A Method for Successive Graphic Integrations.

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The method here described applies to the successive integrations of any function whose graph consists of segments of straight lines.

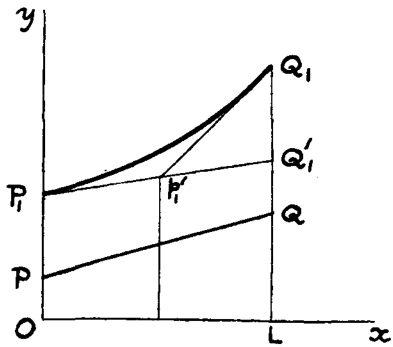

Fig. 1.

The integral curve P_1Q_1 of the straight graph PQ, which extends from $x=0$ to $x=h$, is of course parabolic, with axis parallel to Oy, and hence it follows that the tangents at P_1 and Q_1 will meet at p_1', a point whose abscissa is $h/2$.

This affords a ready means of obtaining the point Q_1 and the tangent at Q_1 by construction, provided the position of P_1 is known, *i.e.* provided we know the value of the ordinate of the integral curve for $x=0$.

Starting from P_1 we draw the line $P_1p_1'Q_1'$ whose gradient is OP/H, where H is the "polar distance" for graphic integration, to meet the ordinate of Q in Q_1'. Then from p_1', the mid point of this line, we draw $p_1'Q_1$ to meet the ordinate of Q in Q_1, the gradient of $p_1'Q_1$ being LQ/H where LQ is the ordinate of Q.

This determines the point Q_1 of the integral curve, as well as the tangent at that point.

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This observation led me to search for a generalization of the process just indicated, and what follows is the result of this search.

Let PQRS... be any graph (see Fig. 2) made up of straight line segments PQ, QR, RS, etc., the abscissae of P, Q, R, S, etc., being 0, h, $h+k$, $h+k+l$, etc.

Draw the ordinates QL, RM, SN, etc., and let lengths OO_0, LL_0, MM_0, NN_0..., each equal to H, the "polar distance," be marked off on the axis, going backwards from the points O, L, M, N, etc.

Let OP_1 be the initial value of the first integral, then for the first integration we proceed as follows:—

Through P_1 draw $P_1p_1'Q_1'$ parallel to O_0P to meet QL in Q_1'.

From p_1', the mid-point of P_1Q_1', draw $p_1'Q_1q_1'R_1'$ parallel to L_0Q to meet QL in Q_1 and RM in R_1'.

From q_1', the mid-point of Q_1R_1', draw $q_1'R_1r_1'S_1'$ parallel to M_0R to meet RM in R_1 and SN in S_1'.

From r_1', the mid-point of R_1S_1', draw r_1S_1 parallel to N_0S to meet SN in S_1.

Then P_1, Q_1, R_1, S_1 are points on the first integral graph, and P_1Q_1', Q_1R_1', R_1S_1' and $r_1'S_1$ are the tangents at these points.

For brevity in what follows let us premise that all points indicated by one capital letter have the same abscissa, whatever suffixes or dashes they may have. Thus Q, Q_1, Q_1', Q_2, etc., have the same abscissa h. As for the small letters, the number of dashes indicates the corresponding abscissa. Thus the abscissae of p_3', p_3'', p_3''', etc., are $h/2$, $2h/3$, $3h/4$, etc., respectively, and that of $p_n^{(r)}$ is $rh/(r+1)$, while the abscissa of $q_n^{(r)}$ is $h+rk/(r+1)$.

Second Integration.

Through P_2 (supposed given) draw $P_2p_2'Q_2'$ parallel to O_0P_1.

,,	p_2'	draw	$p_2'p_2''Q_2''$	parallel	to	L_0Q_1'.
,,	p_2''	,,	$p_2''Q_2q_2'R_2'$	,,	,,	L_0Q_1.
,,	q_2'	,,	$q_2'q_2''R_2''$	,,	,,	M_0R_1'.
,,	q_2''	,,	$q_2''R_2r_2'S_2'$	,,	,,	M_0R_1.
,,	r_2'	,,	$r_2'r_2''S_2''$	,,	,,	N_0S_1'.
,,	r_2''	,,	$r_2''S_2$	,,	,,	N_0S_1.

And so on.

Then P_2, Q_2, R_2, S_2 are points on the second integral graph, and the tangents at these points are respectively P_2Q_2', Q_2R_2', R_2S_2' and $r_2''S_2$.

Third Integral.

