(G)|-\bullet}(\overline{\mathcal{M}}(\!(G)\!), D(\!(G)\!))^{\mathrm{Aut}(G)},\end{aligned}$$

where $D(\!(G)\!) = \overline{\mathcal{M}}(\!(G)\!) \setminus \mathcal{M}(\!(G)\!)$. The component of the last sum corresponding to G is

$$\mathbb{M}\left(\mathfrak{p}^{-1}\mathcal{E}^\bullet(\overline{\mathcal{M}}, D)\right)(\!(G)\!)$$

and we obtain the sought after identification at the level of stable $\mathbb{S}$-modules. In fact, this identification also respects the compositions of the two modular operads. It remains to check that the differentials coincide; we leave this to the reader. □

As with any sort of cobar construction, the equation $\delta^2 = 0$ in the Feynman transform $\mathsf{F}^{\mathrm{top}}_{\mathfrak{K}}\mathfrak{p}\mathsf{Grav}$ is precisely the associativity of the composition in $\mathfrak{p}\mathsf{Grav}$. This gives a simple explanation of why Grav is a modular $\mathfrak{K}^{-1}$-operad.

7. Characteristics of cyclic operads

If $\mathcal{V}$ is a stable $\mathbb{S}$-module, we can associate to it a symmetric function $\mathrm{Ch}(\mathcal{V})$, called its characteristic. In this section and the next, we give formulas for $\mathrm{Ch}(\mathsf{B}\mathcal{A})$ in terms of $\mathrm{Ch}(\mathcal{A})$, where $\mathcal{A}$ is a cyclic operad, and for $\mathrm{Ch}(\mathsf{F}\mathcal{A})$ in terms of $\mathrm{Ch}(\mathcal{A})$, where $\mathcal{A}$ is a modular operad. The first of these formulas involves a generalization of the Legendre transform, and the second a generalization of the Fourier transform, from power series in one variable to symmetric functions in infinitely many variables. Here, symmetric functions arise because of well-known correspondence between the characters of the symmetric group and the ring of symmetric functions. For

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further details on the theory of symmetric functions, see Chapter 1 of Macdonald [26].

(7.1) **Symmetric functions**. Consider the ring

$$\Lambda = \varprojlim \mathbb{Z}[\![x_1, \dots, x_k]\!]^{\mathbb{S}_k}$$

of symmetric functions (power series) in infinitely many variables. The following standard symmetric functions

$$h_n(x_i) = \sum_{i_1 \leqslant \dots \leqslant i_n} x_{i_1} \dots x_{i_n}, \qquad e_n(x_i) = \sum_{i_1 < \dots < i_n} x_{i_1} \dots x_{i_n},$$

$$p_n(x_i) = \sum_{i=1}^{\infty} x_i^n$$

are called respectively the complete symmetric functions, the elementary symmetric functions and the power sums. It is a basic fact that

$$\Lambda = \mathbb{Z}[\![h_1, h_2, \dots]\!] = \mathbb{Z}[\![e_1, e_2, \dots]\!],$$

$$\Lambda \otimes \mathbb{Q} = \mathbb{Q}[\![p_1, p_2, \dots]\!],$$

that is, that each of these three series of symmetric functions freely generates Λ (in the case of the power sums, over $\mathbb{Q}$). In particular, $h_1 = e_1 = p_1$, while $h_2 = \frac{1}{2}(p_1^2 + p_2)$ and $e_2 = \frac{1}{2}(p_1^2 - p_2)$.

Let σ be an element of the symmetric group $\mathbb{S}_n$, with cycles of length $a_1 \geqslant a_2 \geqslant \dots \geqslant a_\ell$; thus $n = a_1 + \dots + a_\ell$. The cycle index of σ is the symmetric function

$$\psi(\sigma) = p_{a_1} \dots p_{a_\ell} \in \Lambda.$$

The characteristic of a finite-dimensional $\mathbb{S}_n$-module V is the symmetric function

$$\mathrm{ch}_n(V) = \frac{1}{n!} \sum_{\sigma \in \mathbb{S}_n} \mathrm{Tr}_V(\sigma)\psi(\sigma).$$

It may be proved that $\mathrm{ch}_n(V)$ is in Λ, although it is only evident from its definition that it is in $\Lambda \otimes \mathbb{Q}$.

We extend the definition of ch_n to graded $\mathbb{S}_n$-modules by

$$\mathrm{ch}_n(V) = \sum_i (-1)^i \mathrm{ch}_n(V_i),$$

where V_i is the degree i component of V. Finally, the characteristic of a graded $\mathbb{S}$-module $\mathcal{V} = \{\mathcal{V}(n) \mid n \geqslant 0\}$ such that $\mathcal{V}(n)$ is finite-dimensional for all n is

$$\mathrm{ch}(\mathcal{V}) = \sum_{n=0}^{\infty} \mathrm{ch}_n(\mathcal{V}(n)).$$

We denote by rk: $\Lambda \to \mathbb{Q}[\![x]\!]$ the ring homomorphism which sends

$$h_n \mapsto \frac{x^n}{n!},$$

or equivalently, $p_1 \mapsto x$ and $p_n \mapsto 0$, $n > 1$. If V is an $\mathbb{S}_n$-module,

$$\mathrm{rk}(\mathrm{ch}_n(V)) = \frac{\dim(V)x^n}{n!}.$$

For this reason, we call rk the rank homomorphism.

(7.2) **Plethysm**. Plethysm is the associative operation on Λ, denoted $f \circ g$, characterized by the formulas

(1) $(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$;
(2) $(f_1 f_2) \circ g = (f_1 \circ g)(f_2 \circ g)$;
(3) if $f = f(p_1, p_2, \ldots)$, then $p_n \circ f = f(p_n, p_{2n}, \ldots)$.

Note that under the rank homomorphism, plethysm is carried into composition of power series.

There is a monoidal structure on the category of $\mathbb{S}$-modules, with tensor product

$$(\mathcal{V} \circ \mathcal{W})(n) = \bigoplus_{k=0}^{\infty} \left(\mathcal{V}(k) \otimes \bigoplus_{f:\{1,\ldots,n\}\to\{1,\ldots,k\}} \bigotimes_{i=1}^{k} \mathcal{W}(f^{-1}(i)) \right)_{\mathbb{S}_k}.$$

(An operad $\mathcal{V}$ is just an $\mathbb{S}$-module with an associative composition $\mathcal{V} \circ \mathcal{V} \to \mathcal{V}$.)

(7.3) PROPOSITION. $\mathrm{ch}(\mathcal{V} \circ \mathcal{W}) = \mathrm{ch}(\mathcal{V}) \circ \mathrm{ch}(\mathcal{W})$

When $\mathcal{V}$ and $\mathcal{W}$ are ungraded, this is proved in Macdonald [26]. In the general case, the proof depends on an analysis of the interplay between the minus signs in the Euler characteristic and the action of symmetric groups on tensor powers of graded vector spaces.

(7.4) **Characteristic of $\mathbb{S}$-modules**. If $\mathcal{V} = \{\mathcal{V}(\!(n)\!) \mid n \geqslant 1\}$ is a cyclic $\mathbb{S}$-module, its characteristic is

$$\mathrm{Ch}(\mathcal{V}) = \sum_{n=1}^{\infty} \mathrm{ch}_n(\mathcal{V}(\!(n)\!)).$$

There is a forgetful functor from cyclic $\mathbb{S}$-modules to $\mathbb{S}$-modules, obtained by restricting the action of $\mathcal{V}(n) = \mathcal{V}(\!(n+1)\!)$ from $\mathbb{S}_{n+}$ to the subgroup $\mathbb{S}_n$. The characteristics of $\mathcal{V}$ considered as a cyclic $\mathbb{S}$-module and an $\mathbb{S}$-module are related by

$$\mathrm{ch}(\mathcal{V}) = \frac{\partial \mathrm{Ch}(\mathcal{V})}{\partial p_1}. \tag{7.5}$$

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(7.6) **Examples of characteristics**. To illustrate the above definitions, let us give some examples of characteristics of cyclic operads.

(7.6.1) *The commutative operad*. For the commutative operad $\mathcal{C}om$, $\mathcal{C}om(\!(n)\!)$ is the trivial representation of $\mathbb{S}_n$ for all $n \geqslant 3$, see (1.8.3). It follows that $\mathrm{ch}_n(\mathcal{C}om(\!(n)\!)) = h_n$ for $n \geqslant 3$, and hence that

$$\mathrm{Ch}(\mathcal{C}om) = \mathrm{Exp}\left(\sum_{n=1}^{\infty} \frac{p_n}{n}\right) - (1 + h_1 + h_2)\,.$$

(7.6.2) *The associative operad* Since (1.8.4) $\mathcal{A}ss(\!(n)\!) \cong \mathrm{Ind}_{\mathbb{Z}_n}^{\mathbb{S}_n} \mathbf{k}$,

$$\mathrm{ch}_n(\mathcal{A}ss(\!(n)\!)) = \sum_{d|n} \frac{\phi(d)}{n} p_d^{n/d},$$

where $\phi(d)$ is the Euler function (the number of units in $\mathbb{Z}/d$). Summing over $n \geqslant 3$, we see that

$$\mathrm{Ch}(\mathcal{A}ss) = -\sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log(1 - p_n) - (h_1 + h_2).$$

(7.6.3) *The Lie operad*. In (7.24) below, we will prove that the characteristic of the Lie operad is

$$\mathrm{Ch}(\mathcal{L}ie) = (1 - p_1) \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(1 - p_n) + p_1,$$

where $\mu(n)$ is the Möbius function.

(7.7) **The Legendre transform**. Classically, the Legendre transform of a convex function $f : \mathbb{R} \to \mathbb{R}$ is the function

$$(\mathcal{L}f)(\xi) = g(\xi) = \max_x \left(x\xi - f(x)\right).$$

(See Section 3.3 of Arnold [1].) Setting $\xi = f'(x)$, we see that

$$g \circ f' + f = xf'. \tag{7.8}$$

Suppose that, instead of being a convex function, $f(x)$ is a formal power series of the form

$$f(x) = \sum_{n=2}^{\infty} \frac{a_n x^n}{n!} \in \mathbb{Q}[\![x]\!], \tag{7.9}$$

where $a_2 \neq 0$; we denote this set of power series by $\mathbb{Q}[\![x]\!]_*$. The equation (7.8) defines a unique power series $(\mathcal{L}f)(\xi) = g(\xi) \in \mathbb{Q}[\![\xi]\!]_*$, which we again call the Legendre transform.

