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CONTINUUM-WISE EXPANSIVE HOMEOMORPHISMS

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ABSTRACT The notion of expansive homeomorphism is important in topological dynamics and continuum theory In this paper, a new kind of homeomorphism will be introduced and studied, namely the continuum-wise expansive homeomorphism The class of continuum-wise expansive homeomorphisms is much larger than the one of expansive homeomorphisms In fact, the class of continuum-wise expansive homeomorphisms contains many important homeomorphisms which often appear in "chaotic" topological dynamics and continuum theory, but which are not expansive homeomorphisms For example, the shift maps of Knaster's indecomposable chainable continua are continuum-wise expansive homeomorphisms, but they are not expansive homeomorphisms Also, there is a continuum-wise expansive homeomorphism on the pseudoarc We study several properties of continuum-wise expansive homeomorphisms Many theorems concerning expansive homeomorphisms will be generalized to the case of continuum-wise expansive homeomorphisms

1. **Introduction.** By a *continuum*, we mean a compact metric connected nondegenerate space. Let X be a compact metric space with metric d. Let $\mathbf{Z}$ be the set of integers. A homeomorphism $f: X \longrightarrow X$ is *expansive* (see [6, p. 86]) if there is a positive number $c > 0$ such that if $x, y \in X$ and $x \neq y$, then there is an integer $n = n(x, y) \in \mathbf{Z}$ such that

$$d\big(f^n(x), f^n(y)\big) > c.$$

This property has frequent applications in topological dynamics, ergodic theory and continuum theory.

A homeomorphism $f: X \longrightarrow X$ is *continuum-wise expansive* if there is a positive number $c > 0$ such that if A is a nondegenerate subcontinuum of X, then there is an integer $n = n(A) \in \mathbf{Z}$ such that $\operatorname{diam} f^n(A) > c$, where $\operatorname{diam} S = \sup\{d(x, y) \mid x, y \in S\}$ for any subset S of X. Such $c > 0$ is called an *expansive constant for f*. Clearly, every expansive homeomorphism is continuum-wise expansive. Since a continuum-wise expansive homeomorphism of a continuum is "chaotic", the continuum admitting such a homeomorphism may contain considerably complicated subspaces.

In this paper, we study several properties of continuum-wise expansive homeomorphisms. We will know that there are many important homeomorphisms of continua which are continuum-wise expansive, but not expansive. However, many theorems concerning expansive homeomorphisms will be generalized to the case of continuum-wise expansive homeomorphisms.

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We use hyperspace theory, which is very convenient for this study. We refer readers to [1], [6] and [27] for some basic properties of expansive homeomorphisms and to [21] for hyperspace theory.

2. **Definitions and preliminaries.** In this section, we give some definitions and results which will be needed in the sequel.

Let X be a compact metric space with metric d. By the *hyperspace* of X, we mean $2^X = \{A \mid A \text{ is a nonempty closed subset of } X\}$ and $C(X) = \{A \in 2^X \mid A \text{ is a nonempty subcontinuum of } X\}$ with the *Hausdorff metric* d_H, *i.e.*, $d_H(A,B) = \inf\{\varepsilon > 0 \mid U_\varepsilon(A) \supset B \text{ and } U_\varepsilon(B) \supset A\}$, where $U_\varepsilon(A)$ denotes the ε-neighborhood of A in X. It is well-known that if X is a continuum, then 2^X and $C(X)$ are arcwise connected continua (*e.g.*, see [21]). For any subsets A and B of X, let $d(A,B) = \inf\{d(a,b) \mid a \in A \text{ and } b \in B\}$.

Let $f: X \longrightarrow X$ be a homeomorphism of a compact metric space X. For any $\varepsilon > 0$, let W^s_ε and W^u_ε be the *local stable* and *unstable families of subcontinua of* X defined by

$$W^s_\varepsilon = \{A \in C(X) \mid \operatorname{diam} f^n(A) \leq \varepsilon \text{ for each } n \geq 0\} \text{ and}$$
$$W^u_\varepsilon = \{A \in C(X) \mid \operatorname{diam} f^{-n}(A) \leq \varepsilon \text{ for each } n \geq 0\}.$$

Also, define *stable* and *unstable families* W^s and W^u of $C(X)$ as follows:

$$W^s = \{A \in C(X) \mid \lim_{n\to\infty} \operatorname{diam} f^n(A) = 0\} \text{ and}$$
$$W^u = \{A \in C(X) \mid \lim_{n\to\infty} \operatorname{diam} f^{-n}(A) = 0\}.$$

Then we have

PROPOSITION 2.1 (*cf.* [20, P. 315]). *Let $f: X \longrightarrow X$ be a continuum-wise expansive homeomorphism of a compact metric space X with an expansive constant $c > 0$ and let $c \geq \varepsilon > 0$. If $A \in W^s_\varepsilon$ (resp. $A \in W^u_\varepsilon$), then $A \in W^s$ (resp. $A \in W^u$). In particular, $W^s = \{f^{-n}(A) \mid A \in W^s_\varepsilon, n \geq 0\}$ and $W^u = \{f^n(A) \mid A \in W^u_\varepsilon, n \geq 0\}$.*

PROOF. Let $A \in W^s_\varepsilon$. Suppose, on the contrary, that there is a sequence $n(1) < n(2) < \cdots$, of natural numbers such that $\operatorname{diam} f^{n(i)}(A) \geq \delta$ for some positive number $\delta > 0$. Since $C(X)$ is compact (see [21]), we may assume that $\lim_{i\to\infty} f^{n(i)}(A) = B \in C(X)$. Since $\lim_{i\to\infty} n(i) = \infty$ and $A \in W^s_\varepsilon$, we see that $\delta \leq \operatorname{diam} B \leq \varepsilon$ and $\operatorname{diam} f^n(B) \leq \varepsilon$ for any integer $n \in \mathbf{Z}$. Since f is a continuum-wise expansive homeomorphism with the expansive constant c, this is a contradiction. Hence $A \in W^s$. Similarly, if $A \in W^u_\varepsilon$, then $A \in W^u$.

PROPOSITION 2.2 (*cf.* [20, P. 318]). *Let $f: X \longrightarrow X$ be a continuum-wise expansive homeomorphism of a compact metric space X with an expansive constant $c > 0$. Let $0 < 2\varepsilon \leq c$. Then there is $\delta > 0$ such that if $A \in C(X)$, $\operatorname{diam} A \leq \delta$ and for some $n > 0$, $\varepsilon \leq \sup\{\operatorname{diam} f^j(A) \mid j = 0, 1, \ldots, n\} \leq 2\varepsilon$, then $\operatorname{diam} f^n(A) \geq \delta$.*

PROOF. Suppose, on the contrary, that there are a sequence $\{\delta_i\}$ of positive numbers with $\lim_{i\to\infty} \delta_i = 0$, a sequence $\{A_i\}$ of subcontinua of X and a sequence $\{n(i)\}$ of natural numbers such that $\operatorname{diam} A_i \leq \delta_i$ for each $i = 1, 2, \ldots$, $\varepsilon \leq \sup\{\operatorname{diam} f^j(A_i) \mid j = 0, 1, \ldots, n(i)\} \leq 2\varepsilon$ and $\operatorname{diam} f^{n(i)}(A_i) < \delta_i$. We may assume that $n(1) < n(2) < \cdots$. Choose $0 \leq m(i) \leq n(i)$ such that $\varepsilon \leq \operatorname{diam} f^{m(i)}(A_i) \leq 2\varepsilon$. Then $\lim_{i\to\infty}\big(n(i) - m(i)\big) = \infty = \lim_{i\to\infty} m(i)$. We may assume that $\lim_{i\to\infty} f^{m(i)}(A_i) = B \in C(X)$. It is easily checked that B is nondegenerate and $\operatorname{diam} f^n(B) \leq c$ for any integer $n \in \mathbf{Z}$. This is a contradiction.

PROPOSITION 2.3 (*cf.* [16, (2.8)]). *Let f, c, ε, δ be as in Proposition 2.2. If A is any nondegenerate subcontinuum of X such that that* $\operatorname{diam} A \leq \delta$ *and* $\operatorname{diam} f^m(A) \geq \varepsilon$ *for some integer m, then one of the following conditions holds:*

(*a*) *If $m \geq 0$, then* $\operatorname{diam} f^n(A) \geq \delta$ *for any $n \geq m$. More precisely, there is a subcontinuum B of A such that* $\sup\{\operatorname{diam} f^j(B) \mid j = 0, 1, \ldots, n\} \leq \varepsilon$ *and* $\operatorname{diam} f^n(B) = \delta$.

(*b*) *If $m < 0$, then* $\operatorname{diam} f^{-n}(A) \geq \delta$ *for any $n \geq -m$. More precisely, there is a subcontinuum B of A such that* $\sup\{\operatorname{diam} f^{-j}(B) \mid j = 0, 1, \ldots, n\} \leq \varepsilon$ *and* $\operatorname{diam} f^{-n}(B) = \delta$.

PROOF. Let A be as in Proposition 2.3. Choose a point a from A. Suppose $m \geq 0$. It is well-known that there is an arc $\alpha: [0, 1] \longrightarrow C(X)$ from $\{a\}$ to A in $C(X)$ such that if $x \leq y$, then $\alpha(x) \subset \alpha(y)$ (see [21, (1.26)]). Let $n \geq m$. Define a map $F: [0, 1] \longrightarrow [0, \infty)$ by

$$F(x) = \sup\Big\{\operatorname{diam} f^j\big(\alpha(x)\big) \mid j = 0, 1, \ldots, n\Big\}.$$

Choose $x_0 \in [0, 1]$ such that $x_0 \in F^{-1}(\varepsilon)$. Put $B' = \alpha(x_0)$. By Proposition 2.2, $\operatorname{diam} f^n(A) \geq \operatorname{diam} f^n\big(\alpha(x_0)\big) \geq \delta$. Define a map $D: C(B') \longrightarrow [0, \infty)$ by $D(C) = \operatorname{diam} f^n(C)$. Since $C(B')$ is connected, we can choose $B \in D^{-1}(\delta)$. Clearly, B satisfies the desired conditions. The case $m < 0$ is the same as before.

