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SOME NEW EXAMPLES OF SMASH-NILPOTENT ALGEBRAIC CYCLES

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Abstract. Voevodsky has conjectured that numerical equivalence and smash-equivalence coincide for algebraic cycles on any smooth projective variety. Building on work of Vial and Kahn–Sebastian, we give some new examples of varieties where Voevodsky's conjecture is verified.

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1. Introduction. Let X be a smooth projective variety over $\mathbb{C}$. There exist numerous adequate equivalence relations (in the sense of [**36**]) on the group of algebraic cycles on X, ranging from rational equivalence (the finest) to numerical equivalence (the coarsest). Rational equivalence gives rise to the Chow groups $A^j(X) := CH^j(X)_{\mathbb{Q}}$ (i.e., codimension j cycles with rational coefficients modulo rational equivalence). The other equivalence relations give rise to subgroups $A^j_{\sim}$; for example, there are subgroups

$$A^j_{alg}(X) \subset A^j_{\otimes}(X) \subset A^j_{hom}(X) \subset A^j_{num}(X)$$

of cycles algebraically resp. smash-nilpotent resp. homologically resp. numerically trivial. Here, the first inclusion is a theorem of Voevodsky [**47**] and Voisin [**48**], and the last inclusion is the subject of one of the standard conjectures [**28**]. More ambitiously, Voevodsky has conjectured that $A^j_{\otimes}(X)$ and $A^j_{num}(X)$ should coincide [**47**].

Not a great deal is known about this conjecture of Voevodsky's; most results focus on 1-cycles. For instance, Voevodsky's conjecture has been proven for 1-cycles on varieties rationally dominated by products of curves [**38**], [**39**, Proposition 2] (this is further generalized by [**44**, Theorem 3.17]).

In this note (which is inspired by [**38**, **39**] and particularly [**44**]), we aim for results for cycles in other dimensions by restricting attention to very special varieties. The main result is as follows:

THEOREM (≈ Theorem 3.1[1]). *Let X be a smooth projective variety. Assume that X is dominated by a product of curves, and that the even cohomology of X verifies*

$$H^{2i}(X, \mathbb{Q}) = \widetilde{N}^{i-1} H^{2i}(X, \mathbb{Q})\,.$$

[1]The actual statement of Theorem 3.1 is somewhat more general, but this simplified version suffices for many applications.

Then,

$$A^j_{\otimes}(X) = A^j_{num}(X) \quad \text{for all } j\ .$$

Here, $\widetilde{N}^*$ denotes Vial's niveau filtration, which is a variant of the coniveau filtration (cf. [**45**] and Section 2.3 below). Conjecturally, the condition $H^{2i}(X,\mathbb{Q}) = \widetilde{N}^{i-1}$ is equivalent to have $H^{2i}(X,\mathbb{C}) = F^{i-1}H^{2i}(X,\mathbb{C})$, where F^* is the Hodge filtration.

Examples of varieties to which Theorem 3.1 applies include the following: Fermat hypersurfaces of odd dimension; products of type $X^2_d \times X^n_{d'}$ with n odd (where X^n_d denotes a Fermat hypersurface of dimension n and degree d). Some more examples where Theorem 3.1 applies are given in Corollary 4.1.

CCONVENTION. In this note, the word variety will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.

All Chow groups will be with rational coefficients: We denote by A_jX the Chow group of j-dimensional cycles on X with $\mathbb{Q}$-coefficients; for X smooth of dimension n the notations A_jX and $A^{n-j}X$ will be used interchangeably.

The notations $A^j_{hom}(X)$, $A^j_{num}(X)$, $A^j_{AJ}(X)$, $A^j_{alg}(X)$ and $A^j_{\otimes}(X)$ will be used to indicate the subgroups of homologically trivial, resp. numerically trivial, resp. Abel–Jacobi trivial resp. algebraically trivial, resp. smash-nilpotent cycles. The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [**32, 37**]) will be denoted $\mathcal{M}_{\text{rat}}$.

We will write $H^j(X)$ and $H_j(X)$ to indicate singular cohomology $H^j(X,\mathbb{Q})$, resp. Borel–Moore homology $H_j(X,\mathbb{Q})$.