(7.10) PROPOSITION. *If f and g are series of the form* (7.9), *then $\mathcal{L}f = g$ if and only if f' and g' are inverse under composition, that is,*

$$g' \circ f' = x.$$

Proof. Taking the derivative of (7.8), we see that

$$(g' \circ f')f'' + f' = xf'' + f'.$$

Cancelling f' from each side and dividing by f'', which is invertible in $\mathbb{Q}[\![x]\!]$ by hypothesis, we find that $g' \circ f' = x$. The same reasoning proves the converse. □

As a consequence of this proposition, we see that $\mathcal{L}$ is involutive: $\mathcal{L}(\mathcal{L}f) = f$.

(7.11) $\mathcal{L}$ **and trees**. Let $\mathcal{V}$ be a cyclic $\mathbb{S}$-module with $\mathcal{V}(\!(n)\!) = 0$ for $n \leqslant 2$. The cyclic $\mathbb{S}$-module $\mathbb{T}\mathcal{V}$ was defined in (3.1).

(7.12) PROPOSITION. *Let $a_n = \chi(\mathcal{V}(\!(n)\!))$ and $b_n = \chi(\mathbb{T}\mathcal{V}(\!(n)\!))$ be the Euler characteristics of the components of $\mathcal{V}$ and $\mathbb{T}\mathcal{V}$. If*

$$f(x) = \frac{x^2}{2} - \sum_{n=3}^{\infty} \frac{a_n x^n}{n!} \quad \textit{and} \quad g(x) = \frac{x^2}{2} + \sum_{n=3}^{\infty} \frac{b_n x^n}{n!},$$

then $g = \mathcal{L}f$.

Proof. It is a corollary of Theorem 3.3.2 of [13] that $g' \circ f' = x$. The results follows by (7.10). □

With the notation of the proposition

$$b_n = \sum_{n\text{-trees}\,T} \prod_{v \in \mathrm{Vert}(T)} a_{n(v)}. \tag{7.13}$$

In fact, the proposition remains true for an arbitrary sequence of rational numbers $\{a_3, a_4, \ldots\}$, if we define $\{b_3, b_4, \ldots\}$ by (7.13).

(7.14) **The Legendre transform for symmetric functions**. Denote by Λ_* the set of symmetric functions such that $\mathrm{rk}(f) \in \mathbb{Q}[\![x]\!]_*$.

(7.15) THEOREM. (a) *If $f \in \Lambda_*$, there is a unique element $g = \mathcal{L}f \in \Lambda_*$ such that*

$$g \circ \frac{\partial f}{\partial p_1} + f = p_1 \frac{\partial f}{\partial p_1}. \tag{7.16}$$

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We call $\mathcal{L} : \Lambda_* \to \Lambda_*$ the Legendre transform.

(b) *The Legendre transform of symmetric functions is compatible with that of power series, in the sense that the following diagram commutes*

$$\begin{array}{ccc} \Lambda_* & \xrightarrow{\mathcal{L}} & \Lambda_* \\ {\scriptstyle \mathrm{rk}}\downarrow & & \downarrow{\scriptstyle \mathrm{rk}} \\ \mathbb{Q}[\![x]\!]_* & \xrightarrow{\mathcal{L}} & \mathbb{Q}[\![x]\!]_* . \end{array}$$

(c) *The symmetric functions*

$$\frac{\partial(\mathcal{L}f)}{\partial p_1} \quad \textit{and} \quad \frac{\partial f}{\partial p_1}$$

are plethystic inverses. (*Note that, unlike for power series, this equation does not determine $\mathcal{L}f$.*)

(d) *The transformation $\mathcal{L}$ is an involution, that is, $\mathcal{L}\mathcal{L} = \mathrm{Id}$.*

Proof. If $f \in \Lambda_*$, then $\partial f/\partial p_1$ is invertible with respect to plethysm. Thus (7.16) defines $g \in \Lambda_*$ uniquely, proving (a). Part (b) is obvious, since rk transforms plethysm into composition.

In proving (c), we need an analogue of the chain rule for $\partial/\partial p_1$ acting on Λ

$$\frac{\partial}{\partial p_1}(u \circ v) = \left(\frac{\partial u}{\partial p_1} \circ v\right) \frac{\partial v}{\partial p_1}.$$

This formula is proved by checking that both sides are compatible with the rules (1-3) defining plethysm (7.2).

Using this, the reasoning needed to prove (c) is formally identical to that in the proof of (7.10).

To prove (d), we note that (c) implies

$$p_1 \frac{\partial f}{\partial p_1} = \left(p_1 \frac{\partial g}{\partial p_1}\right) \circ \frac{\partial f}{\partial p_1}.$$

This shows that

$$g \circ \frac{\partial f}{\partial p_1} \circ \frac{\partial g}{\partial p_1} + f \circ \frac{\partial g}{\partial p_1} = \left(p_1 \frac{\partial f}{\partial p_1}\right) \circ \frac{\partial g}{\partial p_1} = \left(p_1 \frac{\partial g}{\partial p_1}\right) \circ \frac{\partial f}{\partial p_1} \circ \frac{\partial g}{\partial p_1}.$$

Cancellation proves that

$$g + f \circ \frac{\partial g}{\partial p_1} = p_1 \frac{\partial g}{\partial p_1}$$

and hence that $f = \mathcal{L}g$. □

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For example, $\mathcal{L}h_2 = e_2$ and vice versa.

The following theorem is related to results of Otter [29] and Hanlon–Robinson [15] on the enumeration of unrooted trees.

(7.17) THEOREM. *Let $\mathcal{V}$ be a cyclic $\mathbb{S}$-module such that $\mathcal{V}(\!(n)\!) = 0$ for $n \leqslant 2$ and $\mathcal{V}(\!(n)\!)$ is finite dimensional for all n. Define the elements of Λ_**

$$f = e_2 - \mathrm{Ch}(\mathcal{V}) \quad \textit{and} \quad g = h_2 + \mathrm{Ch}(\mathbb{T}\mathcal{V}).$$

Then $g = \mathcal{L}f$.

Proof. By definition of $\mathcal{L}$, we must prove that

$$\big(h_2 + \mathrm{Ch}(\mathbb{T}\mathcal{V})\big) \circ \big(p_1 - \mathrm{ch}(\mathcal{V})\big) + \big(e_2 - \mathrm{Ch}(\mathcal{V})\big) = p_1\,\big(p_1 - \mathrm{ch}(\mathcal{V})\big)\,,$$

since by (7.5), $f' = p_1 - \mathrm{ch}(\mathcal{V})$. A little rearrangement shows that this formula is equivalent to

$$\mathrm{Ch}(\mathbb{T}\mathcal{V}) \circ \big(p_1 - \mathrm{ch}(\mathcal{V})\big) = \mathrm{Ch}(\mathcal{V}) - e_2 \circ \mathrm{ch}(\mathcal{V}). \tag{7.18}$$

Indeed, the formulas $h_2 \circ (a+b) = h_2 \circ a + h_2 \circ b + ab$ and $h_2 \circ (-a) = e_2 \circ a$ show that

$$h_2 \circ \big(p_1 - \mathrm{ch}(\mathcal{V})\big) = h_2 + e_2 \circ \mathrm{ch}(\mathcal{V}) - p_1\mathrm{ch}(\mathcal{V}).$$

Equation (7.18) now follows from the formula $p_1^2 = h_2 + e_2$.

We prove (7.18) by constructing a differential graded $\mathbb{S}$-module $\mathcal{C} = \{\mathcal{C}(n)\}$ such that the left-hand side of (7.18) equals $\mathrm{ch}(\mathcal{C})$, and the right-hand side equals $\mathrm{ch}(H_\bullet(\mathcal{C}))$. Define the $\mathbb{S}$-module underlying $\mathcal{C}$ to be the plethysm $\mathcal{X} \circ \mathcal{W}$, where the $\mathbb{S}$-modules $\mathcal{X}$ and $\mathcal{W}$ are defined by

$$\mathcal{X}(n) = \begin{cases} 0, & n \leqslant 2, \\ (\mathbb{T}\mathcal{V})(\!(n)\!), & n \geqslant 3, \end{cases}$$

$$\mathcal{W}(n) = \begin{cases} 0, & n = 0, \\ \mathbf{k}, & n = 1, \\ \Sigma\,\mathrm{Res}^{\mathbb{S}_{n+}}_{\mathbb{S}_n}\mathcal{V}(\!(n+1)\!), & n \geqslant 2. \end{cases}$$

(Here, Σ is the suspension functor on graded $\mathbb{S}_n$-modules.) It follows from (7.3) that $\mathrm{ch}(\mathcal{C})$ equals the right-hand side of (7.18).

We now construct a differential δ on $(\mathcal{X} \circ \mathcal{W})(n)$. We say that a vertex v of a tree T is a boundary vertex if exactly one of its flags forms part of an edge; denote by $\beta(T) \subset \mathrm{Vert}(T)$ the set of boundary vertices of T. Then

$$(\mathcal{X} \circ \mathcal{W})(n) = \bigoplus_{n\text{-trees}\,T}\ \bigoplus_{B \subset \beta(T)} \left(\bigotimes_{v \in \mathrm{Vert}(T)\setminus B} \mathcal{V}(\!(v)\!) \otimes \bigotimes_{v \in B} \Sigma\mathcal{V}(\!(v)\!) \right). \tag{7.19}$$

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On the summand of (7.19) associated to (T, B), define the differential

$$\delta = \sum_{v \in B} \delta_v,$$

where δ_v is the natural identification, of degree -1, between this summand and the summand associated to $(T, B \setminus \{v\})$.

Clearly $(\mathcal{X} \circ \mathcal{W})(n)$ splits into a sum of subcomplexes $\mathcal{C}_T$ indexed by the n-trees T. If T has at least one non-boundary vertex, the complex $\mathcal{C}_T$ is isomorphic to the tensor product of the graded vector space $\mathcal{V}((T))$ and the augmented chain complex of the simplex with vertices $\beta(T)$, and is thus contractible. As observed by Jordan [20], there remain trees with either one vertex or one edge. We consider each of these cases separately.

(1) The characteristic $\mathrm{ch}(\mathcal{C}_T)$ summed over trees T with one vertex equals $\mathrm{Ch}(\mathcal{V})$.
(2) The characteristic $\mathrm{ch}(\mathcal{C}_T)$ summed over trees T with one edge has two contributions: the terms with B empty, which sum to $h_2 \circ \mathrm{ch}(\mathcal{V})$, and the terms in which $|B| = 1$, which sum to $-\mathrm{ch}(\mathcal{V})^2$. The sum of these two terms is $-e_2 \circ \mathrm{ch}(\mathcal{V})$. □

(7.20) **The involution $\tilde{\omega}$ and the characteristic of the cobar operad**. Using the theorem just proved, we now write a formula for $\mathrm{Ch}(\mathsf{B}\mathcal{A})$, where $\mathcal{A}$ is a cyclic operad. Up to differential, $\mathsf{B}\mathcal{A}$ is the cyclic operad $\mathbb{T}\mathfrak{s}^{-1}\Sigma^{-1}\mathcal{A}^*$, and thus $\mathrm{Ch}(\mathsf{B}\mathcal{A}) = \mathrm{Ch}(\mathbb{T}\mathfrak{s}^{-1}\Sigma^{-1}\mathcal{A}^*)$. Since $\mathrm{Ch}(\Sigma^{-1}\mathcal{A}^*) = -\mathrm{Ch}(\mathcal{A})$, it suffices to determine the effect of $\mathfrak{s}$ and Σ on $\mathrm{Ch}(\mathcal{V})$.