COROLLARY 2.4. *Let f, c, ε, δ be as in Proposition 2.2. Then for each $\gamma > 0$ there is $N > 0$ such that if $A \in C(X)$ and* $\operatorname{diam} A \geq \gamma$, *then* $\operatorname{diam} f^n(A) \geq \delta$ *for all $n \geq N$ or* $\operatorname{diam} f^{-n}(A) \geq \delta$ *for all $n \geq N$.*

PROPOSITION 2.5 (*cf.* [20, P. 315]). *If $f: X \longrightarrow X$ is a continuum-wise expansive homeomorphism of compact metric space X and* $\dim X > 0$, *then there is a nondegenerate subcontinuum A of X such that $A \in W^u$ or $A \in W^s$.*

PROOF. Let C be a nondegenerate subcontinuum of X with $\operatorname{diam} C \leq \delta$, where c, ε and δ are positive numbers as in Proposition 2.2. Suppose that any nondegenerate subcontinuum C' of C is not contained in W^s_c. Choose a sequence $C_1 \supset C_2 \supset \cdots$, of nondegenerate subcontinua of C and a sequence $n(1) < n(2) < \cdots$, of natural numbers such that $\lim_{i\to\infty} \operatorname{diam} C_i = 0$, $\sup\{\operatorname{diam} f^j(C_i) \mid j = 0, 1, \ldots, n(i)\} \leq \varepsilon$ and $\operatorname{diam} f^{n(i)}(C_i) \geq \delta$ for each $i = 1, 2, \ldots$, (see 2.3). We may assume that $\lim_{i\to\infty} f^{n(i)}(C_i) = A$. Then $A \in W^u_c$ and A is nondegenerate. This completes the proof.

From continuum theory in topology, we know that inverse limit spaces yield powerful techniques for constructing complicated spaces and maps from simple ones. Let $f: X \longrightarrow X$ be a map of a compact metric space X. Consider the following inverse limit space:

$$(X, f) = \{(x_n)_{n=0}^{\infty} \mid x_n \in X \text{ and } f(x_{n+1}) = x_n\}.$$

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The set (X,f) is a compact metric space with metric

$$d(\tilde{x},\tilde{y}) = \sum_{n=0}^{\infty} d(x_n,y_n)/2^n, \text{ where } \tilde{x} = (x_n)_{n=0}^{\infty}, \quad \tilde{y} = (y_n)_{n=0}^{\infty} \in (X,f).$$

Define a map $\tilde{f}\colon (X,f) \longrightarrow (X,f)$ by $\tilde{f}\big((x_0,x_1,\ldots)\big) = \big(f(x_0),f(x_1),\ldots\big)$ $\big(= \big(f(x_0),x_0,x_1,\ldots\big)\big)$, where $(x_0,x_1,\ldots) \in (X,f)$. The map $\tilde{f}$ is homeomorphism and it is called the *shift map of* f.

Let A be a subset of a compact metric space X. A map $f\colon X \longrightarrow X$ is called *positively expansive* (or *separated*) *on* A if there is a positive number $c > 0$ such that if $x,y \in A$ and $x \neq y$, then there is a natural number $n \geq 0$ such that $d\big(f^n(x),f^n(y)\big) > c$. A homeomorphism $f\colon X \longrightarrow X$ of a compact metric space X is *expansive* (or *separated*) *on* A if there is a positive number $c > 0$ such that if $x,y \in A$ and $x \neq y$, then there is an integer $n \in \mathbf{Z}$ such that $d\big(f^n(x),f^n(y)\big) > c$ (see [6, p. 38]). As before, such $c > 0$ is called an *expansive constant for* $f|A$.

A map $f\colon X \longrightarrow X$ is called *positively continuum-wise expansive* if there is a positive number $c > 0$ such that if A is a nondegenerate subcontinuum of X, then there is a natural number $n \geq 0$ such that $\operatorname{diam} f^n(A) > c$.

A compact connected 1-dimensional polyhedron is called a *graph*. Let $\mathbb{F}$ be a family of compact polyhedra. A continuum X is $\mathbb{F}$*-like* if for any $\varepsilon > 0$ there is a map f from X onto some member F of $\mathbb{F}$ such that $\operatorname{diam} f^{-1}(y) < \varepsilon$ for each $y \in F$. A continuum X is *arc-like* (= *chainable*) if X is $\{I\}$-like, where I is the unit interval $[0,1]$ (see [2]). A continuum X is *tree-like* if X is $\mathbb{T}$-like, where $\mathbb{T} = \{$ all trees$\}$. A continuum X is *decomposable* if X is the union of two subcontinua different from X. A continuum X is *indecomposable* if X is not decomposable. A continuum X is *hereditarily decomposable* (resp. *hereditarily indecomposable*) if each nondegenerate subcontinuum of X is decomposable (resp. indecomposable).

Let $f\colon X \longrightarrow Y$ be an onto map of compact metric spaces. Then f is said to be *light* if $\dim f^{-1}(y) = 0$ for each $y \in Y$. Also, f is said to be *weakly confluent* if for any subcontinuum B of Y, there is a subcontinuum A of X such that $f(A) = B$.

By the definition of continuum-wise expansive homeomorphism, we can easily see the following proposition.

PROPOSITION 2.6. *(1) If $f\colon X \longrightarrow X$ is a continuum-wise expansive homeomorphism, then for any integer $k \in \mathbf{Z}$, f^k is also continuum-wise expansive.*

(2) If $f\colon X \longrightarrow X$ and $g\colon Y \longrightarrow Y$ are continuum-wise expansive homeomorphisms, then the product $f \times g\colon X \times Y \longrightarrow X \times Y$ is also continuum-wise expansive.

(3) Let $f\colon X \longrightarrow X$, $h\colon Y \longrightarrow Y$ be homeomorphisms and let $\varphi\colon X \longrightarrow Y$ be an onto map making the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ Y & \xrightarrow[h]{} & Y \end{array}$$

If φ is light and h is continuum-wise expansive, then f is also continuum-wise expansive. If φ is light and weakly confluent and f is continuum-wise expansive, then h is also continuum-wise expansive.

EXAMPLE 2.7. Let $G = S^1 \cup A$, where $S^1 = \{(x,y) \in \mathbf{R}^2 \mid x^2+y^2 = 1\}, A = \{(x,0) \in \mathbf{R}^2 \mid 1 \leq x \leq 2\}$ and $\mathbf{R}$ is the set of real numbers. Define a map $g\colon G \longrightarrow G$ such that $g|S^1\colon S^1 \longrightarrow S^1$ is the natural covering map with degree 2 and $g|A\colon A \longrightarrow G$ is a positively expansive map such that $g\big((1,0)\big) = (1,0), g\big((2,0)\big) = (2,0), g(A_1) = S^1$ and $g(A_2) = A$, where $A_1 = \{(x,0) \mid 1 \leq x \leq 3/2\}$ and $A_2 = \{(x,0) \mid 3/2 \leq x \leq 2\}$. Put $X = (G,g)$ and $f = \tilde{g}$. Then $f\colon X \longrightarrow X$ is an expansive homeomorphism (see [13,(4.1)] or [16,(2.6)]); in particular it is a continuum-wise expansive homeomorphism. Set $Z = (g|S^1, S^1)$ and consider the quotient map $\varphi\colon X \longrightarrow Y = X/Z$, where X/Z is obtained from X by shrinking the subcontinuum Z to a point. Then Y is an arc and there is a homeomorphism $h\colon Y \longrightarrow Y$ such that $\varphi f = h\varphi$. Note that φ is a monotone map (*i.e.*, $\varphi^{-1}(D)$ is connected if D is a subcontinuum of Y), but $h\colon Y \longrightarrow Y$ is not continuum-wise expansive. In fact, φ is not a light map.

3. **Continuum-wise expansiveness of shift maps of inverse limits of graphs.** In this section, we give a characterization of continuum-wise expansiveness of shift maps of inverse limits of graphs (see 3.2), and we give some examples in order to clarify the difference between expansive homeomorphisms and continuum-wise expansive homeomorphisms. The similar characterization concerning expansiveness of shift maps of inverse limits of graphs is more complicated (see [16, (2.6)]).

First, we show the following proposition.

PROPOSITION 3.1. *If $f\colon X \longrightarrow X$ is a positively continuum-wise expansive map of a compact metric space X, then the shift map $\tilde{f}$ of f is a positively continuum-wise expansive homeomorphism.*

PROOF. Note that if C is any nondegenerate subcontinuum of X, $f(C)$ is nondegenerate, *i.e.*, f is a light map. This implies that for any $n \geq 0$, there is $m \geq n$ such that $\operatorname{diam} f^m(C) > c'$, where c' is an expansive constant for f. Put $Y = (X,f)$. Let $A \in C(Y)$ be nondegenerate. Choose a natural number n such that $p_n(A)$ is nondegenerate, where $p_n\colon Y \longrightarrow X$ is the natural projection. Choose $m \geq 0$ such that $\operatorname{diam} f^m\big(p_n(A)\big) > c'$. Choose a positive number $c > 0$ such that if $B \in C(Y)$ and $\operatorname{diam} B \leq c$, then $\operatorname{diam} p_0(B) \leq c'$. Since $p_0\big(\tilde{f}^{m-n}(A)\big) = f^{m-n}\big(p_0(A)\big) = f^m\big(p_n(A)\big) > c'$, $\operatorname{diam} \tilde{f}^{m-n}(A) > c$. Since we can choose $m \geq n$, $\tilde{f}$ is a positively continuum-wise expansive homeomorphism.

THEOREM 3.2. *Let $f\colon G \longrightarrow G$ be an onto map of a graph G. Then the following are equivalent.*

(1) $\tilde{f}\colon (G,f) \longrightarrow (G,f)$ is a continuum-wise expansive homeomorphism.