2. Preliminary.

2.1. Motives of abelian type. We refer to [**1, 17, 22, 24, 32**] for the definition of finite-dimensional motive. An essential property of varieties with finite-dimensional motive is embodied by the nilpotence theorem:

THEOREM 2.1 (Kimura [**24**]). *Let X be a smooth projective variety of dimension n with finite-dimensional motive. Let $\Gamma \in A^n(X \times X)$ be a correspondence which is numerically trivial. Then, there is $N \in \mathbb{N}$ such that*

$$\Gamma^{\circ N} = 0 \quad \text{in} \quad A^n(X \times X)$$

(here, $\circ$ indicates composition of correspondences).

Actually, the nilpotence property (for all powers of X) could serve as an alternative definition of finite-dimensional motive, as shown by Jannsen [**22**, Corollary 3.9].

CONJECTURE 2.2 (Kimura [**24**]). Every smooth projective variety has finite-dimensional motive.

We are still far from knowing this, but at least there are quite a few non-trivial examples:

REMARK 2.3. The following varieties have finite-dimensional motive: abelian varieties, varieties dominated by products of curves [**24**], $K3$ surfaces with Picard

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number 19 or 20 [**34**], surfaces not of general type with vanishing geometric genus [**16**, Theorem 2.11], Godeaux surfaces [**16**], certain surfaces of general type with $p_g = 0$ [**35**, **49**], Hilbert schemes of surfaces known to have finite-dimensional motive [**10**], generalized Kummer varieties [**52**, Remark 2.9(ii)], 3-folds with nef tangent bundle [**18**] (an alternative proof is given in [**44**, Example 3.16]), 4-folds with nef tangent bundle [**19**], log-homogeneous varieties in the sense of [**9**] (this follows from [**19**, Theorem 4.4]), certain 3-folds of general type [**46**, Section 8], varieties of dimension ≤ 3 rationally dominated by products of curves [**44**, Example 3.15], varieties X with $A^i_{AJ}(X) = 0$ for all i [**43**, Theorem 4], products of varieties with finite-dimensional motive [**24**].

DEFINITION 2.4. Let X be a smooth projective variety of dimension n. We say that X has *motive of abelian type* if $h(X) \in \mathcal{M}_{\text{rat}}$ is in the subcategory generated by the motives of curves.

REMARK 2.5. It follows from the fact that curves have finite-dimensional motive that "motive of abelian type" implies "finite-dimensional motive". The converse is probably not true (many motives are *not* of abelian type, cf. [**13**, 7.6]), yet it is a (somewhat embarassing) fact that all known finite-dimensional motives happen to be of abelian type.

Various characterizations of motives of abelian type are given in [**44**]. One of these is as follows:

PROPOSITION 2.6 (Vial [**44**]). *Let X be a smooth projective variety of dimension n. The motive of X is of abelian type if and only if $A^j_{alg}(X)$ is generated, via correspondences, by Chow groups of products of curves, for all $j > \lceil \frac{n}{2} \rceil$.*

Proof. This follows from [**44**, Theorem 5]. □

PROPOSITION 2.7 (Vial [**44**]). *Let X be a smooth projective variety of dimension n, and assume X has motive of abelian type. Then, the motive of X is isomorphic to a direct summand*

$$h(X) \subset \bigoplus_j h(A_j)(m_j) \quad \text{in } \mathcal{M}_{\text{rat}} ,$$

where the A_j are abelian varieties.

Proof. It suffices to note that for motives of abelian type there is an inclusion

$$h(X) \subset \bigoplus_j h(M_j)(m_j) \quad \text{in } \mathcal{M}_{\text{rat}} ,$$

where M_j is a product of curves $C_1 \times \cdots \times C_{r_j}$ (this follows from [**44**, Theorem 4], plus [**44**, Theorem 3.11] applied with $l = d := \dim X$). It is well-known this implies Proposition 2.7.

(Indeed, for some $n_i \geq 2g(C_i)$ let $C_i^{[n_i]}$ denote the n_i-th symmetric product, and let J_i denote the Jacobian of C_i. There exist morphisms

$$M := C_1 \times \cdots C_r \to C_1^{[n_1]} \times \cdots \times C_r^{[n_r]} \to J_1 \times \cdots \times J_r .$$

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The first arrow identifies $h(M)$ with a direct summand of $h(C_1^{[n_1]} \times \cdots \times C_r^{[n_r]})$ [**26**]. The second arrow is a composition of projective bundles, so the motive $h(C_1^{[n_1]} \times \cdots \times C_r^{[n_r]})$ identifies with a sum of shifted motives of $J_1 \times \cdots \times J_r$). □

2.2. Lefschetz standard conjecture.

NOTATION 2.8. Let X be a smooth projective variety of dimension n, and $h \in H^2(X, \mathbb{Q})$ the class of an ample line bundle. The hard Lefschetz theorem asserts that the map

$$L^{n-i}\colon H^i(X) \to H^{2n-i}(X)$$

obtained by cupping with h^{n-i} is an isomorphism, for any $i < n$.