Denote by $\omega\colon \Lambda \to \Lambda$ the ring homomorphism such that $\omega(h_n) = e_n$, $n \geqslant 1$. If V is a finite-dimensional $\mathbb{S}_n$-module,

$$\omega(\mathrm{ch}_n(V)) = \mathrm{ch}_n(\varepsilon_n \otimes V)$$

and thus ω is an involution. Note also that $\omega(p_n) = (-1)^{n-1}p_n$.

We also need a modified involution $\tilde{\omega}$, defined by $\tilde{\omega}(h_n) = (-1)^n e_n$, or equivalently $\tilde{\omega}(p_n) = -p_n$. Thus, if $\mathcal{V}$ is a cyclic $\mathbb{S}$-module such that $\mathcal{V}((n))$ is finite-dimensional for each n,

$$\mathrm{Ch}(\mathfrak{s}\mathcal{V}) = \tilde{\omega}(\mathrm{Ch}(\mathcal{V})). \tag{7.21}$$

(7.22) COROLLARY. *Let $\mathcal{A}$ be a cyclic operad such that $\mathcal{A}((n)) = 0$ for $n \leqslant 2$ and $\mathcal{A}((n))$ is finite-dimensional for each n, and let $\mathsf{B}\mathcal{A}$ be its cobar operad. Then*

$$h_2 + \mathrm{Ch}(\mathsf{B}\mathcal{A}) = \mathcal{L}\tilde{\omega}(h_2 + \mathrm{Ch}(\mathcal{A})).$$

Recall [13] that $\mathsf{BB}\mathcal{A}$ is weakly equivalent to $\mathcal{A}$, which suggests that the transform $\mathcal{L}\tilde{\omega}\colon \Lambda_* \to \Lambda_*$ should be an involution. This follows from the next result.

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(7.23) PROPOSITION. *If $f \in \Lambda_*$, then $-\mathcal{L}\tilde{\omega}f = \mathcal{L}(-f)$.*

Proof. By Example 8.1 of Macdonald [26], if u and v are symmetric functions, $u \circ (-v) = (\tilde{\omega}u) \circ v$. If $g = \mathcal{L}(\tilde{\omega}f)$, we see that the defining equation

$$(\tilde{\omega}f) \circ \frac{\partial g}{\partial p_1} + g = p_1 \frac{\partial g}{\partial p_1}$$

is equivalent to

$$-f \circ \frac{\partial(-g)}{\partial p_1} + (-g) = -p_1 \frac{\partial(-g)}{\partial p_1},$$

that is, $-g = \mathcal{L}(-f)$.

(7.24) EXAMPLE: *the Lie operad*. The Lie operad $\mathcal{L}ie$ is weakly equivalent to the cobar operad $\mathsf{B}\mathcal{C}om$ of the commutative operad, see [13], and thus $h_2 + \mathrm{Ch}(\mathcal{L}ie)$ is the Legendre transform of

$$\begin{aligned}\tilde{\omega}(h_2 + \mathrm{Ch}(\mathcal{L}ie)) &= \tilde{\omega}\left(\mathrm{Exp}\left(\sum_{n=1}^{\infty} \frac{p_n}{n}\right) - (1 + h_1)\right) \\ &= \mathrm{Exp}\left(\sum_{n=1}^{\infty} -\frac{p_n}{n}\right) - (1 - p_1).\end{aligned}$$

The Legendre transform of this symmetric function is

$$(1 - p_1) \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(1 - p_n) + p_1.$$

It follows that

$$\mathrm{Ch}(\mathcal{L}ie) = (1 - p_1) \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(1 - p_n) + h_1 - h_2,$$

as was promised in (7.6.3).

8. Characteristics of modular operads

(8.1) **The ring** $\Lambda(\!(\hbar)\!)$. Consider the ring $\Lambda(\!(\hbar)\!)$ of Laurent series with coefficients in Λ. This ring has a descending filtration

$$F^m \Lambda(\!(\hbar)\!) = \left\{ \sum f_i \hbar^i \mid f_i \in F^{m-2i} \Lambda \right\},$$

inducing a topology on $\Lambda(\!(\hbar)\!)$. If $f \in \Lambda$, the plethysm $f \circ (-) \colon \Lambda \to \Lambda$ extends to $\Lambda(\!(\hbar)\!)$ by retaining axioms (1) and (2) of (7.2) and replacing (3) by

(3′) $p_n \circ f(\hbar, p_1, p_2, \ldots) = f(\hbar^n, p_n, p_{2n}, \ldots)$.

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(8.2) **The characteristic of a stable $\mathbb{S}$-module**. The characteristic of a stable $\mathbb{S}$-module $\mathcal{V}$ is the element of $\Lambda(\!(\hbar)\!)$ given by the formula

$$\mathbb{C}\mathrm{h}(\mathcal{V}) = \sum_{2(g-1)+n>0} \hbar^{g-1} \mathrm{ch}_n(\mathcal{V}(\!(g,n)\!)).$$

The stability condition ensures that $\mathbb{C}\mathrm{h}(\mathcal{V}) \in F^1\Lambda(\!(\hbar)\!)$. If $\mathcal{V}$ is a cyclic $\mathbb{S}$-module, the characteristic $\mathbb{C}\mathrm{h}(\mathcal{V})$ of $\mathcal{V}$ considered as a stable $\mathbb{S}$-module, equals $\hbar^{-1}\mathrm{Ch}(\mathcal{V})$.

Note that as in the case of cyclic $\mathbb{S}$-modules, we have

$$\mathbb{C}\mathrm{h}(\mathfrak{s}\mathcal{V}) = \tilde{\omega}(\mathbb{C}\mathrm{h}(\mathcal{V})). \tag{8.3}$$

Our goal in this section is to present formulas for $\mathbb{C}\mathrm{h}(\mathbb{M}\mathcal{A})$ and $\mathbb{C}\mathrm{h}(\mathbb{M}_{\mathrm{Det(Edge)}}\mathcal{A})$ in terms of $\mathbb{C}\mathrm{h}(\mathcal{V})$. This will also permit us to give formulas for $\mathbb{C}\mathrm{h}(\mathsf{F}\mathcal{A})$ and $\mathbb{C}\mathrm{h}(\mathsf{F}^{-1}\mathcal{A})$.

(8.4) **Plethystic exponential**. For $f \in F^1\Lambda(\!(\hbar)\!)$, let

$$\mathrm{Exp}(f) = \left(\sum_{n=0}^{\infty} h_n\right) \circ f = \mathrm{Exp}\left(\sum_{n=1}^{\infty} \frac{p_n}{n}\right) \circ f.$$

Note that

$$\mathrm{Exp}(f+g) = \mathrm{Exp}(f)\,\mathrm{Exp}(g)$$

and that under specialization rk: $\Lambda(\!(\hbar)\!) \to \mathbb{Q}[\![x]\!](\!(\hbar)\!)$, the map Exp goes into exponentiation

$$f(\hbar,x) \mapsto e^{f(\hbar,x)}.$$

(8.5) PROPOSITION. *If $\mathcal{V}$ is a stable $\mathbb{S}$-module, let $\mathrm{Exp}_n(\mathcal{V})$ be the stable $\mathbb{S}$-module such that*

$$\mathrm{Exp}_n(\mathcal{V})(\!(g,n)\!) = \left(\bigoplus_{\substack{f:\mathrm{I}\to\{1,\dots,n\}\\ g_1+\dots+g_n=g}} \mathrm{Ind}^{\mathrm{Aut(I)}}_{\mathrm{Aut}(f)} \left(\bigotimes_{i=1}^{n} \mathcal{V}(\!(g_i, f^{-1}(i))\!) \right)\right)_{\mathbb{S}_n},$$

where $\mathrm{Aut}(f) = \mathrm{Aut}(f^{-1}(1)) \times \ldots \times \mathrm{Aut}(f^{-1}(n))$. Then

$$\mathrm{Exp}(\mathbb{C}\mathrm{h}(\mathcal{V})) = \sum_{n=0}^{\infty} \hbar^{-n}\mathbb{C}\mathrm{h}(\mathrm{Exp}_n(\mathcal{V})).$$

Proof. This follows from (7.3) and the definition of $\mathrm{Exp}(f)$, $f \in \Lambda(\!(\hbar)\!)$.

Informally, the stable $\mathbb{S}$-module $\mathrm{Exp}_n(\mathcal{V})$ may be thought of as representing disconnected graphs with n vertices and no edges: all of its flags are legs.

The following proposition is essentially due to Cadogan [4], although he does not use the notation Exp.

(8.6) PROPOSITION. *The map* Exp: $F^1\Lambda((\hbar)) \to 1 + F^1\Lambda((\hbar))$ *is invertible over* $\mathbb{Q}$, *with inverse*

$$\operatorname{Log}(f) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(p_n) \circ f = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(p_n \circ f).$$

Proof.

$$\begin{aligned} \operatorname{Log}(\operatorname{Exp}(f)) &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(p_n) \circ \operatorname{Exp}\left(\sum_{m=1}^{\infty} \frac{p_m}{m}\right) \circ f \\ &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \operatorname{Exp}\left(\sum_{m=1}^{\infty} \frac{p_{nm}}{m}\right) \circ f \\ &= \sum_{n=1}^{\infty} \sum_{d|n} \frac{\mu(d) p_n \circ f}{n} = f. \end{aligned}$$

□

(8.7) **The inner product on** Λ. To a partition $\lambda = (1^{m_1} 2^{m_2} \dots)$, where $m_k = 0$ for $k \gg 0$, is associated a monomial

$$p_\lambda = p_1^{m_1} p_2^{m_2} \dots .$$

These monomials form a topological basis of Λ. Let Λ_{alg} be the space of finite linear combinations of the p_λ. The standard inner product on Λ_{alg} is determined by the formula

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} \prod_{i=1}^{\infty} i^{m_i} m_i!$$

Note in particular that $\langle p_i, p_j \rangle = i\delta_{ij}$; the inner product on Λ_{alg} is the standard extension of the inner product on a vector space to its symmetric algebra (Fock space).