(2) $\tilde{f}\colon (G,f) \longrightarrow (G,f)$ is a positively continuum-wise expansive homeomorphism.

(3) $f: G \longrightarrow G$ is a positively continuum-wise expansive map.

PROOF. Suppose that $f: G \longrightarrow G$ is a positively continuum-wise expansive map. By Proposition 3.1, we see that $\tilde{f}$ is a positively continuum-wise expansive homeomorphism. This implies that (3) $\longrightarrow$ (2). (2) $\longrightarrow$ (1) is trivial.

We shall show that (1) $\longrightarrow$ (3). Suppose that $\tilde{f}$ is a continuum-wise expansive homeomorphism. First, we shall show that there is $\tau > 0$ such that if A is a nondegenerate subcontinuum of $X = (G,f)$ with $\operatorname{diam} A \leq \tau$, then $A \in W^u_c$, where c is an expansive constant for $\tilde{f}$. Suppose, on the contrary, that there are subcontinua A_n of X such that

(1) $\operatorname{diam} A_n \leq 1/n$, and

(2) A_n is not contained in W^u_c.

We may assume that $\lim_{n\to\infty} A_n = \{x\}$. Let ε and $\delta > 0$ be as in Proposition 2.3. We may assume that $\operatorname{diam} A_n \leq \delta$ for all n. Since A_1 is not contained in W^u_c, there is a natural number $m(1) > 0$ such that $\operatorname{diam} \tilde{f}^{-m(1)}(A_1) > c > 2\varepsilon$. Choose a neighborhood U_1 of x in X such that there is a subcontinuum B_1 of A_1 such that $\operatorname{diam} \tilde{f}^{-m(1)}(B_1) \geq \varepsilon$ and $B_1 \cap U_1 = \emptyset$. Choose $n(2) > 0$ such that $A_{n(2)} \subset U_1$. Since $A_{n(2)}$ is not contained in W^u_c, there is a natural number $m(2)$ such that $\operatorname{diam} \tilde{f}^{-m(2)}(A_{n(2)}) > c > 2\varepsilon$. Choose a neighborhood U_2 of x in X such that $U_2 \cap B_1 = \emptyset$ and for some subcontinuum $B_{n(2)}$ of $A_{n(2)}$, $\operatorname{diam} \tilde{f}^{-m(2)}(B_{n(2)}) \geq \varepsilon$ and $B_{n(2)} \cap U_2 = \emptyset$. If we continue this procedure, we obtain two sequences $\{m(i)\}_{i=1}^{\infty}$ and $1 = n(1) < n(2) < \cdots$, of natural numbers, and a sequence $\{B_{n(i)}\}_{i=1}^{\infty}$ of subcontinua of X such that

(3) $B_{n(i)} \cap B_{n(j)} = \emptyset$ $(i \neq j)$ and

(4) $\operatorname{diam} \tilde{f}^{-m(i)}(B_{n(i)}) \geq \varepsilon$ for each $i = 1, 2, \ldots$.

By 2.3, we see that

(5) $\operatorname{diam} \tilde{f}^{-n}(B_{n(i)}) \geq \delta$ for all $n \geq m(i)$.

Let n_0 be a natural number and let $\eta = \eta(n_0, \delta)$ be a positive number such that if E is a subset of (G,f) and $\operatorname{diam} p_{n_0}(E) < \eta$, then $\operatorname{diam} E < \delta$. Since G is a finite graph, we can choose a natural number $N(\eta) > 0$ such that if $\{D_j\}_{j=1}^{N(\eta)}$ is a mutually disjoint family of subcontinua of G, then there is D_j with $\operatorname{diam} D_j < \eta$. Consider the family $\mathcal{B} = \{B_{n(i)} \mid i = 1, 2, \ldots, N(\eta)\}$. Choose a natural number N so that $p_N(B_{n(i)}) \cap p_N(B_{n(j)}) = \emptyset$ $(i \neq j)$ and $N \geq \max\{m(i) \mid i = 1, 2, \ldots, N(\eta)\}$. Then $\operatorname{diam} p_{n_0+N}(B_{n(i)}) = \operatorname{diam} p_{n_0}\big(\tilde{f}^{-N}(B_{n(i)})\big) \geq \eta$ $\big(i = 1, 2, \ldots, N(\eta)\big)$ and $p_{n_0+N}(B_{n(i)}) \cap p_{n_0+N}(B_{n(j)}) = \emptyset$ $(i \neq j)$. This is a contradiction. Therefore there is a positive number $\tau > 0$ such that if $A \in C(X)$ and $\operatorname{diam} A \leq \tau$, then $A \in W^u_c$. Consider the set $C(X; \tau) = \{A \in C(X) \mid \operatorname{diam} A = \tau\}$. Note that $C(X; \tau)$ is compact and for any $x \in X$, there is some $A \in C(X; \tau)$ such that $x \in A$, because there is an arc $\alpha: I \longrightarrow C(X)$ such that from $\{x\}$ to X such that $\alpha(t) \subsetneq \alpha(t')$ if $t < t'$ (see [21]). Choose a natural number N and a positive number $\gamma > 0$ such that if C is a subset of X with $\operatorname{diam} p_N(C) < \gamma$, then $\operatorname{diam} C < c$. We may assume that $\operatorname{diam} p_N(A) \geq \beta$ for any $A \in C(X; \tau)$, where β is some positive number. Note that if $x_p \in G$, $\lim_{p\to\infty} x_p = x \in G$, $x_p \in A_{x_p} \in C(G)$, $A_{x_p} \subset p_N(A_p)$ for some $A_p \in C(X; \tau)$, and $\lim_{p\to\infty} A_{x_p} = C$, then $x \in C$ and $C \subset p_N(A)$ for some $A \in C(X; \tau)$. Note that $p_N: X \longrightarrow G$ is an onto map. By using these facts, we can see that there is a simplicial complex K such that $|K| = G$

and if E is any edge of K, then there is $A \in C(X;\tau)$ such that $p_N(A) \supset E$. Let D be a nondegenerate subcontinuum of G. Choose an edge E of K such that $\mathrm{Int}_G(E \cap D) \neq \emptyset$. Choose $A \in C(X;\tau)$ with $p_N(A) \supset E$. Then we can choose a subcontinuum B of A such that $p_N(B)$ is nondegenerate and $p_N(B) \subset \mathrm{Int}_G(E \cap D)$. Since $B \in W^u_c$, we see that there is a natural number n such that $\mathrm{diam} f^n(D) \geq \mathrm{diam} f^n\big(p_N(B)\big) = \mathrm{diam}\, p_N \tilde{f}^n(B) \geq \gamma$. Hence f is a positively continuum-wise expansive map with an expansive constant $\gamma > 0$. This completes the proof.

EXAMPLE 3.3. The converse of Proposition 3.1 is not true. Let $Y = C \cup I$, where C is a Cantor set and I is the unit interval such that $C \cap I = \emptyset$. Let $g_1: I \longrightarrow I$ be a constant map, *i.e.*, $g_1(I)$ is a one point set, and $g_2: C \longrightarrow Y$ be an onto map. Define a map $g: Y \longrightarrow Y$ by $g(x) = g_1(x)$ for $x \in I$ and $g(x) = g_2(x)$ for $x \in C$. Let X be the cone of Y, *i.e.*, $X = (Y \times I)/(Y \times \{0\})$, which is obtained from $Y \times I$ by shrinking $Y \times \{0\}$ to a point. Define a map $f: X \longrightarrow X$ by $f([y,t]) = [g(y), f_3(t)]$ for $[y,t] \in X$, where $f_3: I \longrightarrow I$ is the map as in Example 3.5 (see below). Then it is checked that the shift map $\tilde{f}$ of f is a positively continuum-wise expansive homeomorphism of a continuum (X,f), but f is not a positively continuum-wise expansive map. Also, in the statement of Theorem 3.2, we can not replace "a graph" by "n-dimensional polyhedron ($n \geq 2$)". In fact, by [25] there is an expansive homeomorphism $f: T^2 \longrightarrow T^2$, but f and f^{-1} are not positively continuum-wise expansive, where T^2 is the 2-torus. Note $\tilde{f}$ is also expansive.

REMARK 3.4. Let $\tilde{f}$ be a continuum-wise expansive homeomorphism as in Theorem 3.2. Then it is positively continuum-wise expansive. But, the situation of expansive homeomorphisms of compact metric spaces is different. In fact, it is well-known that if $f: X \longrightarrow X$ is a homeomorphism of a compact metric space X and positively expansive, then X is a finite set (see [1] or [6]). In [16], we proved that for any onto map $f: G \longrightarrow G$ of a graph G, $\tilde{f}$ is expansive if and only if f is a positively pseudo-expansive map (see [16] for the definition of positively pseudo-expansive map).

EXAMPLE 3.5. We will consider an interesting class of inverse limit spaces called *Knaster's chainable continua*. Let I denote the unit interval $[0,1]$. For each natural number $n = 2, 3, \ldots$, let $f_n: I \longrightarrow I$ be a map defined

$$f_n(t) = \begin{cases} nt - s, & \text{if } s \text{ is even,} \\ -nt + s + 1, & \text{if } s \text{ is odd,} \end{cases}$$

for $t \in [s/n, s + 1/n]$ and $s = 0, 1, \ldots, n-1$. Then $K(n) = (I, f_n)$ is the Knaster's chainable continuum of order n. It is clear that the map f_n is a positively continuum-wise expansive map; hence the shift map $\tilde{f}_n$ is a (positively) continuum-wise expansive homeomorphism. Since every chainable continuum can be embedded into the plane $\mathbf{R}^2$, there are many nonseparating plane continua admitting (positively) continuum-wise expansive homeomorphisms. On the other hand, $\tilde{f}_n$ is not an expansive homeomorphism. In fact, it is not known whether or not there exists a nonseparating plane continuum admitting an expansive homeomorphism (see [15], [23] and [24]).