One of the standard conjectures asserts that the inverse isomorphism is algebraic:

DEFINITION 2.9. Given a variety X, we say that $B(X)$ holds if for all ample h, and all $i < n$ the isomorphism

$$(L^{n-i})^{-1}\colon H^{2n-i}(X) \xrightarrow{\cong} H^i(X)$$

is induced by a correspondence.

REMARK 2.10. It is known that $B(X)$ holds for the following varieties: curves, surfaces, abelian varieties [**27**, **28**], 3-folds not of general type [**41**], hyperkähler varieties of $K3^{[n]}$-type [**11**], n-dimensional varieties X which have $A_i(X)$ supported on a subvariety of dimension $i+2$ for all $i \le \frac{n-3}{2}$ [**42**, Theorem 7.1], n-dimensional varieties X which have $H_i(X) = N^{\llcorner \frac{i}{2} \lrcorner} H_i(X)$ for all $i > n$ [**43**, Theorem 4.2], products and hyperplane sections of any of these [**27**, **28**].

REMARK 2.11. Let X be a variety with motive of abelian type. Then, $B(X)$ holds. This is because the standard conjecture B can also be formulated for motives. Since $B(A)$ holds for abelian varieties, it also holds for direct summands of a sum of twisted motives of abelian varieties, hence for varieties with motive of abelian type. It follows that the standard conjectures $C(X)$ (i.e., algebraicity of the Künneth components) and $D(X)$ (i.e., homological and numerical equivalence coincide on X and on $X \times X$) also hold [**27**, **28**].

2.3. Niveau filtration.

DEFINITION 2.12 (Coniveau filtration [**6**]). Let X be a quasi-projective variety. The *coniveau filtration* on cohomology and on homology is defined as

$$N^c H^i(X, \mathbb{Q}) = \sum \operatorname{Im}\big(H^i_Y(X, \mathbb{Q}) \to H^i(X, \mathbb{Q})\big)\ ;$$
$$N^c H_i(X, \mathbb{Q}) = \sum \operatorname{Im}\big(H_i(Z, \mathbb{Q}) \to H_i(X, \mathbb{Q})\big)\ ,$$

where Y runs over codimension $\ge c$ subvarieties of X, and Z over dimension $\le i - c$ subvarieties.

Vial introduced the following variant of the coniveau filtration:

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DEFINITION 2.13 (Niveau filtration [**45**]). Let X be a smooth projective variety. The *niveau filtration* on homology is defined as

$$\widetilde{N}^j H_i(X) = \sum_{\Gamma \in A_{i-j}(Z\times X)} \mathrm{Im}\big(H_{i-2j}(Z) \to H_i(X)\big) ,$$

where the union runs over all smooth projective varieties Z of dimension $i-2j$, and all correspondences $\Gamma \in A_{i-j}(Z \times X)$. The niveau filtration on cohomology is defined as

$$\widetilde{N}^c H^i X := \widetilde{N}^{c-i+n} H_{2n-i} X .$$

REMARK 2.14. The niveau filtration is included in the coniveau filtration:

$$\widetilde{N}^j H^i(X) \subset N^j H^i(X) .$$

These two filtrations are expected to coincide; indeed, Vial shows this is true if and only if the Lefschetz standard conjecture is true for all varieties [**45**, Proposition 1.1].