(8.8) PROPOSITION. *If V and W are $\mathbb{S}_n$-modules,*

$$\langle \operatorname{ch}_n(V), \operatorname{ch}_n(W) \rangle = \dim \operatorname{Hom}_{\mathbb{S}_n}(V, W).$$

Proof. This statement is well-known in the theory of symmetric functions: it follows from the fact that the Schur functions form an orthonormal basis of Λ_{alg}. □

We extend the inner product on Λ_{alg} to a $\mathbb{Q}((\hbar))$-valued inner product on $\Lambda_{\text{alg}}((\hbar))$ by $\mathbb{Q}((\hbar))$-bilinearity. If $f \in \Lambda_{\text{alg}}((\hbar))$, let $D(f)\colon \Lambda((\hbar)) \to \Lambda((\hbar))$ be the adjoint of

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multiplication by f with respect to this inner product. The following proposition is Example. 5.3 of Macdonald [26].

(8.9) PROPOSITION. *If* $f = f(\hbar, p_1, p_2, \dots) \in \Lambda_{\mathrm{alg}}(\!(\hbar)\!)$, *then*

$$D(f) = f\left(\hbar, \frac{\partial}{\partial p_1}, 2\frac{\partial}{\partial p_2}, 3\frac{\partial}{\partial p_3}, \dots\right).$$

(8.10) PROPOSITION. *Let* $k \leqslant n$, V *be an* $\mathbb{S}_k$*-module, and* W *be an* $\mathbb{S}_n$*-module. Then*

$$D(\mathrm{ch}_k(V))\mathrm{ch}_n(W) = \mathrm{ch}_{n-k}\mathrm{Hom}_{\mathbb{S}_k}\left(V, \mathrm{Res}^{\mathbb{S}_n}_{\mathbb{S}_k\times\mathbb{S}_{n-k}} W\right).$$

Proof. This follows by taking adjoints on both sides of the formula

$$\mathrm{ch}_j(U)\mathrm{ch}_k(V) = \mathrm{ch}_{j+k}\mathrm{Ind}^{\mathbb{S}_{j+k}}_{\mathbb{S}_j\times\mathbb{S}_k}(U \otimes V). \qquad \square$$

(8.11) **A Laplacian on** $\Lambda(\!(\hbar)\!)$. We now introduce an analogue of the Laplacian on $\Lambda(\!(\hbar)\!)$, given by the formula

$$\Delta = \sum_{n=1}^{\infty} \hbar^n \left(\frac{n}{2}\frac{\partial^2}{\partial p_n^2} + \frac{\partial}{\partial p_{2n}}\right).$$

Note that Δ is homogeneous of degree zero, and thus preserves the filtration of $\Lambda(\!(\hbar)\!)$. Under specialization rk: $\Lambda(\!(\hbar)\!) \to \mathbb{Q}[\![x]\!](\!(\hbar)\!)$, the operator Δ corresponds to the Laplacian $\frac{\hbar}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on the line.

(8.12) PROPOSITION. $D(\mathrm{Exp}(\hbar h_2)) = \mathrm{Exp}(\Delta)$.

Proof. By (8.9), it suffices to substitute $n\partial/\partial p_n$ for p_n on the right-hand side of

$$\begin{aligned} \mathrm{Exp}(\hbar h_2) &= \mathrm{Exp}\left(\sum_{n=1}^{\infty}\frac{p_n}{n}\right) \circ \left(\frac{\hbar}{2}(p_1^2 + p_2)\right) \\ &= \mathrm{Exp}\left(\sum_{n=1}^{\infty}\frac{\hbar^n}{2n}(p_n^2 + p_{2n})\right). \qquad \square \end{aligned}$$

(8.13) THEOREM. *If* $\mathcal{V}$ *is a stable* $\mathbb{S}$*-module, then*

$$\mathbb{C}\mathrm{h}(\mathbb{M}\mathcal{V}) = \mathrm{Log}\left(\mathrm{Exp}(\Delta)\,\mathrm{Exp}(\mathbb{C}\mathrm{h}(\mathcal{V}))\right).$$

Proof. We start by neglecting $\hbar$ and explain the appearance of the sum over graphs on the right-hand side of the formula. Formally, we set $\hbar = 1$; this is legitimate if $\mathcal{V}(\!(g, n)\!) = 0$ for $g \gg 0$.

Applying Exp to $\mathbb{Ch}(\mathcal{V})$, we obtain the stable $\mathbb{S}$-module representing possibly disconnected graphs each component of which has one vertex. Applying $D(h_2)$ to $\operatorname{Exp}(\mathbb{Ch}(\mathcal{V}))$ gives the sum over all ways of joining two legs (or flags) of such a graph; h_2 arises because the two ends of an edge are indistinguishable. (If edges carried a direction, we would replace h_2 by p_1^2, the characteristic of the regular representation of $\mathbb{S}_2$.)

Similarly, applying $D(\operatorname{Exp}(\hbar h_2))$ to $\operatorname{Exp}(\mathbb{Ch}(\mathcal{V}))$ gives the sum over all ways of joining together any number N of pairs of legs by edges. In this way, we see (recall that $\hbar$ temporarily equals 1) that

$$\operatorname{Exp}(\Delta)\operatorname{Exp}(\mathbb{Ch}(\mathcal{V})) = \mathbb{Ch}(\mathcal{W}),$$

where $\mathcal{W}$ is the stable $\mathbb{S}$-module such that

$$\mathcal{W}(\!(g,n)\!) = \bigoplus_G \mathcal{V}(\!(G)\!)_{\operatorname{Aut}(G)}, \tag{8.14}$$

where G runs over all possibly disconnected, labelled n-graphs such that each component is stable. But $\mathcal{W} = \operatorname{Exp}(\mathbb{M}\mathcal{V})$, since $\mathbb{M}\mathcal{V}$ is defined in a similar way, but summing only over connected graphs.

To finish the proof, we must account for the powers of $\hbar$ in each term of (8.14). Each term $\operatorname{ch}_n(\mathcal{V}(\!(g,n)\!))$ in $\mathbb{Ch}(\mathcal{V})$ comes with a factor of $\hbar^{g-1}$. The term of $\operatorname{Exp}(\mathbb{Ch}(\mathcal{V}))$ corresponding to a labelled graph G (with each component having one vertex) comes with a factor of $\hbar$ raised to the power

$$-|\operatorname{Vert}(G)| + \sum_{v\in\operatorname{Vert}(G)} g(v).$$

Each new edge introduced by the action of $D(\operatorname{Exp}(\hbar h_2))$ contributes a factor of $\hbar$. Therefore, the term in (8.14) corresponding to a labelled graph G comes with a factor of $\hbar$ raised to the power

$$-\chi(G) + \sum_{v\in\operatorname{Vert}(G)} g(v).$$

Applying Log has the effect of discarding all the disconnected graphs G. If G is connected, the power of $\hbar$ in question equals $g(G) - 1$, where $g(G)$ is defined in (2.9). □

Recall that F denotes the Feynman transform $\mathsf{F}_{\mathbb{1}}$ associated to the trivial cocycle.

(8.15) COROLLARY. $\mathbb{Ch}(\mathsf{F}\mathcal{A}) = \operatorname{Log}\left(\operatorname{Exp}(-\Delta)\right)\operatorname{Exp}(\mathbb{Ch}(\mathcal{A})))$

Proof. If $\mathcal{A}$ is a modular $\mathfrak{K}$-operad, the stable $\mathbb{S}$-modules $\mathbb{M}\mathcal{A}$ and $\mathsf{F}^{-1}\mathcal{A}$ have the same characteristic, showing that

$$\mathbb{Ch}(\mathsf{F}^{-1}\mathcal{A}) = \operatorname{Log}\left(\operatorname{Exp}(\Delta)\right)\operatorname{Exp}(\mathbb{Ch}(\mathcal{A}))) .$$

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Since F is a homotopy inverse of F^{-1}, and characteristics are homotopy invariant, the result follows.

Note that (8.15) may also be proved by calculating $\mathbb{C}\mathrm{h}(\mathbb{M}_{\mathfrak{K}}\mathcal{V})$; this is given by the stated formula, since the effect of twisting by $\mathrm{Det}(\mathrm{Edge})$ is to attach a suspension to each edge, which changes the operator $D(\mathrm{Exp}(\hbar h_2))$ into $D(\mathrm{Exp}(-\hbar h_2)) = \exp(-\Delta)$.

(8.16) **Plethystic Fourier transform**. Let us give a formal interpretation of the previous theorem in terms of the Fourier transform on the infinite-dimensional vector space $\mathrm{Spec}(\Lambda_{\mathbb{R}}) \cong \mathbb{R}^\infty$, with coordinates $p_1, p_2, \dots$, where $\Lambda_{\mathbb{R}} = \Lambda_{\mathrm{alg}} \otimes \mathbb{R}$. This space has a translation invariant Riemannian metric

$$\langle p_i, p_j \rangle = i\delta_{ij}. \tag{8.17}$$

We denote the function $p_n \otimes 1 \in \Lambda_{\mathbb{R}} \otimes \Lambda_{\mathbb{R}}$ by p_n, and the function $1 \otimes p_n$ by q_n.

Let $\mathrm{d}\mu$ be the formal Gaussian measure

$$\mathrm{d}\mu = \prod_{n \text{ odd}} e^{-p_n^2/2n\hbar^n} \prod_{n \text{ even}} e^{-p_n^2/2n\hbar^n + p_n/n\hbar^{n/2}} \frac{\mathrm{d}p_n}{e^{c(n)/2n}\sqrt{2\pi n\hbar^n}},$$

on $\mathrm{Spec}(\Lambda_{\mathbb{R}})$, where $c(n) = \frac{1}{2}((-1)^n + 1)$. We may rewrite this measure as

$$\mathrm{d}\mu = \mathrm{Exp}(-e_2/\hbar) \prod_{n=1}^{\infty} \frac{\mathrm{d}p_n}{e^{c(n)/2n}\sqrt{2\pi n\hbar^n}};$$

it is the translate of the Gaussian measure associated to the metric (8.17) by the vector $(p_1, p_2, \dots) = (0, \hbar, 0, \hbar^2, 0, \hbar^3, 0, \dots)$.