REMARK 3.6. The pseudoarc P is a hereditarily indecomposable chainable continuum. Then there is a positively continuum-wise expansive homeomorphism f on P. In [19, Section 2], J. Kennedy proved that there exist a homeomorphism $h: P \longrightarrow P$ and an onto map $\theta: P \longrightarrow I$ such that $f_2\theta = \theta h$ and θ satisfies that if P' is a nondegenerate subcontinuum of P, then $\theta(P')$ is also nondegenerate, *i.e.*, θ is a light map, where f_2 is as in Example 3.5. Since f_2 is positively continuum-wise expansive, it is easily seen that $h: P \longrightarrow P$ is also a positively continuum-wise expansive homeomorphism (*cf.* 2.6).

By 3.1, we have the following corollary.

COROLLARY 3.7. *For any graph G, there is a positively continuum-wise expansive map f from G onto G, and hence $\tilde{f}: (G,f) \longrightarrow (G,f)$ is a continuum-wise expansive homeomorphism. Moreover, for any graph G there is a $\{G\}$-like and indecomposable continuum X and a continuum-wise expansive homeomorphism on X.*

OUTLINE OF PROOF. Take a simplicial complex K of G, *i.e.*, $|K| = G$. We can choose an onto map $g: G \longrightarrow G$ such that for each edge $e = \langle V, V' \rangle$ of K, g is positively continuum-wise expansive on e and $g(\langle V, V_0 \rangle) = g(\langle V_0, V' \rangle) = G$, where V_0 is the middle point of $\langle V, V' \rangle$. Let $X = (G, g)$ and $f = \tilde{g}$. Then f is a continuum-wise expansive homeomorphism and X is $\{G\}$-like. Suppose, on the contrary, that X is decomposable, *i.e.*, there are two proper subcontinua A and B of X such that $A \cup B = X$. Choose a natural number N such that $p_n(A)$ and $p_n(B)$ are proper subcontinua of G if $n \geq N - 1$. We may assume that $V_0 \in A_N = p_N(A)$. Consider the following cases:

CASE (I). A_N contains V or V'. We may assume that $A_N \supset \langle V, V_0 \rangle$. Then $p_{N-1}(A) = g(A_N) \supset g(\langle V, V_0 \rangle) = G$. This is a contradiction.

CASE (II). A_N does not contain V and V'. In this case, $V, V' \in B_N$. Since B_N is connected, there is an edge e' of K such that $B_N \supset e'$. Then $p_{N-1}(B) = g(B_N) \supset g(e') = G$. This is a contradiction.

An onto map $f: X \longrightarrow X$ of a compact metric space X has *sensitive dependence on initial conditions* if there is $\tau > 0$ such that if $x \in X$ and U is any open set that contains x, then there are some point y in U and a natural number $n \geq 0$ such that $d\big(f^n(x), f^n(y)\big) > \tau$.

Then we have

PROPOSITION 3.8. *(1) If $f: X \longrightarrow X$ is a positively continuum-wise expansive map of a continuum X, then f has sensitive dependence on initial conditions.*

(2) Let G be a graph. Then a map $f: G \longrightarrow G$ is a positively continuum-wise expansive map if and only if f has sensitive dependence on initial conditions.

PROOF. Let $f: X \longrightarrow X$ be a positively continuum-wise expansive map with an expansive constant $c > 0$. Put $\tau = c/2$. Let $x \in X$ and U be an open set such that $x \in U$. Choose a nondegenerate subcontinuum A of X such that $x \in A \subset U$. Take a natural number $n \geq 0$ such that $\operatorname{diam} f^n(A) > c$. Clearly, there is a point $y \in A \subset U$ such that $d\big(f^n(x), f^n(y)\big) > c/2 = \tau$. This proves (1). Next, we shall prove (2). Let $f: G \longrightarrow G$ be an onto map of a graph G which has sensitive dependence on initial conditions. Let A

be a nondegenerate subcontinuum of G Choose an open interval J in A and $x \in J$ Then there is $y \in J$ and a natural number $n \geq 0$ such that $d\big(f^n(x), f^n(y)\big) > \tau$ This implies that $\operatorname{diam} f^n(A) > \tau$ Hence f is a positively continuum-wise expansive map with an expansive constant $\tau > 0$

It is easily seen that an onto map f $X \longrightarrow X$ has sensitive dependence on initial conditions if and only if the shift map $\tilde{f}$ has sensitive dependence on initial conditions As a corollary of Theorem 3 2 and Proposition 3 8, we have

COROLLARY 3 9 *Let f $G \longrightarrow G$ be an onto map of a graph G Then the following are equivalent*

(1) The shift map $\tilde{f}$ of f is a continuum-wise expansive homeomorphism
(2) f is a positively continuum-wise expansive map
(3) f has sensitive dependence on initial conditions
(4) $\tilde{f}$ has sensitive dependence on initial conditions

REMARK 3 10 We can easily see that if an onto map f $X \longrightarrow X$ of a compact metric space X has sensitive dependence on initial conditions, then X is perfect, *i e* , $x \in \mathrm{Cl}(X - \{x\})$ for any $x \in X$, and $f \times g$ $X \times Y \longrightarrow X \times Y$ has also sensitive dependence on intial conditions for any onto map g $Y \longrightarrow Y$ Let f_3 $I \longrightarrow I$ be as in Example 3 5 Then $f = f_3 \times 1$ $I \times I \longrightarrow I \times I$ has sensitive dependence on initial conditions, but it is not positively continuum-wise expansive, where 1 $I \longrightarrow I$ is the identity map Hence (2) in Proposition 3 8 is not true for the case of 2-dimensional polyhedra Also, consider the Cantor middle-third set C in the unit interval I Note that $f_3(C) = C$ Put $X = I \times \{0\} \cup C \times I$ Then $\dim X = 1$ and $f|X$ $X \longrightarrow X$ has sensitive dependence on initial conditions, but $f|X$ is not positively continuum-wise expansive Hence (2) in Proposition 3 8 is not true for the case of 1-dimensional continua

4 Topological entropy and expansiveness of subsets of continuum-wise expansive homeomorphisms.

Let $\mathcal{A}$ and $\mathcal{B}$ be finite open covers of a compact metric space X, and let $N(\mathcal{A})$ denote the minimum cardinality of a subcover of $\mathcal{A}$ For any map f $X \longrightarrow X$, put $f^{-k}(\mathcal{A}) = \{f^{-k}(U) \mid U \in \mathcal{A}\}$ Define $\mathcal{A} \vee \mathcal{B}$ by $\mathcal{A} \vee \mathcal{B} = \{U \cap V \mid U \in \mathcal{A}, V \in \mathcal{B}\}$ Consider the following

$$h(f, \mathcal{A}) = \lim_{n \to \infty} (1/n) \cdot \log N\big(\mathcal{A} \vee f^{-1}(\mathcal{A}) \vee \cdots \vee f^{-(n-1)}(\mathcal{A})\big)$$

The *topological entropy of f* is then

$$h(f) = \sup\{h(f, \mathcal{A}) \mid \mathcal{A} \text{ is an open cover of } X\}$$

A subset E of X is (n, ε)-*separated* if for each $x, y \in E$, $x \neq y$, there is k $(0 \leq k < n)$ such that $d\big(f^k(x), f^k(y)\big) > \varepsilon$ Let $S(n, \varepsilon)$ denote the maximum cardinality of (n, ε)-separated sets in X Consider the following

$$h(f, \varepsilon) = \limsup_{n \to \infty} (1/n) \cdot \log S(n, \varepsilon)$$

Note that if $\varepsilon > \varepsilon'$, then $h(f,\varepsilon) \leq h(f,\varepsilon')$. Then the topological entropy is given by $h(f) = \lim_{\varepsilon\to 0} h(f,\varepsilon)$ (*e.g.*, see [27]).

In this section, we show that if $f: X \longrightarrow X$ is a continuum-wise expansive homeomorphism of compact metric space X and $\dim X > 0$, then the topological entropy $h(f) > 0$. As a corollary, every expansive homeomorphism of any n-dimensional compact metric space ($n > 0$) has a positive entropy. It is well-known that the topological entropy of expansive homeomorphism is finite (see [27, Theorem 7.11]). We give an example in which some continuum-wise expansive homeomorphism of a chainable continuum has an infinite topological entropy. Also, we investigate expansiveness of subsets of continuum-wise expansive homeomorphisms.

THEOREM 4.1. *If $f: X \longrightarrow X$ is a continuum-wise expansive homeomorphism of a compact metric space X with* $\dim X > 0$, *then $h(f)$ is positive.*

PROOF. Note that $h(f) = h(f^{-1})$. Let $c > 0$ be an expansive constant for f. By Proposition 2.5, there is a nondegenerate subcontinuum A of X such that $A \in W^s_c$ or $A \in W^u_c$. We assume that $A \in W^u_c$. Let $\delta > 0$ be as in Proposition 2.3. By Proposition 2.3, we may assume $\operatorname{diam} A = \delta/3$. Choose a natural number N such that if $D \in C(X)$ and $\operatorname{diam} D \geq \delta/3$, then $\max\{\operatorname{diam} f^j(D) \mid |j| \leq N\} > c$. Hence we see that if $D \in W^u_c$ and $\operatorname{diam} D = \delta/3$, then $\operatorname{diam} f^N(D) \geq \delta$ (see Proposition 2.3). Let $n(m) = m \cdot N$. By Proposition 2.3, it is checked that there is a finite collection $\{A_{i_1,i_2,\ldots,i_j}\}$ ($i_k = 0$ or 1, and $j \leq m$) of subcontinua of X satisfying the following conditions:

(1) A_{i_1} ($i_1 = 0$ or 1) is a subcontinuum of $f^N(A)$ such that $\delta/3 = \operatorname{diam} A_{i_1}$ and $d(A_0, A_1) \geq \delta/3$.

(2) $A_{i_1,i_2,\ldots,i_k}$ is a subcontinuum of $f^N(A_{i_1,i_2,\ldots,i_{k-1}})$ such that $\operatorname{diam} A_{i_1,i_2,\ldots,i_k} = \delta/3$ and $d(A_{i_1,i_2,\ldots,i_{k-1},0}, A_{i_1,i_2,\ldots,k-1,1}) \geq \delta/3$.