Using the truth of the Lefschetz standard conjecture in degree ≤ 1, it can be checked [**45**, page 415 “Properties"] that the two filtrations coincide in a certain range:

$$\widetilde{N}^j H^i(X) = N^j H^i X \quad \text{for all } j \geq \frac{i-1}{2} .$$

LEMMA 2.15. *Let X be a smooth projective variety of dimension n such that $B(X)$ holds. Suppose*

$$H^{2i}(X) = \widetilde{N}^{i-1} H^{2i}(X)$$

for some i. Then, there exists a smooth projective surface S and correspondences $\Gamma_{2i} \in A^{n+1-i}(X \times S)$, $\Psi_{2i} \in A^{i+1}(S \times X)$ such that

$$\pi_{2i} = \Psi_{2i} \circ \Gamma_{2i} \quad \textit{in } H^{2n}(X \times X) .$$

Proof. This follows readily from the arguments contained in [**45**]. Indeed, by assumption there exists a surface S and a correspondence $\Psi_{2i} \in A^{i+1}(S \times X)$ such that

$$H^{2i}(X) = (\Psi_{2i})_* H^2(S) .$$

This means that the homomorphism of motives

$$\Psi_{2i}\colon \ (S, \pi_2, 0) \ \to \ (X, \pi_{2i}, 0) \quad \text{in } \mathcal{M}_{\mathrm{hom}}$$

is surjective (i.e.,

$$(\Psi_{2i} \times \Delta_M)_*\colon H^*(S \times M) \ \to \ (\pi_{2i} \times \Delta_M)_* H^*(X \times M)$$

is surjective for all smooth projective varieties M). On the other hand, the motives $(S, \pi_2, 0)$ and $(X, \pi_{2i}, 0)$ lie in a subcategory $\mathcal{M}^{\circ}_{\mathrm{hom}} \subset \mathcal{M}_{\mathrm{hom}}$ which is semi-simple (one can define $\mathcal{M}^{\circ}_{\mathrm{hom}}$ as the smallest full subcategory containing the motives of all varieties

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M for which $B(M)$ is known). As such, there is a left-inverse to Ψ_{2i}; this gives the correspondence Γ_{2i} with the property that $\Psi_{2i} \circ \Gamma_{2i} = \pi_{2i}$. □

2.4. Smash-nilpotence.

DEFINITION 2.16. Let X be a smooth projective variety. A cycle $a \in A^r(X)$ is called *smash-nilpotent* if there exists $m \in \mathbb{N}$ such that

$$a^m := \underbrace{a \times \cdots \times a}_{(m \text{ times})} = 0 \ \text{ in } A^{mr}(X \times \cdots \times X)\,.$$

We will write $A^r_{\otimes}(X) \subset A^r(X)$ for the subgroup of smash-nilpotent cycles.

CONJECTURE 2.17 (Voevodsky [**47**]). Let X be a smooth projective variety. Then,

$$A^r_{num}(X) \subset A^r_{\otimes}(X) \quad \text{for all } r\,.$$

REMARK 2.18. It is known [**1**, Théorème 3.33] that conjecture 2.17 implies (and is strictly stronger than) conjecture 2.2.

The most general result concerning smash-nilpotence is the following:

THEOREM 2.19 (Voevodsky [**47**], Voisin [**48**]). *Let X be a smooth projective variety. Then,*

$$A^r_{alg}(X) \subset A^r_{\otimes}(X) \quad \textit{for all } r\,.$$

In particular, it follows from Theorem 2.19 that conjecture 2.17 is true for $r = 1$ and for $r = \dim X$. Another useful result is the following (this is [**23**, Proposition 1], which builds on results of Kimura's [**24**]):

THEOREM 2.20 (Kahn–Sebastian [**23**]). *Let A be an abelian variety. Assume $a \in A^r(A)$ is* skew*, i.e., $(-1)^*(a) = -a$ in $A^r(A)$. Then, $a \in A^r_{\otimes}(A)$.*

3. Main result. This section contains the proof of our main result (stated in somewhat more general form than in the introduction):

THEOREM 3.1. *Let X be a smooth projective variety of dimension n. Assume*

(*i*) *X has motive of abelian type;*
(*ii*) *$H^{2i}(X) = \widetilde{N}^{i-1}H^{2i}(X)$ for all $i \leq n/2$.*
Then, Voevodsky's conjecture is true for X, i.e.,

$$A^r_{\otimes}(X) = A^r_{num}(X) \quad \textit{for all } r\,.$$

Proof. Let us denote

$$Z^r(X) := \frac{A^r_{num}(X)}{A^r_{\otimes}(X)}\,.$$

By assumption (i), the Künneth components π_i of X are algebraic (Remark 2.11). By assumption (ii) and Lemma 2.15, any "even" Künneth component π_{2i} with $i \leq$