If p^α is a monomial in the variables $p = (p_1, p_2, \dots)$, where α is a multi-index $(\alpha_1, \alpha_2, \dots)$, define a power series

$$\int_{\mathbb{R}^\infty}^{*} p^\alpha \, \mathrm{d}\mu(p) \in \mathbb{Z}[\![\hbar]\!]$$

by the formula

$$\int_{\mathbb{R}^\infty}^{*} p^\alpha \, \mathrm{d}\mu(p) = \prod_{n \text{ odd}} \int_{-\infty}^{\infty} p_n^{\alpha_n} e^{-p_n^2/2n\hbar^n} \frac{\mathrm{d}p_n}{\sqrt{2\pi n\hbar^n}}$$

$$\times \prod_{n \text{ even}} \int_{-\infty}^{\infty} p_n^{\alpha_n} e^{-p_n^2/2n\hbar^n + p_n/n\hbar^{n/2}} \frac{\mathrm{d}p_n}{e^{1/2n}\sqrt{2\pi n\hbar^n}}.$$

This formula makes sense because almost all terms equal 1, and it may be defined in purely algebraic way by induction on $|\alpha|$: for $\alpha = 0$ it equals 1, and the induction step is performed by means of integration by parts in one of the variables p_n. Extend the operation $f \mapsto \int_{\mathbb{R}^\infty}^{*} f \, \mathrm{d}\mu(p)$ to a map from $\Lambda(\!(\hbar)\!)$ to $\mathbb{Z}(\!(\hbar)\!)$ by linearity.

We may now restate (8.13) in the form of a (Gaussian) Fourier transform. In the course of the proof, we make use of another formal integral $\int^*_{\mathbb{R}^\infty} f\,\mathrm{d}\nu(p)$, whose definition is similar to $\int^*_{\mathbb{R}^\infty} f\,\mathrm{d}\mu(p)$ and is given by the formula

$$\int^*_{\mathbb{R}^\infty} p^\alpha\,\mathrm{d}\nu(p) = \prod_n \int_{-\infty}^{\infty} p_n^{\alpha_n} e^{-p_n^2/2n\hbar^n} \frac{\mathrm{d}p_n}{\sqrt{2\pi n\hbar^n}}.$$

(8.18) THEOREM. *As a function of* $\hbar$ *and* $q = (q_1, q_2, \ldots)$

$$\hbar^{-1}h_2 + \mathbb{C}\mathrm{h}(\mathbb{M}\mathcal{V}) = \mathrm{Log} \int^*_{\mathbb{R}^\infty} \mathrm{Exp}(\hbar^{-1}p_1q_1 + \mathbb{C}\mathrm{h}(\mathcal{V}))\,\mathrm{d}\mu(p).$$

Proof. Using the explicit heat kernel

$$\exp\left(\tfrac{1}{2}t\frac{\partial^2}{\partial q^2}\right) f(q) = \int_{-\infty}^{\infty} f(p)\exp(-(p-q)^2/2t)\,\frac{\mathrm{d}p}{\sqrt{2\pi t}},$$

we see that, at a formal level

$$\begin{aligned}&\exp(\Delta) f(\hbar, q_1, q_2, \ldots)\\ &\quad = \int_{\mathbb{R}^\infty} \exp\left(\sum_{n=1}^{\infty} \hbar^n \frac{\partial}{\partial p_{2n}}\right) f(\hbar, p) \prod_{n=1}^{\infty} \frac{e^{-(p_n-q_n)^2/2n\hbar^n}\mathrm{d}p_n}{\sqrt{2\pi n\hbar^n}}.\end{aligned}$$

Using the formal integral $\int^*_{\mathbb{R}^\infty} f\,\mathrm{d}\mu(p)$, we may rewrite this rigourously as

$$\exp\left(-\sum_{n=1}^{\infty}\frac{q_n^2}{2n\hbar^n}\right)\int^*_{\mathbb{R}^\infty}\left\{\exp\left(\sum_{n=1}^{\infty}\hbar^n\frac{\partial}{\partial p_{2n}}\right) f(\hbar,p)\right\}\exp\left(\sum_{n=1}^{\infty}\frac{p_nq_n}{n\hbar^n}\right)\,\mathrm{d}\nu(p).$$

Integrating by parts, we obtain

$$\begin{aligned}&\exp\left(-\sum_{n=1}^{\infty}\frac{q_n^2}{2n\hbar^n}\right)\int^*_{\mathbb{R}^\infty} f(p)\exp\left(\sum_{n=1}^{\infty}\frac{p_nq_n}{n\hbar^n} + \sum_{n \text{ even}}\left\{\frac{p_n-q_n}{n\hbar^{n/2}} - \frac{1}{2n}\right\}\right)\,\mathrm{d}\nu(p)\\ &\quad = \exp\left(-\sum_{n=1}^{\infty}\frac{q_n^2+q_{2n}}{2n\hbar^n}\right)\int^*_{\mathbb{R}^\infty} f(p)\exp\left(\sum_{n=1}^{\infty}\frac{p_nq_n}{n\hbar^n}\right)\,\mathrm{d}\mu(p)\\ &\quad = \mathrm{Exp}(-\hbar^{-1}h_2)\int^*_{\mathbb{R}^\infty} f(p)\mathrm{Exp}(\hbar^{-1}p_1q_1)\,\mathrm{d}\mu. \qquad \square\end{aligned}$$

Although it is possible that (7.17), the analogue for cyclic operads, can be obtained from (8.18) by the principle of stationary phase, we do not know how to do this.

In the next section, we need the following consequence of (8.18).

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(8.19) COROLLARY.

$$-\hbar^{-1}e_2 + \mathbb{C}\mathrm{h}(\mathsf{F}_{\mathrm{Det}}\mathcal{A}) = -\mathrm{Log}\int^*_{\mathbb{R}^\infty} \mathrm{Exp}\Big(\hbar^{-1}p_1q_1 - \mathbb{C}\mathrm{h}(\mathcal{A})\Big)\, \mathrm{d}\mu(p).$$

Proof. By (4.14), we have $\mathrm{Det}^\vee \cong \mathfrak{D}_{\mathfrak{s}} \otimes \mathfrak{D}_\Sigma$. Applying (8.3), we see that

$$\mathbb{C}\mathrm{h}(\mathsf{F}_{\mathrm{Det}}\mathcal{A}) \cong \mathbb{C}\mathrm{h}(\Sigma\mathfrak{s}\mathbb{M}\mathfrak{s}^{-1}\Sigma^{-1}\mathcal{A}^*) \cong -\tilde{\omega}\mathbb{C}\mathrm{h}(\mathbb{M}\mathfrak{s}\Sigma\mathcal{A}).$$

Since $\tilde{\omega}\,\mathrm{Log} = \mathrm{Log}\,\tilde{\omega}$, the result follows from (8.18). □

9. Euler characteristics of moduli spaces of curves and $\mathbb{C}\mathrm{h}(\mathsf{F}_{\mathrm{Det}}\mathcal{A}ss)$

In this section, we apply the results of Section 8 to calculate $\mathbb{C}\mathrm{h}(\mathsf{F}_{\mathrm{Det}}\mathcal{A}ss)$ explicitly. Using the decompositions of moduli spaces of curves found by Harer, Mumford, Penner and others, we obtain new information on the Euler characteristics of these moduli spaces (9.19).

(9.1) DEFINITION. A ribbon graph is a graph G, each vertex of which has valence at least 3, together with a cyclic order on the set of flags v making up each vertex $v \in \mathrm{Vert}(G)$. (This is what Penner [30] calls a fat graph.)

Equivalently, a ribbon structure on a graph G is the same as an isotopy class of embeddings of the CW complex $|G|$ into a compact oriented Riemann surface $\Sigma(G)$ with boundary, such that

(1) the intersection of the image of $|G|$ with the boundary $\partial\Sigma(G)$ is the set of endpoints of the legs of $|G|$;
(2) the image of $|G|$ is a deformation retract of $\Sigma(G)$.

The cyclic orders of the sets v are then induced by the embedding $|G| \hookrightarrow \Sigma(G)$ and the orientation of $\Sigma(G)$.

Denote by $\gamma(G)$ and $\nu(G)$ the genus and number of boundary components of $\Sigma(G)$. Note that $2(\gamma(G)-1)+\nu(G) = g(G)-1$, where $g(G) = \dim H_1(|G|)$.

Ribbon graphs are related to the operad $\mathcal{A}ss$ in the following way. Since $\mathcal{A}ss(\!(n)\!) \cong \mathrm{Ind}^{\mathbb{S}_n}_{\mathbb{Z}_n}(\mathbf{k})$, it has a basis $\{e_\sigma\}$ labelled by the cyclic orders on $\{1,\dots,n\}$ (see (1.8.4)). It follows that $\mathsf{F}\mathcal{A}ss(\!(g,n)\!)$ has a natural basis $\{e_G\}$ labelled by all ribbon n-graphs G with $g(G) = g$.

(9.2) PROPOSITION. *If $\mathsf{F}\mathcal{A}ss(\!(\gamma,\nu,n)\!)$ is the subcomplex spanned by ribbon graphs G with $\gamma(G)=\gamma$ and $\nu(G)=\nu$, there is a splitting of chain complexes*

$$\mathsf{F}\mathcal{A}ss(\!(g,n)\!) = \bigoplus_{2(\gamma-1)+\nu=g-1} \mathsf{F}\mathcal{A}ss(\!(\gamma,\nu,n)\!).$$

Proof. If e is an edge of a ribbon graph G, there is a natural ribbon graph structure on G/e. If e is a loop, the vertex of G/e corresponding to e has genus 1, and since $\mathcal{A}ss(\!(g,n)\!) = 0$ for $g > 0$, we see that $\mathcal{A}ss(\!(G/e)\!) = 0$; thus, the term of the differential on $\mathsf{F}\mathcal{A}ss$ associated to this edge vanishes. On the other hand, if both ends of e are distinct, then $\gamma(G/e) = \gamma(G)$ and $\nu(G/e) = \nu(G)$, so that the corresponding term of the differential preserves the splitting. □

We may also consider the cyclic operad $\mathcal{A}ss$ as a modular Det-operad, where Det is the cocycle $\mathrm{Det}(G) = \mathrm{Det}(H^1(G))$. (This uses the fact that the cocycle Det is canonically trivial on trees; see (4.13).) The Feynman transform $\mathsf{F}_{\mathrm{Det}}\mathcal{A}ss$ also has a basis labelled by ribbon graphs, and we have a decomposition of $\mathsf{F}_{\mathrm{Det}}\mathcal{A}ss$ into a sum of subcomplexes $\mathsf{F}_{\mathrm{Det}}\mathcal{A}ss(\!(\gamma,\nu,n)\!)$ similar to (9.2).

(9.3) **$\mathsf{F}\mathcal{A}ss$, $\mathsf{F}_{\mathrm{Det}}\mathcal{A}ss$ and moduli spaces of curves**. In this section, we relate the complexes $\mathsf{F}\mathcal{A}ss(\!(\gamma,\nu,n)\!)$ and $\mathsf{F}_{\mathrm{Det}}\mathcal{A}ss(\!(\gamma,\nu,n)\!)$ to cell decompositions of moduli spaces of punctured curves (which we learnt about from Penner). These moduli spaces are differentiable orbifolds, by which we mean, by analogy to the algebraic case (6.2), a proper étale differentiable groupoid $\mathcal{G}$.