Through	P_3	draw	$P_3p_3'Q_3'$	parallel	to	O_0P_2.
,,	p_3'	,,	$p_3'p_3''Q_3''$	,,	,,	L_0Q_2'.
,,	p_3''	,,	$p_3''p_3'''Q_3'''$	,,	,,	L_0Q_2''.
,,	p_3'''	,,	$p_3'''Q_3q_3'R_3'$	,,	,,	L_0Q_2.
,,	q_3'	,,	$q_3'q_3''R_3''$	,,	,,	M_0R_2'.
,,	q_3''	,,	$q_3''q_3'''R_3'''$	,,	,,	M_0R_2''.
,,	q_3'''	,,	$q_3'''R_3r_3'S_3'$	,,	,,	M_0R_2.
,,	r_3'	,,	$r_3'r_3''S_3''$	,,	,,	N_0S_2'.
,,	r_3''	,,	$r_3''r_3'''S_3'''$	,,	,,	N_0S_2''.
,,	r_3'''	,,	$r_3'''S_3$	,,	,,	N_0S_2.

And so on

Then $P_3Q_3R_3S_3$ are points on the third integral graph, and the tangents at these points are respectively P_3Q_3', Q_3R_3', R_3S_3' and $r_3'''S_3$.

The law of construction of the successive integral graphs will now be obvious, but for completeness we may add the construction of the

n^{th} Integral.

Through	P_n	draw	$P_np_n'Q_n'$	parallel	to	O_0P_{n-1}.
,,	p_n'	,,	$p_n'p_n''Q_n''$	,,	,,	$L_0Q'_{n-1}$.
,,	p_n''	,,	$p_n''p_n'''Q_n'''$	,,	,,	$L_0Q''_{n-1}$.
			....			
,,	$p_n^{(r)}$	,,	$p_n^{(r)}p_n^{(r+1)}Q_n^{(r+1)}$	,,	,,	$L_0Q_{n-1}^{(r)}$.
			....			
,,	$p_n^{(n)}$	,,	$p_n^{(n)}Q_n$	,,	,,	L_0Q_{n-1}.

And so on.

P_n, Q_n, R_n... are points on the n^{th} integral graph, and the tangents at these points are respectively P_nQ_n', Q_nR_n', R_nS_n'....

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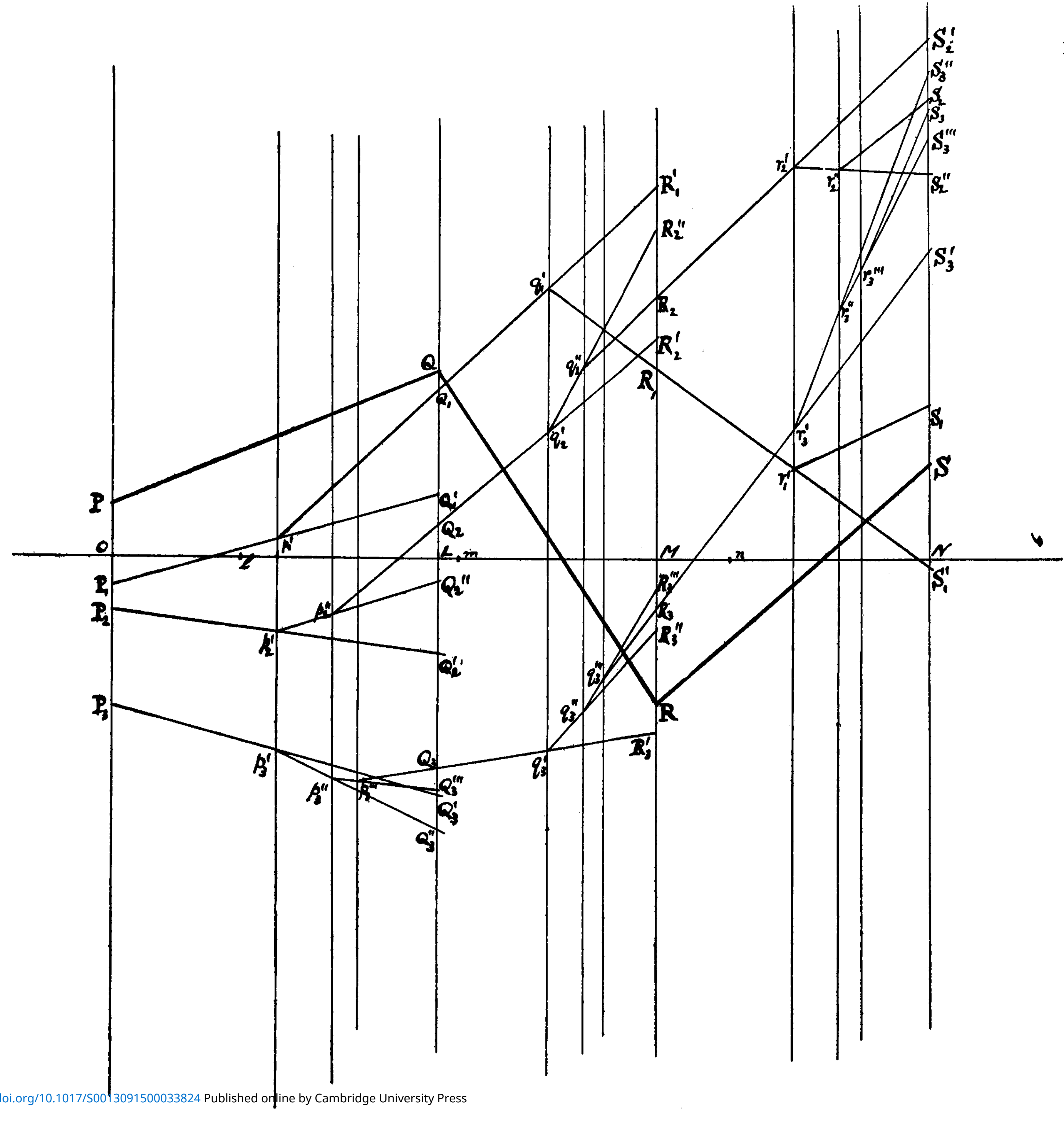