(3) $A_{i_1,i_2,\ldots,i_j} \in W^u_c$ (see Proposition 2.3).

Choose a point $a_{i_1,i_2,\ldots,i_m} \in f^{-mN}(A_{i_1,i_2,\ldots,i_m})$. Then the set $E_m = \{a_{i_1,i_2,\ldots,i_m} \mid i_j = 0 \text{ or } 1\}$ is $\big(\delta/3, n(m)+1\big)$-separated. Hence $S\big(n(m)+1, \delta/3)\big) \geq 2^m$, which implies that

$$\begin{aligned} h(f,\delta/3) &= \limsup_{n\to\infty} (1/n)\cdot \log S(n,\delta/3) \\ &\geq \limsup_{m\to\infty} \Big(1/\big(n(m)+1\big)\Big)\cdot \log S\big(n(m)+1, \delta/3\big) \\ &\geq \lim_{m\to\infty} \big(m/(mN+1)\big)\cdot \log 2 = (1/N)\cdot \log 2 > 0. \end{aligned}$$

Then $h(f) > 0$. This completes the proof.

COROLLARY 4.2. *If $f: X \longrightarrow X$ is an expansive homeomorphism of a compact metric space X with* $\dim X > 0$, *then the topological entropy $h(f)$ is positive.*

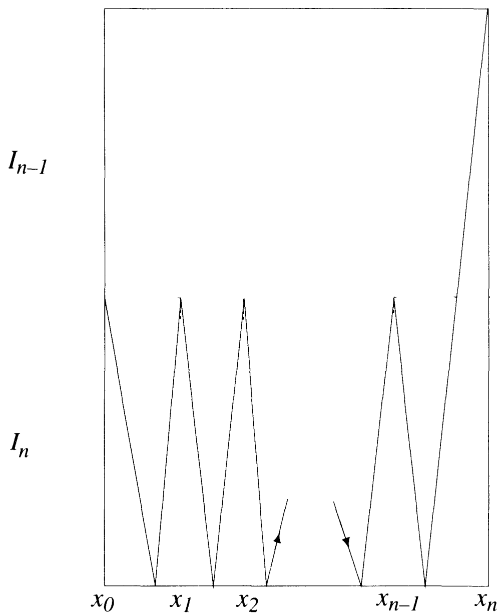
REMARK 4.3. In [4], Fathi has already proved Corollary 4.2. However, his proof is completely different from one of this paper and our proof is more general and elementary.

EXAMPLE 4.4. We will construct a map $f: I \longrightarrow I$ such that $\tilde{f}$ is a continuum-wise expansive homeomorphism and $h(\tilde{f}) = \infty$. Put $I_n = [1/(n+1), 1/n]$ ($n = 1, 2, \ldots$). Let

$g_1\colon [1/2, 1] \longrightarrow [1/2, 1]$ be the map defined by

$$g_1(x) = \begin{cases} -2x + 2, & x \in [1/2, 3/4] \\ 2x - 1, & x \in [3/4, 1]. \end{cases}$$

Let $g_n\colon I_n \longrightarrow I_n \cup I_{n-1}$ $(n \geq 2)$ denote the piecewise linear map defined by the following figure:



where $1/(n+1) = x_0 < x_1 < \cdots < x_n = 1/n$ and $x_i - x_{i-1} = 1/\big(n^2(n+1)\big)$ for each i. Define a map $f\colon I \longrightarrow I$ by

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ g_n(x), & \text{if } x \in I_n. \end{cases}$$

It is checked that f is a positively continuum-wise expansive map and $h(\tilde{f}) \geq \log n$ for all n. Hence $h(\tilde{f}) = \infty$.

THEOREM 4.5. *Let $f\colon X \longrightarrow X$ be a homeomorphism of a compact metric space X. Then f is continuum-wise expansive if and only if there is a positive number τ such that for any nondegenerate subcontinuum Y of X, there is a dense and uncountable subset D of Y such that f is expansive on D with an expansive constant τ.*

PROOF. Suppose that f is continuum-wise expansive. Let Y be any nondegenerate subcontinuum of X. Let $\{U_i\}_{i=1}^{\infty}$ be a countable open base of Y. Choose a point p_1 of U_1. Note that f is expansive on $E_1 = \{p_1\}$. We assume that there are points p_i of U_i $(1 \leq i \leq k)$ such that if $p, q \in E_k = \{p_1, p_2, \ldots, p_k\}$ and $p \neq q$, there is an integer n such that $d\big(f^n(p), f^n(q)\big) \geq \delta/4$, where c, ε and δ are as in Proposition 2.3. By Corollary 2.4, there is a natural number $N > 0$ such that if $B \in C(X)$ and $\operatorname{diam} B \geq \delta/4$,

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then $\operatorname{diam} f^n(B) \geq \delta$ for all $n \geq N$ or $\operatorname{diam} f^{-n}(B) \geq \delta$ for all $n \geq N$. Take a nondegenerate subcontinuum A of Y such that $A \subset U_{k+1}$ and $\operatorname{diam} A < \delta$. We may assume that for some $m \geq N$, $\operatorname{diam} f^m(A) \geq \delta$ and $\operatorname{diam} f^j(A) \leq \varepsilon$ $(0 \leq j \leq m)$ (see Proposition 2.3). Since $\operatorname{diam} f^m(A) \geq \delta$, there is a subcontinuum A_1 of $f^m(A)$ such that $d\big(f^m(p_1), A_1\big) \geq \delta/4$ and $\operatorname{diam} A_1 = \delta/4$. Next, consider the set $f^N(A_1)$. Since $\operatorname{diam} A_1 < \delta$, we see that $\operatorname{diam} f^N(A_1) \geq \delta$ (see Corollary 2.4). Then there is a subcontinuum A_2 of $f^N(A_1)$ such that $d\big(f^{m+N}(p_2), A_2\big) \geq \delta/4$ and $\operatorname{diam} A_2 = \delta/4$. If we continue this procedure, we obtain $A_1, A_2, \ldots, A_k$. Choose a point $p_{k+1} \in f^{-(m+(k-1)N)}(A_k)$. If $p, q \in E_{k+1} = \{p_i \mid i = 1, 2, \ldots, k+1\}$ and $p \neq q$, then there is an integer n such that $d\big(f^n(p), f^n(q)\big) \geq \delta/4$. Put $D_1 = \bigcup E_k = \{p_1, p_2, \ldots\}$. Then we see that $\operatorname{Cl}(D_1) = Y$ and f is expansive on D_1 with an expansive constant $\delta/5 = \tau$. For a countable ordinal number $\lambda \geq 2$, we assume that for any $\alpha < \lambda$, there is a countable subset D_α of Y such that $D_\alpha \underset{\neq}{\subset} D_\beta$ if $\alpha < \beta < \lambda$, and f is expansive on D_α with an expansive constant $\delta/5$. We shall construct D_λ as follows:

CASE (I). $\lambda = \alpha + 1$. Let $D_\alpha = \{q_1, q_2, \ldots\}$. As before, we can choose a sequence $A_1, A_1, A_2, \ldots,$ of subcontinua of X such that $A_0 \subset Y$, $\operatorname{diam} A_0 < \delta$, $\operatorname{diam} A_i = \delta/4$ $(i \geq 1)$, $d\big(f^{m+(i-1)N}(q_i), A_i\big) \geq \delta/4$ and $f^N(A_i) \supset A_{i+1}$ for some integers m, N. Choose a point $p \in \bigcap_{i=1}^{\infty} f^{-(m+(i-1)N)}(A_i)$. Put $D_\lambda = D_\alpha \cup \{p\}$.

CASE (II). λ is a limit ordinal. Put $D_\lambda = \bigcup_{\alpha<\lambda} D_\alpha$. Note that D_λ is a countable set.

Hence we obtain subset D_λ of Y for any countable ordinal λ. Put $D = \bigcup\{D_\lambda \mid \lambda$ is a countable ordinal$\}$. Then D is uncountable and f is expansive with expansive constant $\delta/5$. Clearly $\operatorname{Cl}(D) = \operatorname{Cl}(D_1) = Y$.

The converse is obvious.

THEOREM 4.6. *Let $f: X \longrightarrow X$ be a homeomorphism of a compact metric space X. Then f is continuum-wise expansive if and only if there is a positive number τ such that for any nondegenerate subcontinuum Y of X, Y contains a Cantor set C such that f or f^{-1} is positively expansive on C with an expansive constant τ.*

PROOF. The proof is similar to the one for (4.1). Suppose that f is continuum-wise expansive. Let c, ε and δ be as in Proposition 2.3. Choose a natural number N such that if $A \in C(X)$ and $\operatorname{diam} A \geq \delta/3$, then $\operatorname{diam} f^n(A) \geq \delta$ for all $n \geq N$ or $\operatorname{diam} f^{-n}(A) \geq \delta$ for all $n \geq N$ (see Corollary 2.4). We may assume that $\operatorname{diam} Y < \delta$. By Proposition 2.3, we may assume that there is a natural number $m > N$ such that $\operatorname{diam} f^m(Y) \geq \delta$. Take two subcontinua A_0, A_1 of $f^m(Y)$ such that $d(A_0, A_1) \geq \delta/3$ and $\operatorname{diam} A_i = \delta/3$ for $i = 0, 1$. Note that $\operatorname{diam} f^N(A_i) \geq \delta$ $(i = 0, 1)$. For each $i = 0, 1$, take two subcontinua A_{i0}, A_{i1} of $f^N(A_i)$ such that $d(A_{i0}, A_{i1}) \geq \delta/3$ and $\operatorname{diam} A_{ij} = \delta/3$ $(j = 0, 1)$. If we continue this procedure, we obtain $A_{i_1 i_2, \ldots, i_n}$ $(i_k = 0, 1$ and $n = 1, 2, \ldots)$. Put $A_{i_1 i_2, \ldots} = \bigcap_{n=1}^{\infty} f^{-(m+(n-1)N)}(A_{i_1 i_2, \ldots, i_n})$. If for each sequence $i_1, i_2, \ldots, A_{i_1 i_2, \ldots}$ is a one point set, $C = \{A_{i_1 i_2, \ldots} \mid i_k = 0, 1\}$ is a Cantor set in Y and f is positively expansive on C with an expansive constant $\tau = \delta/4$. If for some sequence $i_1 i_2, \ldots, A_{i_1 i_2, \ldots}$ is nondegenerate, $A_{i_1 i_2, \ldots} \in W_c^s$. By the same proof

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as before, we see that $A_{i_1 i_2,\ldots}$ contains a Cantor set C such that f^{-1} is positively expansive on C with an expansive constant τ. The converse is obvious.