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$n/2$ factors over a surface, i.e., there exists a surface S_{2i} and correspondences $\Gamma_{2i} \in A^{n+1-i}(X \times S_{2i})$, $\Psi_{2i} \in A^{i+1}(S_{2i} \times X)$ such that

$$\pi_{2i} = \Psi_{2i} \circ \Gamma_{2i} \quad \text{in } H^{2n}(X \times X)\,.$$

We now lift the π_i to the level of rational equivalence in the following way: For the even components, we choose

$$\Pi_{2i} := \begin{cases} \Psi_{2i} \circ \Gamma_{2i} & \text{in } A^n(X \times X) \quad \text{if } i \leq n/2\,; \\ {}^t(\Psi_{2n-2i} \circ \Gamma_{2n-2i}) & \text{in } A^n(X \times X) \quad \text{if } i > n/2\,, \end{cases}$$

Here, Ψ_{2i}, Γ_{2i} are correspondences to and from a surface S_{2i} as above, and ${}^t()$ denotes the transpose of a correspondence. For the odd Künneth components π_{2i+1}, we take arbitrary lifts $\Pi_{2i+1} \in A^n(X \times X)$ of the π_{2i+1}, subject only to the condition that

$$\Delta_X = \sum_{i=0}^{2n} \Pi_i \ \text{ in } A^n(X \times X)$$

(i.e., we define the last Π_{2i+1} as a difference of cycle classes). Note that our $\Pi_i \in A^n(X \times X)$ need *not* be idempotents.

We now remark that

$$(\Pi_{2i})_*\colon \ Z^r(X) \ \to \ Z^r(X)$$

factors over $Z^*(S_{2i})$, which is 0 since S_{2i} is a surface, and so

$$(\Pi_{2i})_* = 0\colon \ Z^r(X) \ \to \ Z^r(X) \quad \text{for all } i \text{ and all } r\,.$$

It follows that

$$(\Delta_X)_* = \Big(\sum_{i \text{ odd}} \Pi_i\Big)_* \ : \ Z^r(X) \ \to \ Z^r(X)\,.$$

For later use, let us note that this last equality also implies

$$\Big(\sum_{i \text{ odd}} \Pi_i\Big)_* = \Big(\Big(\sum_{i \text{ odd}} \Pi_i\Big)^{\circ m}\Big)_* = \mathrm{id}\colon \ Z^r(X) \ \to \ Z^r(X)\,, \quad \text{for all } m \in \mathbb{N}\,. \tag{1}$$

Assumption (i) implies the motive of X identifies with a direct summand

$$h(X) \ \subset \ \bigoplus_{j=1}^{s} h(A_j)(m_j) \ \text{ in } \mathcal{M}_{\mathrm{rat}}\,,$$

where the A_j are abelian varieties (Proposition 2.7). This formally implies that there exist correspondences

$$\Gamma_1 = \sum_j \Gamma_1^j \ \in \bigoplus_j A^*(X \times A_j)\,,$$
$$\Gamma_2 = \sum_j \Gamma_2^j \ \in \bigoplus_j A^*(A_j \times X)$$

such that

$$\Gamma_2 \circ \Gamma_1 = \sum_j \Gamma_2^j \circ \Gamma_1^j = \Delta_X \ \text{ in } A^n(X \times X)\,.$$

In particular, for any i, we also have that the composition

$$H^{2i+1}(X) \xrightarrow{(\Gamma_1)_*} \bigoplus_j H^{2i+1+2c_j}(A_j) \xrightarrow{(\Gamma_2)_*} H^{2i+1}(X)$$

is equal to the identity (here c_j is some integer, dependent on n and $\dim A_j$ and m_j). But this composition is the same as

$$H^{2i+1}(X) \xrightarrow{(\Gamma_1)_*} \bigoplus_j H^{2i+1+2c_j}(A_j) \xrightarrow{\left((\pi^{A_1}_{2i+1+2c_1})_*,\ldots,(\pi^{A_s}_{2i+1+2c_s})_*\right)} \bigoplus_j H^{2i+1+2c_j}(A_j) \xrightarrow{(\Gamma_2)_*} H^{2i+1}(X)\,,$$

where the $\pi_i^{A_j}$ denote the Chow–Künneth decomposition of [**31**] for the abelian variety A_j.