A ν-punctured curve is a pair (Σ, A), where Σ is a smooth projective algebraic curve over $\mathbb{C}$, and $A \subset \Sigma$ is a finite subset. An isomorphism of two punctured curves $(\Sigma_1, A_1) \to (\Sigma_2, A_2)$ is an isomorphism $\Sigma_1 \to \Sigma_2$ inducing a bijection $A_1 \to A_2$. A frame λ of a punctured curve (Σ, A) is an element of the circle bundle over A whose fibre at $z \in A$ is the quotient of $T_z\Sigma\backslash\{0\}$ by the dilatation group $\mathbb{R}_+^\times$.

An n-framed punctured curve is an object $(\Sigma, A, \lambda_1, \dots, \lambda_n)$, where (Σ, A) is a punctured curve, and $(\lambda_1, \dots, \lambda_n)$ are n distinct frames in (Σ, A). An isomorphism of n-framed punctured curves is defined in the obvious way.

The groupoid of n-framed pointed curves of genus γ such that $|A| = \nu$, and their isomorphisms, represents a differentiable orbifold $Q_{\gamma,\nu,n}$. In fact, using the method of level structures, this groupoid is seen to be equivalent to a transformation groupoid (the quotient, in the sense of orbifolds, of a space by a group action). Note that for $n = 0$,

$$Q_{\gamma,\nu,0} = \mathcal{M}_{\gamma,\nu}/\mathbb{S}_\nu.$$

A cellular decomposition of an orbifold $\mathcal{G}$ is a cellular decomposition of $\mathrm{Ob}(\mathcal{G})$ whose inverse images in $\mathrm{Mor}(\mathcal{G})$ under the étale maps s and t: $\mathrm{Mor}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G})$ coincide. Associated to this decomposition is a cochain complex $C^\bullet(|\mathcal{G}|)$, the invariants of the action of $\mathrm{Mor}(\mathcal{G})$ on the cellular cochain complex of $C^\bullet(\mathrm{Ob}(\mathcal{G}))$. This complex may be thought of as the cochain complex associated to the decomposition of $|\mathcal{G}|$ into orbicells: these are the image in $|\mathcal{G}|$ of cells in $\mathrm{Ob}(\mathcal{G})$, and are quotients of cells by finite groups.

In the cellular decompositions which we study, the cells will not necessarily be relatively compact; thus, the cellular cochain complex $C^\bullet(\mathcal{G})$ calculates the cohomology with compact supports $H_c^\bullet(\mathcal{G})$; this is isomorphic to the cohomology with compact supports of $|\mathcal{G}|$, as long as we work over a field of characteristic zero.

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The following result was communicated to us by R. Penner. Only the cases $g = 0$ and $n = 0$ may be found in the literature, and it is only the case $n = 0$ which we will need.

(9.4) THEOREM. *For all* $\gamma \geqslant 1$, $\nu \geqslant 0$ *and* $n \geqslant 0$, *there is an orbifold* $P_{\gamma,\nu,n}$, *fibred over* $Q_{\gamma,\nu,n}$ *with fibres* $\mathbb{R}^{\nu}_{+}$, *with a natural* $\mathbb{S}_n$*-equivariant cell decomposition, and a natural identification* $C^{\bullet}(|P_{\gamma,\nu,n}|) \cong \mathsf{F}\mathcal{A}ss(\!(\gamma,\nu,n)\!)$.

Informally, $P_{\gamma,\nu,n}$ parametrizes Riemann surfaces of genus γ with ν boundary circles, together with n numbered points on the boundary; the circumferences of these circles are labelled by points in the fibre $\mathbb{R}^{\nu}_{+}$.

(9.4.1) $\boldsymbol{g} = \mathbf{0}$. In this case, only ribbon graphs with the topology of trees contribute to $\mathsf{F}\mathcal{A}ss(\!(0,n)\!) = \mathsf{F}\mathcal{A}ss(\!(0,1,n)\!) \cong \Sigma\mathfrak{s}\mathsf{B}\mathcal{A}ss(\!(n)\!)$. Every ribbon n-graph with the topology of a tree can be embedded into the plane, inducing a cyclic order on the set $\{1,\ldots,n\}$ of legs of the graph. Thus, $\mathsf{B}\mathcal{A}ss(\!(n)\!)$ splits into a sum of subcomplexes $\mathsf{B}\mathcal{A}ss(\!(n)\!)_\sigma$ labelled by cyclic orders σ.

On the other hand, $Q_{0,1,n}$ is the quotient of the configuration space of n distinct points in S^1 by rotations, and is the union of cells K_σ corresponding to cyclic orders σ as above. Each cell K_σ may be identified with the interior of the Stasheff polytope K_{n+1} [33], and the cellular decomposition K_σ of (9.4) is (Poincaré) dual to the face decomposition of K_{n+1}, since faces of K_{n+1} correspond to planar trees.

(9.4.2) $\boldsymbol{n} = \mathbf{0}$. For this case, we mention the references [30] and [24]. In the first of these, the orbifold $P_{\gamma,\nu}$ (decorated Teichmüller space) is constructed, and a cellular decomposition given, whose cells are in bijection with isotopy classes of so-called ‘ideal cell decompositions’ of a fixed Riemann surface Σ of genus γ with ν punctures. As remarked on page 40 of Penner [31], the (Poincaré) dual of an ideal cell decomposition is a ‘spine’ on Σ, i.e. a graph G in Σ together with a deformation retraction of Σ to G. This shows that the cells in Penner’s decomposition of $P_{\gamma,\nu}$ are in bijection with ribbon graphs G. As for the differential on $C^{\bullet}(|P_{\gamma,\nu})$, we only need the following result, which we establish explicitly.

(9.5) PROPOSITION. $H^i_c(\mathcal{M}_{\gamma,\nu}/\mathbb{S}_\nu) \cong \Sigma^{1-2\gamma} H_{-i}(\mathsf{F}_{\mathrm{Det}}\mathcal{A}ss(\!(\gamma,\nu,0)\!))$.

Proof. Let $\mathcal{L} = \mathsf{R}^* p_! \underline{\mathbb{C}}$ be the direct image with proper supports along the fibres of the projection $p\colon P_{\gamma,\nu,0} \to \mathcal{M}_{\gamma,\nu}/\mathbb{S}_\nu$. The graded sheaf $\mathcal{L}$ is an invertible local system on the orbifold $\mathcal{M}_{\gamma,\nu}/\mathbb{S}_\nu$, concentrated in cohomological degree ν. The Thom isomorphism shows that

$$H^{\bullet}_c(Q_{\gamma,\nu,0},\mathbb{C}) \cong H^{\bullet}_c(P_{\gamma,\nu,0}, p^*\mathcal{L}^{-1});$$

thus, the cohomology with compact supports of $Q_{\gamma,\nu,0} \cong \mathcal{M}_{\gamma,\nu}/\mathbb{S}_\nu$ may be identified with the homology of the cellular cochains on $P_{\gamma,\nu,0}$ with coefficients in the graded local system $p^*\mathcal{L}^{-1}$.

Let G be a ribbon graph with no legs such that $2(\gamma-1)+\nu=g-1$, and let C_G be the corresponding cell of $P_{\gamma,\nu,0}$. We will prove the natural identification

$$H_c^\bullet(C_G, p^*\mathcal{L}^{-1}) \cong \Sigma^{1-2\gamma}\left(\omega(G)\otimes \mathrm{Det}(G)^{-1}\right), \tag{9.6}$$

which identifies the complex of cellular cochains on $P_{\gamma,\nu,0}$ with coefficients in $p^*\mathcal{L}^{-1}$ with $\Sigma^{1-2\gamma}\mathsf{F}_{\mathrm{Det}}\mathcal{A}ss(\!(\gamma,\nu,0)\!)$, proving the result.

To prove (9.6), let $\Sigma(G)$ be the surface with boundary constructed from the ribbon graph G. In Penner's theory, a point z of C_G is represented by a hyperbolic metric on $\Sigma(G)$. The fibre $p^{-1}(z)$ is homeomorphic to $\mathbb{R}^{\pi_0(\partial\Sigma(G))}$, the coordinates being logarithms of the lengths of the components of $\partial\Sigma(G)$. Thus

$$H_c^\bullet(C_G, p^*\mathcal{L}^{-1}) \cong H_c^\bullet(C_G,\mathbb{C})\otimes H_c^\bullet(p^{-1}(p(z)),\mathbb{C})^{-1}.$$

The graded vector space $H_c^\bullet(C_G,\mathbb{C})$ is naturally identified with $\omega(G)$, while the graded vector space $H_c^\bullet(p^{-1}(z),\mathbb{C})$ is naturally identified with $\mathrm{Det}(\pi_0(\partial\Sigma(G)))$. Thus, (9.6) follows from the formula

$$\mathrm{Det}(G)^{-1}\otimes\mathrm{Det}(\pi_0(\partial\Sigma(G)))\cong\Sigma^{2\gamma-1}\mathbb{C},$$

which we now prove.

Let $\overline{\Sigma}(G)$ be the compact surface obtained by gluing a disk D_i along each component S_i of $\partial\Sigma(G)$. Define the determinant of a finite-dimensional graded vector space $V_\bullet$ to be

$$\mathrm{Det}(V)=\mathrm{Det}(V_0)\otimes\mathrm{Det}(V_1)^{-1}\otimes\mathrm{Det}(V_2)\otimes\dots.$$

If M is a closed manifold, Poincaré duality gives a canonical identification $\mathrm{Det}(H^\bullet(M,\mathbb{C}))\cong\Sigma^{-e(M)}\mathbb{C}$, where $e(M)$ is the Euler characteristic of M.