Let $f\colon X \longrightarrow X$ be a homeomorphism of a compact metric space X and let $x, y \in X$. The points x and y are said to be *doubly asymptotic under f* (see [6, p. 84]) if

$$\lim_{n\to\infty} d\big(f^n(x), f^n(y)\big) = 0 = \lim_{n\to\infty} d\big(f^{-n}(x), f^{-n}(y)\big).$$

THEOREM 4.7. *Let $f\colon G \longrightarrow G$ be an onto map of a graph G. Suppose that the shift map $\tilde{f}$ of f is a continuum-wise expansive homeomorphism and let $X = (G, f)$. Then the following are true:*

(1) For any nondegenerate subcontinuum A of X, there are two points $\tilde{x}$ and $\tilde{y}$ ($\tilde{x} \neq \tilde{y}$) of A such that $\tilde{x}$ and $\tilde{y}$ are doubly asymptotic under $\tilde{f}$.

(2) If f is null-homotopic, for any nondegenerate subcontinuum A of X and any $\gamma > 0$, there are two points $\tilde{x}$ and $\tilde{y}$ ($\tilde{x} \neq \tilde{y}$) of A such that $d\big(\tilde{f}^m(\tilde{x}), \tilde{f}^m(\tilde{y})\big) < \gamma$ for each integer $m \in \mathbf{Z}$ and $\tilde{x}$ and $\tilde{y}$ are doubly asymptotic under $\tilde{f}$. In particular, $\tilde{f}$ is not expansive on any nondegenerate subcontinuum A of X.

OUTLINE OF PROOF. (1) Let A be a nondegenerate subcontinuum of X. By the proof of Theorem 3.2, there is a positive number $\tau > 0$ such that if $B \in C(X)$ and $\operatorname{diam} B \leq \tau$, then $B \in W^u_c$, where c is an expansive constant for $\tilde{f}$. Let $A_n = p_n(A)$ and $A_{-n} = f^n(A_0)$ $(n = 1, 2, \ldots)$. We may assume that $\operatorname{diam} A \leq \tau$, and hence $A \in W^u_c$. Note that $\lim_{n\to\infty} \operatorname{diam} A_n = 0$. Note that A_n is nondegenerate and f is a positively continuum-wise expansive map (see Theorem 3.2). Hence, we see that for some natural number k, $f^k|A_0\colon A_0 \longrightarrow A_{-k}$ is not injective (see the proof of Theorem 3.2). Choose two points $\tilde{x}$ and $\tilde{y}$ of A such that $p_0(\tilde{x}) \neq p_0(\tilde{y})$ and $f^k\big(p_0(\tilde{x})\big) = f^k\big(p_0(\tilde{y})\big)$. Clearly, $\tilde{x}$ and $\tilde{y}$ are doubly asymptotic under $\tilde{f}$.

(2) Next, suppose that $f\colon G \longrightarrow G$ is null-homotopic. Let k be as before. Let n be a sufficiently large natural number. Note that $f^{n+k}|A_n\colon A_n \longrightarrow A_{-k}$ is not injective. Consider the universal covering $p\colon \tilde{G} \longrightarrow G$ of G. Since f is null-homotopic, there is a lifting $g\colon G \longrightarrow \tilde{G}$ of f, *i.e.*, $pg = f$. Since $gf^{n+k}|A_n$ is not injective and $g(G)$ is a compact tree, we can choose two points a and a' of A_n such that $a \neq a'$, a and a' are sufficiently near and $gf^{n+k}(a) = gf^{n+k}(a')$ (see the proof of [13, (3.5)]). Hence $f^{n+k+1}(a) = f^{n+k+1}(a')$. Take two points $\tilde{x}$ and $\tilde{y}$ of A such that $p_n(\tilde{x}) = a$ and $p_n(\tilde{y}) = a'$. Since n is sufficiently large, we see that $\tilde{f}^m(x)$ and $\tilde{f}^m(\tilde{y})$ is near for each integer m, because of $A \in W^u_c$. Clearly, $\tilde{x}$ and $\tilde{y}$ are doubly asymptotic under $\tilde{f}$.

THEOREM 4.8. *Let X be a compact metric space and let $f\colon X \longrightarrow X$ be an onto map of X. Then the following are equivalent.*

(1) f has sensitive dependence on initial conditions.

(2) There is a dense and uncountable subset D of X such that f is positively expansive on D.

(3) There is a positive number $c > 0$ such that for any $x \in X$ and any open subset U of X, there is a Cantor subset C of U such that $x \in C$ and f is positively expansive on C with an expansive constant c.

OUTLINE OF PROOF. The proof is similar to those of Theorem 4.5 and Theorem 4.6. We shall prove that (1) implies (2). Let $\tau > 0$ be a positive number as in the definition of sensitive dependence on initial conditions. Put $c = \tau/2$. Let $\{U_\iota\}_{\iota=1}^{\infty}$ be a countable base of X. Suppose that f is positively expansive on a countable set $E = \{p_n\}$ with an expansive constant c. Let i be an any natural number. Choose a point $p \in U_\iota$. If $d\big(f^j(p_1), f^j(p)\big) \leq c$ for each $j \geq 0$, we can choose $p' \in U_\iota$ such that for some $m \geq 0$ such that $d\big(f^m(p), f^m(p')\big) > \tau$. Then $d\big(f^m(p_1), f^m(p')\big) > c$. Choose a closed neighborhood W_1 of p' in U_ι such that $d\big(f^m(W_1), f^m(p_1)\big) > c$. Otherwise, put $p = p'$ and choose a closed neighborhood W_1 of p' in U_ι such that $d\big(f^m(W_1), f^m(p_1)\big) > c$. Next, we choose $p'' \in W_1$ and a closed neighborhood W_2 of p'' in W_1 such that $d\big(f^m(W_2), f^m(p_2)\big) > c$ for some $m \geq 0$. If we continue this procedure, we have a sequence $U_\iota \supset W_1 \supset W_2 \supset \cdots$. Choose a point $q \in \bigcap_{n=1}^{\infty} W_n$. Then f is positively expansive on $E \cup \{q\}$ with an expansive constant c. The remaining proofs are similar to the proofs of Theorem 4.5 and Theorem 4.6, and we omit them.

Note that any homeomorphism of a 0-dimensional compact metric space is always continuum-wise expansive. We have the following theorem.

THEOREM 4.9. *If $g: Z \longrightarrow Z$ is any homeomorphism of a 0-dimensional compact metric space Z, then there exists an indecomposable chainable continuum X containing Z and a continuum-wise expansive homeomorphism $f: X \longrightarrow X$ such that f is an extension of g.*

PROOF. We may assume that Z is a nowhere dense closed subset of the Cantor middle-third set C. It is well-known that there is a homeomorphism g' of C which is an extension of g. Let I be the unit interval $[0, 1]$. Put $I - C = J \cup \bigcup\{J_{\iota_1,\iota_2,\ldots,\iota_n} \mid i_k = 0, 1 \text{ and } n = 1, 2, \ldots\}$, where $J = (1/3, 2/3), J_0 = (1/9, 2/9), J_1 = (7/9, 8/9), J_{0,0} = (1/27, 2/27), J_{0,1} = (7/27, 8/27), \ldots$, is a sequence of mutually disjoint open intervals such that $\operatorname{diam} J_{\iota_1,\iota_2,\ldots,\iota_n} = \operatorname{diam} J_{k_1,k_2,\ldots,k_n}$ for each n and $\operatorname{diam} J_{\iota_1,\iota_2,\ldots,\iota_{n-1}} > \operatorname{diam} J_{\iota_1,\iota_2,\ldots,\iota_{n-1},\iota_n}$. Choose a map $g_J: \mathrm{Cl}(J) \longrightarrow I$ such that g_J is an extension of $g' | \mathrm{Bd}(J)$, $g_J\big(\mathrm{Cl}(J)\big) = I$ and if A is a closed interval in $\mathrm{Cl}(J)$, then $\operatorname{diam} g_J(A) \geq 2 \cdot \operatorname{diam} A$ (see Example 3.5 and Example 4.4). For each $J_{\iota_1\iota_2,\ldots,\iota_n}$ $(n \geq 1)$, we can choose a map $g_{\iota_1,\iota_2,\ldots,\iota_n}: \mathrm{Cl}(J_{\iota_1,\iota_2,\ldots,\iota_n}) \longrightarrow I$ such that $g_{\iota_1,\iota_2,\ldots,\iota_n}$ is an extension of $g' | \mathrm{Bd}(J_{\iota_1,\iota_2,\ldots,\iota_n})$, the image of $g_{\iota_1,\iota_2,\ldots,\iota_n}$ contains some $J_{k_1,k_2,\ldots,k_{n-1}}$, and if A is a closed interval of $J_{\iota_1,\iota_2,\ldots,\iota_n}$, then $\operatorname{diam} g_{\iota_1,\iota_2,\ldots,\iota_n}(A) \geq 2 \cdot \operatorname{diam} A$ (see Example 3.5 and Example 4.4). Also, with careful constructions, we may assume that $\lim_{n\to\infty} \operatorname{diam} g_{\iota_1,\iota_2,\ldots,\iota_n}\big(\mathrm{Cl}(J_{\iota_1,\iota_2,\ldots,\iota_n})\big) = 0$ for any sequence $i_1, i_2, \ldots$. Define a map $G: I \longrightarrow I$ by $G(x) = g_J(x)$ for $x \in \mathrm{Cl}(J)$, $G(x) = g_{\iota_1,\iota_2,\ldots,\iota_n}(x)$ for $x \in \mathrm{Cl}(J_{\iota_1,\iota_2,\ldots,\iota_n})$, and $G(x) = g'(x)$ for $x \in C$. Now, we shall show that G is a positively continuum-wise expansive map. Let A be any closed interval in I. If there is $m \geq 0$ such that $G^m(A)$ contains some $\mathrm{Cl}(J_{\iota_1,\iota_2,\ldots,\iota_n})$, we see that $G^{m+n+1}(A) \supset I$. Otherwise, for any $m \geq 0$, $G^m(A)$ is contained in some $\mathrm{Cl}(J_{\iota_1,\iota_2,\ldots,\iota_n})$, because C is a perfect set. Then we see that $\operatorname{diam} G^n(A) \geq 2^n \cdot \operatorname{diam} A$. This is a contradiction. Hence $G: I \longrightarrow I$ is a positively continuum-wise expansive map. Consider the shift map $f = \tilde{G}: (I, G) \longrightarrow (I, G)$. By