Fig. 2.

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When I had hit upon the rules given above, I did not at first see my way to a general proof, and was much indebted to Miss Marjory Strathie, who kindly went through the laborious algebra required to find the expression for the ordinate of S_3, in terms of the given quantities. The result agreeing with that arrived at by the ordinary process of integration, my opinion as to the correctness of the rules was confirmed.

I found later, however, that a direct proof of their correctness is not difficult. It turns out, in fact, that the quantities OP_n, $LQ_n' - OP_n$, $Q_n'Q_n''$, $Q_n''Q_n'''$, $\ldots Q_n^{(r)}Q_n^{(r+1} \ldots Q_n^{(n)}Q_n$ are simply the successive terms of Maclaurin's Theorem as applied to the n^{th} integral, for the value of $x = h$.

We have, in fact, if $f(x)$, $f_1(x)$, $f_2(x) \ldots f_n(x)$ denote the original function and its successive integrals, $f(x) = f_1'(x) = f_2''(x) \ldots\ldots = f_n^{(n)}(x)$, and generally, $f_r^{(p)}(x) = f_{r+q}^{(p+q)}(x)$.

Now by the construction,

$$Q_n^{(r)}Q_n^{(r+1)} = \frac{h}{r+1} \cdot \frac{Q_{n-1}^{(r-1)}Q_{n-1}^{(r)}}{H}$$

$$= \frac{h}{r+1} \cdot \frac{h}{r} \cdot \frac{Q_{n-2}^{(r-2)}Q_{n-2}^{(r-1)}}{H^{r-1}} = \ldots$$

$$= \frac{h}{r+1} \cdot \frac{h}{r} \cdot \frac{h}{r-1} \cdot \ldots \cdot \frac{h}{3} \cdot \frac{Q'_{n-r+1}Q''_{n-r+1}}{H^{r-1}}$$

$$= \frac{h^r}{r+1 \,.\, r \ldots 3 \,.\, 2} \cdot \frac{LQ_{n-r} - OP_{n-r}}{H^r}$$

$$= \frac{h^{r+1}}{(r+1)\,!} \cdot \frac{OP_{n-r-1}}{H^{r+1}}.$$

$$= \frac{h^{r+1}}{(r+1)\,!} \cdot \frac{f_{n-r-1}(0)}{H^{r+1}}$$

$$= \frac{h^{r+1}}{(r+1)\,!} \cdot \frac{f_n^{(r+1)}(0)}{H^{r+1}}.$$

Thus if we take the "polar distance" H equal to 1, we have

$$OQ_n = f_n(0) + \frac{h}{1} f_n'(0) + \frac{h^2}{2\,!} f_n''(0) + \ldots + \frac{h^r}{r\,!} f_n^{(r)}(0) + \ldots + \frac{h^{n+1}}{(n+1)\,!} f_n^{(n+1)}(0),$$

which is by Maclaurin's Theorem $=f_n(h)$, since $\overset{(n+1)}{f_n}(x)$ is constant from $x=0$ to $x=h$, so that the series stops at the $n+2^{\text{th}}$ term.

The foregoing graphic method of successive integration, like all others, is liable to lead to ill-conditioned constructions, in some cases, for instance, if the scale of ordinates were such as to keep the 3rd or 4th integral curves within the limits of a sheet of drawing paper, it might be too small for accurate work. This may to a great extent be avoided by changing the x-axis so as to bring the successive integral curves within bounds. The correction to be applied to the result is, of course, easy to make.

By drawing the lines belonging to the successive integrations in different *colours*, the diagram can be made clearer to look at. A check on the accuracy of the construction can be obtained by treating two parts of one line-segment as two successive segments, and modifying the construction accordingly.

Note.—The points referred to in the text as L_0, M_0, N_0, have by inadvertence been marked l, m, n in Figure 2.