Consider the Mayer-Vietoris sequence for the decomposition of $\overline{\Sigma}(G)$ as the union of $\Sigma(G)$ and $\coprod_i D_i$ along $\coprod_i S_i$. From the multiplicativity of Det in long exact sequences, we obtain the identification

$$\mathrm{Det}(H^\bullet(\overline{\Sigma}(G),\mathbb{C}))\cong\frac{\mathrm{Det}(H^\bullet(\Sigma(G),\mathbb{C}))\otimes\mathrm{Det}(H^\bullet(\coprod_i D_i,\mathbb{C}))}{\mathrm{Det}(H^\bullet(\coprod_i S_i,\mathbb{C}))}.$$

By Poincaré duality, the denominator is trivial, while $\mathrm{Det}(H^\bullet(\overline{\Sigma}(G),\mathbb{C}))\cong\Sigma^{2(\gamma-1)}\mathbb{C}$. Since the inclusion $|G|\hookrightarrow\Sigma(G)$ is a homotopy equivalence, we see that

$$\mathrm{Det}(H^\bullet(\Sigma(G),\mathbb{C}))\cong\mathrm{Det}(H^\bullet(|G|,\mathbb{C}))\cong\Sigma^{-1}\mathrm{Det}(G)^{-1},$$

completing the proof. □

(9.7) $\mathbb{C}\mathrm{h}(\mathsf{F}_{\mathrm{Det}}\mathcal{A}ss)$**: factorization of the integral**. By (7.6.2) the characteristic of the cyclic operad $\mathcal{A}ss$ has a special form: it is a sum of terms the nth of which depends on p_n alone, in the sense that

$$h_1+h_2+\mathrm{Ch}(\mathcal{A}ss)=-\sum_{n=1}^{\infty}\frac{\phi(n)}{n}\log(1-p_n).$$

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The formula of (8.19) now gives as

$$-\hbar^{-1}e_2 + \mathbb{C}\mathrm{h}(\mathsf{F}_{\mathrm{Det}}\mathcal{A}ss)$$
$$= -\mathrm{Log}\int_{\mathbb{R}^\infty}^{*} \mathrm{Exp}\frac{1}{\hbar}\left(p_1 q_1 + \sum_{n=1}^{\infty}\frac{\phi(n)}{n}\log(1-p_n)\right) \mathrm{d}\mu(p). \tag{9.8}$$

The special form of this formula will allow us to calculate it by a separation of variables. We will then calculate these integrals separately using the method of stationary phase.

(9.9) **Stationary phase and Wick's Theorem**. Let $f \in \mathbb{Q}[\![x,\hbar]\!]$ be a power series of the form

$$f = \tfrac{1}{2}x^2 + \sum_{2(g-1)+n>0}\frac{f_{g,n}\hbar^g x^n}{n!}.$$

The exponential $\exp(-f/\hbar)$ has the form

$$\sum_{k=-\infty}^{\infty}\sum_{\substack{\ell\geqslant 0\\ 2k+\ell>0}} c_{k,\ell}\,\hbar^k x^\ell \exp\left(\frac{-x^2}{2\hbar}\right), \quad c_{k,\ell}\in\mathbb{Q}. \tag{9.10}$$

(9.11) DEFINITION. Define the formal integral

$$\log\int^{\#}\exp\left(\hbar^{-1}(x\xi - f(x,\hbar))\right)\frac{\mathrm{d}x}{\sqrt{2\pi\hbar}} \in \hbar^{-1}\mathbb{Q}[\![\xi,\hbar]\!]$$

by the formula

$$\log\sum_{k,\ell} c_{k,\ell}\,\hbar^k\int_{-\infty}^{\infty} x^k\,\mathrm{e}^{x\xi/\hbar - x^2/2\hbar}\frac{\mathrm{d}x}{\sqrt{2\pi\hbar}}.$$

This is well-defined by (9.10).

(9.12) REMARK. The coefficients $F_{g,n}$ of the power series

$$\log\int^{\#}\exp(\hbar^{-1}(x\xi - f(x,\hbar)))\frac{\mathrm{d}x}{\sqrt{2\pi\hbar}} = \frac{1}{\hbar}\left(\tfrac{1}{2}\xi^2 + \sum_{2(g-1)+n>0}\frac{F_{g,n}\hbar^g\xi^n}{n!}\right),$$

may be calculated by Wick's formula, mentioned in the introduction

$$F_{g,n} = \sum_{G\in\Gamma(\!(g,n)\!)}\frac{1}{|\mathrm{Aut}(G)|}\prod_{v\in\mathrm{Vert}(G)} f_{g(v),n(v)}.$$

In particular, they are given by universal polynomials $F_{g,n}\in\mathbb{Q}[\![f_{g,n}]\!]$.

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We can now state the formula of stationary phase (a special case of Theorem 7.7.7 of Hörmander [18]). The proof uses nothing more than Taylor's Theorem and integration by parts.

(9.13) THEOREM. *Let ϕ be a differentiable function of $x \in (a,b)$ and $\hbar \in (-\varepsilon,\varepsilon)$, such that the function $x \mapsto \phi(x,0)$ has only the single critical point $x=0$ in the interval (a,b). Let f be the Taylor series of ϕ around $(x,\hbar)=(0,0)$, and suppose that $f_{xx}(0,0)=1$. Let $u \in C_c^\infty(a,b)$ equal 1 in a neighbourhood of the critical point 0. Then there is an asymptotic expansion*

$$\log \int_a^b u(x)\ \exp(\hbar^{-1}(x\xi - \phi(x,\hbar))\, \frac{\mathrm{d}x}{\sqrt{2\pi\hbar}}$$
$$\sim \log \int^{\#} \exp(\hbar^{-1}(x\xi - f(x,\hbar))\, \frac{\mathrm{d}x}{\sqrt{2\pi\hbar}}.$$

Let us give a simple application of Theorem (9.13).

(9.14) PROPOSITION.

$$\log \int^{\#} \exp \frac{1}{\hbar}\left(x\xi + x + \log(1-x)\right)\ \frac{\mathrm{d}x}{\sqrt{2\pi\hbar}}$$
$$= \frac{1}{\hbar}\left(\xi - \log(1+\xi) - \hbar\log(1+\xi) + \sum_{g=2}^{\infty} \frac{\zeta(1-g)}{1-g}\hbar^g\right).$$

Proof. The function $x + \log(1-x)$ has a unique critical point 0 in the interval $(-\infty,1)$. By the theorem of stationary phase, if $u \in C_c^\infty(-\infty,1)$ equals 1 in a neighbourhood of 0,

$$\log \int_{-\infty}^1 u(x)\ \exp \frac{1}{\hbar}\left(x\xi + x + \log(1-x)\right) \frac{\mathrm{d}x}{\sqrt{2\pi\hbar}}$$
$$\sim \log \int^{\#} \exp \frac{1}{\hbar}\left(x\xi + x + \log(1-x)\right)\ \frac{\mathrm{d}x}{\sqrt{2\pi\hbar}}.$$

In fact, we may take $u=1$, since the contribution of the integral away from a neighbourhood of 0 may be shown to vanish to infinite order by repeated integration by parts.

Performing the changes of variables $u=(1-x)(1+\xi)/\hbar$ and $s=\hbar^{-1}$, we see that

$$\log \int_{-\infty}^1 \exp\Big(\hbar^{-1}(x\xi + x + \log(1-x))\Big)\ \frac{\mathrm{d}x}{\sqrt{2\pi\hbar}}$$
$$= \log s^{-s-1/2}(1+\xi)^{-s-1}\mathrm{e}^{s(1+\xi)} \int_0^\infty u^s \mathrm{e}^{-u}\, \frac{\mathrm{d}u}{\sqrt{2\pi}}$$

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$$= \log (2\pi)^{-1/2} s^{-s-1/2}(1+\xi)^{-s-1}\mathrm{e}^{s(1+\xi)}\Gamma(s+1)$$

$$= s(1+\xi) + \log (2\pi)^{-1/2} s^{-s+1/2}(1+\xi)^{-s-1}\Gamma(s).$$

The proposition follows on inserting Stirling's formula

$$\log \Gamma(s) \sim \sum_{k=1}^{\infty} \frac{\zeta(-k)}{-k} s^{-k} + \left(s - \tfrac{1}{2}\right)\log(s) - s + \tfrac{1}{2}\log(2\pi), \quad s \to +\infty,$$

and replacing s by $\hbar^{-1}$. □

(9.15) **Application of stationary phase to $\mathbb{C}\mathrm{h}(\mathsf{F}_{\mathrm{Det}}\mathcal{A}ss)$.** For $n \geqslant 1$, introduce the Laurent polynomial

$$\alpha_n(\hbar) = \frac{1}{n}\sum_{d|n} \frac{\phi(d)}{\hbar^{n/d}} = \frac{1}{n\hbar^n}(1 + O(\hbar^{n/2}))$$

and the formal integral

$$\mathrm{I}_n(q_n,\hbar) = \log \int^{\#} \exp \frac{1}{n\hbar^n}\left(p_n q_n + p_n + n\hbar^n \alpha_n(\hbar)\log(1-p_n)\right) \times \frac{\mathrm{d}p_n}{e^{c(n)/2n}\sqrt{2\pi n\hbar^n}}. \tag{9.16}$$

This power series may be transformed into one of the form which we considered above, by making the changes of variables $p_n = n^{1/2}\hbar^{(n-1)/2}x$ and $q_n = n^{1/2}\hbar^{(n-1)/2}\xi$, which convert it to

$$\mathrm{I}_n(n^{1/2}\hbar^{(n-1)/2}\xi,\hbar)$$
$$= \log \int^{\#} \exp \frac{1}{\hbar}\left(x\xi + n^{-1/2}\hbar^{-(n+1)/2}x + \hbar\alpha_n(\hbar)\log(1 - n^{1/2}\hbar^{(n-1)/2}x)\right)$$
$$\times \frac{\mathrm{d}x}{e^{c(n)/2n}\sqrt{2\pi\hbar}}.$$

Strictly speaking, we need a slight generalization of the formal integral $\log \int^{\#} \ldots$, in which $f(x,\hbar)$ depends not on $\hbar$, but on $\hbar^{1/2}$; this does not present any additional difficulties.

Using the power series I_n, we may rewrite (9.8) as

$$-\hbar^{-1}e_2 + \mathbb{C}\mathrm{h}(\mathsf{F}_{\mathrm{Det}}\mathcal{A}ss)$$
$$= -\sum_{n=1}^{\infty} \mathrm{Log} \int^{\#} \exp \frac{1}{n\hbar^n}\left(p_n q_n + p_n + n\hbar^n \alpha_n(\hbar)\log(1-p_n)\right)$$

$$\times \frac{\mathrm{d}p_n}{e^{c(n)/2n}\sqrt{2\pi n\hbar^n}}$$

$$= -\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{\mu(\ell)}{\ell}\mathrm{I}_n(q_{\ell n},\hbar^\ell). \tag{9.17}$$

The following result resolves a problem left open in [24].

(9.18) THEOREM. *Let* $\Psi_n(\hbar)$, $n \geqslant 1$, *be the power series*

$$\Psi_n(\hbar) = \sum_{k=1}^{\infty}\frac{\zeta(-k)}{-k}\alpha_n^{-k} + (\alpha_n + \tfrac{1}{2})\log(n\hbar^n\alpha_n) - \alpha_n + \frac{1}{n\hbar^n} - \frac{c(n)}{2n}.$$

(1) *The series*

$$\Psi(\hbar) = \sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{\mu(\ell)}{\ell}\Psi_n(\hbar^\ell)$$

is convergent; more precisely, $\Psi_n = O\left(\hbar^{\lceil n/6\rceil}\right)$.