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Proposition 3.1, f is a continuum-wise expansive homeomorphism. Since $G|Z = g$, we have the following commutative diagram:

$$\begin{array}{ccc}
(I,G) & \xrightarrow{f=\tilde{G}} & (I,G) \\
{\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\
(Z,g) & \xrightarrow{\tilde{g}} & (Z,g) \\
{\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p_0} \\
Z & \xrightarrow[g]{} & Z
\end{array}$$

Where $p_0\colon (Z,g) \longrightarrow Z$ is the projection defined by $p_0\big((z_n)_{n=0}^{\infty}\big) = z_0$, and i is the inclusion map. Since g is a homeomorphism, p_0 is also a homeomorphism. Hence $f = \tilde{G}$ is a desired homeomorphism. Also, (I,G) is indecomposable, because for any subinterval A of I, there is a natural number m such that $G^m(A) = I$. This completes the proof.

REMARK 4.10. By using Theorem 4.9, we can prove that there is a chainable continuum X and a continuum-wise expansive homeomorphism f of X such that the topological entropy $h(f) = \infty$. In fact, it is easily seen that there is a homeomorphism g of a Cantor set C such that $h(g) = \infty$ (*e.g.*, see [27, Theorem 7.12]). Theorem 4.9 implies that there is a chainable continuum X containing the Cantor set C, and a continuum-wise expansive homeomorphism f of X which is an extension of g. Hence $h(f) \geq h(g) = \infty$.

REMARK 4.11. If $i\colon C \longrightarrow C$ is the identity map of a Cantor set C, then there is an extension $f\colon X \longrightarrow X$ of i such that X is a chainable continuum and f is a continuum-wise expansive homeomorphism. Then the set $\mathrm{Fix}(f)$ of all fixed points of f is uncountable. Note that if f is an expansive homeomorphism of a compact metric space, then $\mathrm{Fix}(f)$ is finite.

5. **Generalization of Mañé's theorem to continuum-wise expansive homeomorphisms and some properties of positively continuum-wise expansive maps.** In [20], Mañé proved that if $f\colon X \longrightarrow X$ is an expansive homeomorphism of a compact metric space X, then $\dim X < \infty$ and every minimal set of f is 0-dimensional. In this section, we show that this theorem of Mañé concerning expansive homeomorphisms and dimension can be generalized to the case of continuum-wise expansive homeomorphisms. Also, we investigate some properties of positively continuum-wise expansive maps.

Let X be a compact metric space. Then X has dimension $\leq n$, denoted by $\dim X \leq n$, if for every $\gamma > 0$ there is a covering $\mathcal{U}$ of X by open sets with diameter $\leq \gamma$ such that $\mathrm{ord}\,\mathcal{U} \leq n+1$, *i.e.*, every point of X belongs to at most $n+1$ sets of $\mathcal{U}$. Note that for a compact metric space X, $\dim X \leq n$ if and only if for every $\gamma > 0$ there is a covering $\mathcal{F}$ of X by closed sets with diameter $\leq \gamma$ such that $\mathrm{ord}\,\mathcal{F} \leq n+1$. If $\dim X \leq n$ and $\dim X \leq n-1$ is not true, $\dim X = n$. It is known that for a compact metric space X, $\dim X = 0$ if and only if each component of X is a single point.

We refer the reader to [8] for the properties of dimension of separable metric spaces.

Let $f\colon X \longrightarrow X$ be an onto map of a compact metric space X. A closed subset M of X is a *minimal set of* f if M is f-invariant, *i.e.*, $f(M) = M$, and for each proper closed subset C of $M, f(C) \neq C$.

First, we show the following proposition.

PROPOSITION 5.1. *Let $f\colon X \longrightarrow X$ be a homeomorphism of a compact metric space X. Then the following are equivalent.*

(1) f is a continuum-wise expansive homeomorphism.

(2) There is a finite open cover α of X such that if for every bisequence $\{A_n\}_{n=-\infty}^{\infty}$ of members of α, then $\dim\Big(\bigcap_{n=-\infty}^{\infty} f^{-n}\big(\mathrm{Cl}(A_n)\big)\Big) \leq 0$.

(3) There is a finite open cover α such that for each $\gamma > 0$, there is $N > 0$ such that if $A, B \in \alpha$, each component of $f^{-n}\big(\mathrm{Cl}(A)\big) \cap f^{n}\big(\mathrm{Cl}(B)\big)$ has diameter less than γ for each $n \geq N$.

PROOF. First, we shall prove that (1) implies (2). Let $c > 0$ be an expansive constant for f. Choose a finite open cover α such that mesh $\alpha \leq c$. Clearly, α satisfies the condition as in (2). Next, we shall prove that (2) implies (1). Let α be as in (2). Suppose, to the contrary, that f is not continuum-wise expansive. Then there is a nondegenerate subcontinuum A of X such that $f^n(A)$ is contained in some element A_n of α $(-\infty \leq n \leq \infty)$. Hence $\dim \bigcap_{n=-\infty}^{\infty} f^{-n}(A_n) > 0$. This is a contradiction. Clearly, (3) implies (2). Finally, we shall show that (1) implies (3). Let c and δ be as in Proposition 2.2. Choose a finite open cover α such that mesh $\alpha \leq \delta/2$. Let $\gamma > 0$. By Corollary 2.4, there is N as in Corollary 2.4. If $A, B \in \alpha$, C is a subcontinuum of $f^{-n}\big(\mathrm{Cl}(A)\big) \cap f^{n}\big(\mathrm{Cl}(B)\big)$ and $n \geq N$, then diam $C < \gamma$. This implies that (3) is true.

THEOREM 5.2. *If $f\colon X \longrightarrow X$ is a continuum-wise expansive homeomorphism of a compact metric space X, then* $\dim X < \infty$ *and every minimal set of f is* 0*-dimensional.*

PROOF. The proof is similar to one of [20, Theorem], but there are some differences.

First, we shall prove that $\dim X < \infty$. Choose a finite open cover α as in (3) of Proposition 5.1. Let

$$C_{i,j}^{n} = f^{-n}\big(\mathrm{Cl}(A_i)\big) \cap f^{n}\big(\mathrm{Cl}(A_j)\big) \text{ for } A_i, A_j \in \alpha \text{ and } n \geq 1.$$

Let $\gamma > 0$. Choose $N > 0$ as in (3) of Proposition 5.1. Since each component of $C_{i,j}^{N}$ has diameter less than γ, there is a finite closed covering $\beta_{i,j} = \{C_{i,j}^{N,1}, C_{i,j}^{N,2}, \ldots, C_{i,j}^{N,k(i,j,N)}\}$ of $C_{i,j}^{N}$ such that $C_{i,j}^{N,k} \cap C_{i,j}^{N,k'} = \emptyset$ $(k \neq k')$ and mesh $\beta_{i,j} < \gamma$. Consider the closed covering $\beta = \{C \mid C \in \beta_{i,j} \text{ and } A_i, A_j \in \alpha\}$ of X. Then we can easily see that ord $\beta \leq |\alpha|^2$ and mesh $\beta = \max\{\mathrm{diam}\, C \mid C \in \beta\} < \gamma$, where $|\alpha|$ denotes the cardinality of the set α. Therefore $\dim X \leq |\alpha|^2 - 1 < \infty$.

Next, we shall show that every minimal set of f is 0-dimensional. Suppose, on the contrary, that X is a minimal set of f and $\dim X > 0$. We need the following lemma.

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LEMMA 5.3 (*cf.* [20, LEMMA II]). *If there is $A \in W^s$ such that $f^m(A) \cap A \neq \emptyset$ for some integer m, then f has a periodic point.*

PROOF. Take $y \in A \cap f^m(A)$ and let $z = f^{-m}(y)$. Then $z \in A$ and $\lim_{n\to\infty} d\big(f^n(f^m(z)), f^n(z)\big) = 0$. Take a subsequence $\{i(n)\}$ of natural numbers such that $\lim_{n\to\infty} f^{i(n)}(z) = w$. Then $d\big(w, f^m(w)\big) = \lim_{n\to\infty} d\Big(f^{i(n)}(z), f^m\big(f^{i(n)}(z)\big)\Big) = \lim_{n\to\infty} d\Big(f^{i(n)}(z), f^{i(n)}\big(f^m(z)\big)\Big) = 0$. Therefore $w = f^m(w)$.