That is, we have a homological equivalence

$$\Pi_{2i+1} = \Gamma_2 \circ \Gamma_1 \circ \Pi_{2i+1} = \sum_j \Gamma_2^j \circ \pi^A_{2i+1+2c_j} \circ \Gamma_1^j \circ \Pi_{2i+1} \ \text{ in } H^{2n}(X \times X)\,.$$

Taking the sum over all odd Künneth components, we find that

$$\sum_{i\text{ odd}} \Pi_i - \sum_{i\text{ odd}} \sum_j \Gamma_2^j \circ \pi^{A_j}_{2i+1+2c_j} \circ \Gamma_1^j \circ \Pi_{2i+1} \quad \in \ A^n(X \times X)$$

is homologically trivial. But then (since X has finite-dimensional motive), it follows from Theorem 2.1 this cycle is nilpotent: There exists $N \in \mathbb{N}$ such that

$$\left(\sum_{i\text{ odd}} \Pi_i - \sum_{i\text{ odd}} \sum_j \Gamma_2^j \circ \pi^{A_j}_{2i+1+2c_j} \circ \Gamma_1^j \circ \Pi_{2i+1} \right)^{\circ N} = 0 \quad \in \ A^n(X \times X)\,.$$

Developing this expression, we obtain

$$\left(\sum_{i\text{ odd}} \Pi_i \right)^{\circ N} = Q_1 + \cdots + Q_{N'} \ \text{ in } A^n(X \times X)\,,$$

where each Q_s is a composition of correspondences in which some $\pi^{A_j}_{2i+1+2c_j}$ occurs at least once, i.e.,

$$Q_s = (\text{something}) \circ \pi_\ell^{A_j} \circ (\text{something}) \quad \text{in } A^n(X \times X), \text{ with } \ell \text{ odd}.$$

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This equality implies in particular that both sides act in the same way on $Z^r(X)$ for any r, i.e.,

$$\Big(\big(\sum_{i \text{ odd}} \Pi_i\big)^{\circ N}\Big)_* = (\sum_s Q_s)_* = (\text{something})_* (\pi_\ell^{A_j})_* (\text{something})_* : \ Z^r(X) \ \to \ Z^r(X)\,.$$

The right-hand side of this equality is 0, since

$$(\pi_\ell^{A_j})_* \Big(\frac{A^*_{num}(A_j)}{A^*_\otimes(A_j)}\Big)$$

for ℓ odd (this follows from Theorem 2.20, combined with the fact that the $\pi_\ell^{A_j}$ project to odd gradeds of the Beauville filtration on $A^*(A_j)$ [**14**, **37**]). As we have seen in equality (1), the left-hand side is the identity. We conclude that

$$Z^r(X) = 0\,.$$

□

4. Examples. In this section, we aim to give some content to Theorem 3.1, by providing examples of varieties satisfying the assumptions. For convenience, we will write X^n_d for the Fermat hypersurface of dimension n and degree d.

COROLLARY 4.1. *Let X be one of the following:*

(1) *a Fermat hypersurface X^n_d with n odd;*
(2) *a product $Y_1 \times \cdots \times Y_s \times X^n_d$, where the Y_i are varieties with $A^*_{hom}(Y_i) = 0$ (examples of such varieties can be found in [**5**, **35**, **49**]), and n is odd;*
(3) *a product $Y_1 \times \cdots \times Y_s \times Y$, where the Y_i are as in (2), and Y is a Calabi–Yau 3-fold with motive of abelian type (examples of such Y are given in [**29**, Section 2] and in [**30**]);*
(4) *a product $S \times X^n_d$ where n is odd, and S is a regular surface with motive of abelian type (e.g., S can be $X^2_{d'}$, or a double plane branched along six lines in general position [**33**], or a K3 surface with Picard number ≥ 19, or any of the surfaces in [**8**, **15**]);*
(5) *a product $Y \times C$, where C is a curve and $Y = X^4_7/\mu_7$ is the 4-fold studied in [**45**, Proposition 2.17];*
(6) *a product $Y \times S$, where S is a surface with $A^2_{AJ}(S) = 0$, and $Y = X^4_7/\mu_7$ is the 4-fold of [**45**, Proposition 2.17];*
(7) *a product $S \times Y$, where S is a regular surface with motive of abelian type, and Y is a Calabi–Yau 3-fold with motive of abelian type;*
(8) *the Calabi–Yau 5-fold obtained from a product of five elliptic curves as in [**12**, Corollary 2.3].*