(2) *We have*

$$-\hbar^{-1}e_2 + \mathbb{C}\mathrm{h}(\mathsf{F}_{\mathrm{Det}}\mathcal{A}ss)$$
$$= \hbar^{-1}p_1 - \left(\hbar^{-1}+1\right)\sum_{n=1}^{\infty}\frac{\phi(n)}{n}\log(1+p_n) - \Psi(\hbar).$$

Proof. Let $\beta_n = n\hbar^n\alpha_n - 1$. Then $\beta_n = c(n)\hbar^{n/2} + O(\hbar^{\lceil 2n/3\rceil})$ and $\beta_n^2 = c(n)\hbar^{n/2} + O(\hbar^{\lceil 7n/6\rceil})$, so that

$$(\alpha_n + \tfrac{1}{2})\log(n\hbar^n\alpha_n) = \frac{1}{n\hbar^n}(1+\beta_n+\tfrac{1}{2}n\hbar^n)\left(\beta_n - \beta_n^2/2 + \beta_n^3/3 + \dots\right)$$

$$= \frac{1}{n\hbar^n}\left(\beta_n + \beta_n^2/2 + O(\hbar^{3n/2})\right)$$

$$= \alpha_n - \frac{1}{n\hbar^n} + \frac{c(n)}{2n} + O\left(\hbar^{\lceil n/6\rceil}\right).$$

It follows that

$$(\alpha_n + \tfrac{1}{2})\log(n\hbar^n\alpha_n) - \alpha_n + \frac{1}{n\hbar^n} - \frac{c(n)}{2n} = O\left(\hbar^{\lceil n/6\rceil}\right).$$

On the other hand, the term proportional to α_n^{-k} in the definition of Ψ_n has the form $O(\hbar^{kn})$, and hence these terms converge to a power series which does

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not contribute to the leading order behaviour of Ψ_n. This completes the proof of Part (1).

Generalizing (9.14), which is the special case $n = 1$, we see that

$$\begin{aligned} \mathrm{I}_n(q_n,\hbar) &= \frac{q_n}{n\hbar^n} - (\alpha_n+1)\log(1+q_n) \\ &\quad + \sum_{k=1}^\infty \frac{\zeta(-k)}{-k}\alpha_n^{-k} + (\alpha_n+\tfrac12)\log(n\hbar^n\alpha_n) \\ &\quad -\alpha_n + \frac{1}{n\hbar^n} - \frac{c(n)}{2n}, \end{aligned}$$

the only difference in the proof is that we make the substitutions $u = (1-p_n)(1+q_n)/n\hbar^n$ and $s = \alpha_n(\hbar)$. Inserting this formula into (9.17), we see that

$$\begin{aligned} &-\hbar^{-1}e_2 + \mathbb{C}\mathrm{h}(\mathsf{F}_{\mathrm{Det}}\mathcal{A}ss) \\ &\quad = -\sum_{n=1}^\infty\sum_{\ell=1}^\infty \frac{\mu(\ell)q_{\ell n}}{\ell n\hbar^{\ell n}} - \sum_{n=1}^\infty\sum_{\ell=1}^\infty \frac{\mu(\ell)}{\ell}\alpha_n(\hbar^\ell)\log(1+q_{\ell n}) \\ &\qquad -\sum_{n=1}^\infty\sum_{\ell=1}^\infty \frac{\mu(\ell)}{\ell}\log(1+q_{\ell n}) - \Psi(\hbar). \end{aligned}$$

The definition of the Möbius function shows that the first term equals

$$-\sum_{k=1}^\infty\sum_{d|k}\frac{\mu(k/d)q_k}{k\hbar^k} = -\frac{q_1}{\hbar}.$$

Inserting the definition of α_n into the second term, we see that it equals

$$-\sum_{k=1}^\infty \log(1+q_k)\sum_{d|k}\frac{\phi(d)}{k\hbar^{k/d}}\sum_{e|(k/d)}\mu(e) = \sum_{k=1}^\infty \frac{\phi(k)}{k\hbar}\log(1+q_k).$$

The third term equals

$$-\sum_{k=1}^\infty \frac{1}{k}\log(1+q_k)\sum_{d|k} d\mu(k/d) = \sum_{k=1}^\infty\frac{\phi(k)}{k}\log(1+q_k),$$

since by Möbius inversion, $\Sigma_{d|k}\mu(k/d)d = \phi(k)$. □

By (9.5), this theorem implies the following purely topological formula. (The change of sign comes from the fact that $H_c^\bullet(\mathcal{M}_{g,n}/\mathbb{S}_{g,n},\mathbb{C})$ is an odd suspension of $H_\bullet(\mathsf{F}_{\mathrm{Det}}\mathcal{A}ss)$.)

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(9.19) COROLLARY. *The power series* $\Psi(\hbar)$ *has the following topological interpretation*

$$\Psi(\hbar) = \sum_{g=2}^{\infty} \hbar^{g-1} \sum_{2(\gamma-1)+\nu=g-1} \mathrm{e}(|\mathcal{M}_{\gamma,\nu}/\mathbb{S}_\nu|),$$

where $\mathrm{e}(|\mathcal{M}_{\gamma,\nu}/\mathbb{S}_\nu|)$ *is the Euler characteristic of the topological space* $|\mathcal{M}_{\gamma,\nu}/\mathbb{S}_\nu|$.

The first few terms of $\Psi(\hbar)$ are as follows

$$\Psi(\hbar) = 2\hbar + 2\hbar^2 + 4\hbar^3 + 2\hbar^4 + 6\hbar^5 + 6\hbar^6 + 6\hbar^7 + \hbar^8 + O(\hbar^9).$$

(9.20) REMARKS. (a) Observe that the coefficients of $\hbar^n$, $n > 0$, in (9.18) are constant (that is, independent of the power sums p_i). This is in agreement with (9.4), since the Euler characteristic of $|Q_{\gamma,\nu,n}|$ vanishes if $n > 0$, provided $2(\gamma-1)+\nu > 0$. Indeed, there is a circle action on $Q_{\gamma,\nu,n}$, given by the formula

$$(\Sigma, A, \lambda_1, \dots, \lambda_n) \mapsto (\Sigma, A, \mathrm{e}^{it}\lambda_1, \dots, \mathrm{e}^{it}\lambda_n).$$

The isotropy groups of the induced circle action on $|Q_{\gamma,\nu,n}|$ are finite, since a punctured Riemann surface (Σ, A) has finitely many automorphisms fixing the punctures. Thus, all of the orbits of this circle action are circles, and the Euler characteristic of $|Q_{\gamma,\nu,n}|$ vanishes.

As expected, the coefficient of $\hbar^{-1}$ in $\mathbb{C}\mathrm{h}(\mathsf{F}\mathcal{A}ss)$ is just $-\tilde{\omega}\mathrm{Ch}(\mathcal{A}ss)$, consistent with the fact that $\mathrm{Cyc}(\mathsf{F}_{\mathrm{Det}}\mathcal{A}ss) = \Sigma\mathfrak{s}\mathsf{B}\mathcal{A}ss \simeq \Sigma\mathfrak{s}\mathcal{A}ss$.

(b) The virtual Euler characteristic $\chi(\mathcal{G})$ of an orbifold $\mathcal{G}$ may defined using a cellular decomposition of $\mathcal{G}$ as the sum

$$\chi(\mathcal{G}) = \sum_{\text{cells } \mathcal{U} \text{ of } \mathrm{Ob}(\mathcal{G})} \frac{(-1)^{\dim(\mathcal{U})}}{|\mathrm{Aut}(\mathcal{U})|},$$

where $\mathrm{Aut}(\mathcal{U})$ is the group of morphisms of $\mathcal{G}$ fixing $\mathcal{U}$. In general, the virtual characteristic is a rational number. It behaves well under quotienting: if G is a finite group acting on $\mathcal{G}$

$$\chi(\mathcal{G}/G) = \frac{\chi(\mathcal{G})}{|G|}.$$

The virtual Euler characteristic of the orbifolds $\mathcal{M}_{\gamma,1}$ was shown by Harer and Zagier [16] to equal $\zeta(1-2\gamma)$. Since the virtual Euler characteristic is multiplicative for fibrations of orbifolds, the fibrations

$$\begin{array}{ccc} \mathcal{M}_{\gamma,\nu} & \longrightarrow & \mathcal{M}_{\gamma,\nu-1} \\ \downarrow & & \\ \mathcal{M}_{\gamma,\nu}/\mathbb{S}_\nu & & \end{array} \tag{9.21}$$

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immediately imply formulas for the virtual Euler characteristics of $\mathcal{M}_{\gamma,\nu}$ and $\mathcal{M}_{\gamma,\nu}/\mathbb{S}_\nu$. However, on descent to the coarse moduli spaces, the maps of (9.21) are not fibrations, so there is no elementary relation between the Euler characteristics of the topological spaces $|\mathcal{M}_{\gamma,\nu}|$ and $|\mathcal{M}_{\gamma,\nu}/\mathbb{S}_\nu|$ for different ν. (In fact, the Euler characteristics of the coarse moduli spaces $|\mathcal{M}_{\gamma,\nu}|$ and $|\mathcal{M}_{\gamma,\nu}/\mathbb{S}_\nu|$ are unknown for $\nu > 1$.)

Harer and Zagier also calculate the Euler characteristic of $|\mathcal{M}_{\gamma,1}|$ (page 482, [16]), obtaining the formula

$$\sum_{\gamma=1}^{\infty} e(|\mathcal{M}_{\gamma,1}|)\hbar^{2\gamma-1} = \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \sum_{\ell=1}^{\infty} \mu(\ell)\Psi_{n,\ell}(\hbar),$$

where

$$\Psi_{n,\ell}(\hbar) = \sum_{k=1}^{\infty} \zeta(-k)\alpha_{n,\ell}^{-k} + \alpha_{n,\ell}\log(n\hbar^n\alpha_n) + \frac{1}{n\hbar^n} - \alpha_{n,\ell}.$$

Here, $\alpha_{n,\ell}$ is the Laurent polynomial

$$\alpha_{n,\ell}(\hbar) = \frac{1}{n}\sum_{d|n} \mu(d/(d,\ell))\frac{\phi(n/d)}{\phi(\ell/(d,\ell))}\hbar^{-d}.$$

There is a striking formal similarity between our formula (9.19) for $\Psi(\hbar)$ and this formula, representing the contribution of $\nu = 1$.

(c) The idea of replacing an asymptotic integral (over an arbitrarily small neighborhood of zero) by an equivalent integral over a fixed interval, which is then explicitly evaluated through Γ-function and estimated by Stirling formula, already arises in the calculation of the virtual Euler characteristics of moduli spaces: see Harer and Zagier [16], Penner [31], Itzykson and Zuber [19], and Kontsevich [23].

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