Then,

$$A^r_\otimes(X) = A^r_{num}(X) \ \textit{for all } r\,.$$

Proof. Clearly, all these examples have motive of abelian type: for case (1), this follows from Shioda's inductive structure [**40**]; for case (2), this follows from [**43**, Theorem 5] (or, independently, from [**25**]); case (5) follows from [**45**, Proposition 2.17]; the surfaces in case (6) follow from [**43**, Theorem 4]. It remains to check hypothesis (ii) of Theorem 3.1 is verified. For cases (1), (2), (3), this is clear since in these cases the even-degree cohomology is algebraic, and so

$$H^{2i}(X) = N^i H^{2i}(X) = \widetilde{N}^i H^{2i}(X) \subset \widetilde{N}^{i-1} H^{2i}(X) .$$

In case (4), we have

$$H^{2i}(X) = \bigoplus_{k+\ell=2i} H^k(S) \otimes H^\ell(X^n_{d'}) .$$

Any direct summand with $k \neq 2$ consists of algebraic classes. For $k = 2$, we have

$$H^2(X^2_d) \otimes H^{2i-2}(X^n_{d'}) \ \subset\ H^2(X^2_d) \otimes \widetilde{N}^{i-1} H^{2i-2}(X^n_{d'}) \ \subset\ \widetilde{N}^{i-1} H^{2i}(X^2_d \times X^n_{d'}) .$$

In case (5), we have $H^4(Y) = \widetilde{N}^1 H^4(Y)$ [**45**, Proposition 2.17]. It follows that

$$H^4(Y \times C) = H^4(Y) \otimes H^0(C) \oplus H^2(Y) \otimes H^2(C) \ \subset\ \widetilde{N}^1 H^4(Y \times C) .$$

In case (6), we have

$$\begin{aligned} H^4(X) &= H^4(Y) \otimes H^0(S) \oplus H^2(Y) \otimes H^2(S) \oplus H^0(Y) \otimes H^4(S) \\ &\subset\ \widetilde{N}^1 H^4(Y) \otimes H^0(S) \oplus \widetilde{N}^1 H^2(Y) \otimes \widetilde{N}^1 H^2(S) \oplus H^0(Y) \otimes \widetilde{N}^2 H^4(S) \\ &\subset \widetilde{N}^1 H^4(X) , \end{aligned}$$

and likewise

$$\begin{aligned} H^6(X) &= H^6(Y) \otimes H^0(S) \oplus H^4(Y) \otimes H^2(S) \oplus H^2(Y) \otimes H^4(S) \\ &\subset\ \widetilde{N}^2 H^6(Y) \otimes H^0(S) \oplus \widetilde{N}^1 H^4(Y) \otimes \widetilde{N}^1 H^2(S) \oplus \widetilde{N}^1 H^2(Y) \otimes \widetilde{N}^2 H^4(S) \\ &\subset \widetilde{N}^2 H^6(X) . \end{aligned}$$

Cases (7) is similar to case (4).

As to case (8): Let $E_1, \ldots, E_5$ be elliptic curves, and let X be a Calabi–Yau 5-fold obtained as a smooth model of the quotient

$$(E_1 \times \cdots \times E_5)/\mathbb{Z}_2^4$$

as in [**12**, Corollary 2.3]. It is readily checked (using the argument of [**12**, Lemma 2.4]) that

$$H^4(E_1 \times \cdots \times E_5)^{\mathbb{Z}_2^4} \subset N^2 H^4(E_1 \times \cdots \times E_5) .$$

Next, the inductive construction of [**12**, Proposition 2.1] shows X is of the form $Z/\mathbb{Z}_2^4$, where Z is obtained from $E_1 \times \cdots \times E_5$ by blowing up some rational subvarieties. Since rational varieties of dimension ≤ 3 verify the Lefschetz standard conjecture, this implies

$$H^i(Z) \ \subset\ \operatorname{Im}\big(H^i(E_1 \times \cdots \times E_5) \ \to\ H^i(Z)\big) \ \cup\ \widetilde{N}^1 H^i(Z) \quad \text{for all } i .$$

In particular, it follows that

$$H^4(X) = H^4(Z)^{\mathbb{Z}_2^4} \ \subset\ \widetilde{N}^1 H^4(X) .$$

□

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