Choose a nondegenerate subcontinuum A of X such that $A \in W_c^s$ or $A \in W_c^u$ (see Proposition 2.5), where c, ε and δ are as in Corollary 2.4. We may assume that $A \in W_c^s$ and $\operatorname{diam} A = \delta$ (see the proof of Proposition 2.3). Let N be as in Corollary 2.4, where $\gamma = \delta/3$. Now, define

$$\tau = \inf\big\{d\big(f^i(C), f^j(D)\big) \mid C \in W_c^s, D \in W_c^s, d(C,d) \geq \delta/3,$$
$$C \cup D \text{ is contained in some member } E \text{ of } W_c^s, \text{ and } 0 \leq i,j \leq N\big\}.$$

where $d(C,D) = \inf\{d(x,y) \mid x \in C \text{ and } y \in D\}$. Then $\tau > 0$, for otherwise there are $x_n \in C_n \in W_c^s$, $y_n \in D_n \in W_c^s$ and $C_n \cup D_n \subset E_n \in W_c^s$ $(n = 1, 2, \ldots)$ such that $d(x_n, y_n) \geq \delta/3$ and $\lim_{n\to\infty} d\big(f^{i(n)}(x_n), f^{j(n)}(y_n)\big) = 0$ for some $0 \leq i(n) \leq j(n) \leq N$. We may assume that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, $\lim_{n\to\infty} C_n = C$, $\lim_{n\to\infty} D_n = D$, $\lim_{n\to\infty} E_n = E$, and $i(n) = i$, $j(n) = j$ for all $n \geq 0$. Then $f^i(x) = f^j(y)$ and $d(x,y) \geq \delta/3$. Hence $i < j$ and $f^{j-i}(E) \cap E \supset f^{j-i}(D) \cap C \neq \emptyset$. By Theorem 5.3, f has a periodic point. This is a contradiction. Hence $\tau > 0$. Let U be an open set with diameter $\leq \tau/2$ and $U \cap A = \emptyset$. Let $A_0 = A$. Since $A \in W_c^s$ and $\operatorname{diam} A = \delta$, $\operatorname{diam} f^{-N}(A) \geq \delta$. Choose a subcontinuum $E \in W_c^s$ such that $E \subset f^{-N}(A)$ and $\operatorname{diam} E = \delta$ (see Proposition 2.3). Then we have two subcontinua C and D of $f^{-N}(A_0)$ such that $d(C,D) \geq \delta/3$, $\operatorname{diam} C = \operatorname{diam} D = \delta/3$ and $C \cup D \subset E$. By the definition of τ, we may assume that $U \cap \big(\bigcup_{j=0}^N f^j(C)\big) = \emptyset$. Put $A_1 = C$. If we continue this procedure, we have a sequence $A_0, A_1, \ldots,$ of subcontinua such that

(1) $A_n \subset f^{-N}(A_{n-1})$ for each $n = 1, 2, \ldots,$

(2) $U \cap \big(\bigcup_{j=0}^N f^j(A_n)\big) = \emptyset$ for each $n = 1, 2, \ldots.$

Choose a point $p \in \bigcap_{n\geq 0} f^{nN}(A_n)$. Then $f^{-i}(p)$ is not contained in U for each $i = 0, 1, 2, \ldots$. Put $F = \operatorname{Cl}\{f^{-i}(p) \mid i = 0, 1, 2, \ldots\}$. Then $f^{-1}(F) \subset F$. Choose a minimal element F_0 of $\{A \in 2^F \mid f^{-1}(A) \subset A\}$. Then $f^{-1}(F_0) = F_0 \subset F \subset X - U$. This implies that X is not a minimal set of f. This completes the proof.

Similarly, we have the following theorem.

THEOREM 5.3. *If $f: X \to X$ is a positively continuum-wise expansive map of a compact metric space X, then* $\dim X < \infty$ *and every minimal set of f is* 0*-dimensional.*

To prove Theorem 5.3, we need the following Lemma 5.4, Lemma 5.5 and Lemma 5.6, whose proofs parallel those of Proposition 2.2, Proposition 2.3 and Corollary 2.4. We omit the proofs.

LEMMA 5.4. *Let $f: X \longrightarrow X$ be a positively continuum-wise expansive map of a compact metric space X with an expansive constant $c > 0$. Let $0 < 2\varepsilon \leq c$. Then there is a $\delta > 0$ such that if $A \in C(X)$, $\operatorname{diam} A \leq \delta$ and for some $n > 0$, $\varepsilon \leq \sup\{\operatorname{diam} f^j(A) \mid j = 0, 1, \ldots, n\} \leq 2\varepsilon$, then $\operatorname{diam} f^n(A) \geq \delta$.*

LEMMA 5.5. *Let f, c, ε and δ be as in Lemma 5.4. If A is any nondegenerate subcontinuum of X such that $\operatorname{diam} A \leq \delta$ and $\operatorname{diam} f^m(A) \geq \varepsilon$ for some $m > 0$, then $\operatorname{diam} f^n(A) \geq \delta$ for each $n \geq m$. More precisely, there is a subcontinuum B of A such that $\sup\{\operatorname{diam} f^j(B) \mid j = 0, 1, \ldots, n\} \leq \varepsilon$ and $\operatorname{diam} f^n(B) = \delta$.*

By using Lemma 5.4 and Lemma 5.5, we have

LEMMA 5.6. *Let f, c, ε and δ be as in Lemma 5.4. Then for any $\gamma > 0$ there is $N > 0$ such that if $A \in C(X)$ and $\operatorname{diam} A \geq \gamma$, then $\operatorname{diam} f^n(A) \geq \delta$ for all $n \geq N$.*

In general, it is not true that if $g: Y \longrightarrow Y$ is an onto map of a compact metric space Y and $\dim Y > 0$, then $\dim(Y, g) > 0$. For example, let $g: Y \longrightarrow Y$ be a map as in Example 3.3. Then g is an onto map of Y, $\dim Y > 0$, but $\dim(Y, g) = 0$.

However, we have the following proposition.

PROPOSITION 5.7. *If $f: X \longrightarrow X$ is a positively continuum-wise expansive map of a compact metric space X and $\dim X > 0$, then $\dim(X, f) > 0$, where (X, f) is the inverse limit space of f.*

PROOF. Choose a nondegenerate subcontinuum A of X. Consider the set $S = \prod_{i=0}^{\infty} X_i$, where $X_i = X$. Let $(y_i) \in S$. For each $n \geq 0$ consider the following subset of S:

$$A(f, n) = \{(x_i) \in S \mid x_n \in A, x_j = f(x_{j+1}) \text{ for } 0 \leq j \leq n-1 \text{ and } x_i = y_i \text{ for } i \geq n+1\}.$$

Then $A(f, n)$ is a subcontinuum of S. Since $C(S)$ is a compact metric space, there is a sequence $n_1 < n_2 < \cdots$ of natural numbers such that $\lim_{i\to\infty} A(f, n_i) = C$. Then $C \subset (X, f)$. By Lemma 5.6, we can easily see that there is a positive number $\lambda > 0$ such that $\operatorname{diam} A(f, n_i) \geq \lambda$ for almost all i. Then $\operatorname{diam} C \geq \lambda$, and hence C is a nondegenerate subcontinuum of (X, f). Therefore $\dim(X, f) > 0$.

OUTLINE OF PROOF OF THEOREM 5.3. First, we shall prove that $\dim X < \infty$. Choose a finite open cover α of X such that $\operatorname{mesh} \alpha \leq \delta/2$, where δ is as in Lemma 5.6. Let $C_{i,j}^n = \operatorname{Cl}(A_i) \cap f^{-n}(\operatorname{Cl}(A_j))$ for $A_i, A_j \in \alpha$. By Lemma 5.6, for any $\gamma > 0$ there is $N > 0$ such that each component of $C_{i,j}^N$ has diameter less than γ. As in the proof of Theorem 5.2, we can prove that $\dim X < \infty$.

Next, we shall show that every minimal set of f is 0-dimensional. Let M be a minimal set of f and let $f|M$ be the restriction of f to M. Consider the inverse limit space (X, f) and the shift map $\tilde{f}$ of f. Note that $\tilde{f}: (X, f) \longrightarrow (X, f)$ is a continuum-wise expansive homeomorphism (see Proposition 3.1). Then $(M, f|M)$ is also a minimal set of $\tilde{f}$, because $\tilde{f}\big((M, f|M)\big) = (M, f|M)$ and the fact that if C is a closed $\tilde{f}$-invariant subset of (X, f), then $C = \big(p_n(C), f|p_n(C)\big)$, where $p_n: (X, f) \longrightarrow X$ is the projection. By Theorem 5.2, $\dim(M, f|M) = 0$. By Proposition 5.7, we can see that $\dim M = 0$.

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Also, we have the following results (*cf.* Theorem 4.1, Theorem 4.5, Theorem 4.6 and Theorem 4.9). The proofs are similar to those of Theorem 4.1, Theorem 4.5, Theorem 4.6 and Theorem 4.9. We omit the proofs.

THEOREM 5.8. *If $f: X \longrightarrow X$ is a positively continuum-wise expansive map of a compact metric space X with $\dim X > 0$, then the topological entropy $h(f)$ of f is positive.*

THEOREM 5.9. *Let $f: X \longrightarrow X$ be an onto map of a compact metric space X. Then the following are equivalent.*

(1) f is positively continuum-wise expansive.

(2) There is a positive number τ such that if Y is any nondegenerate subcontinuum of X, then there is a dense and uncountable subset D of Y such that f is positively expansive on D with an expansive constant $\tau > 0$.

(3) There is a positive number τ such that if Y is any nondegenerate subcontinuum of X, then there is a Cantor subset C of Y such that f is positively expansive on C with an expansive constant $\tau > 0$.

THEOREM 5.10. *If $g: Z \longrightarrow Z$ is any map of a 0-dimensional compact metric space Z, then there is a positively continuum-wise expansive map $f: I \longrightarrow I$ such that f is an extension of g, where $I = [0, 1]$.*

By Lemma 5.6 and a similar way as in Theorem 4.5, we can strengthen (1) of Proposition 3.8 as follows.

COROLLARY 5.11. *If $f: X \longrightarrow X$ is a positively continuum-wise expansive map of a continuum X, then there is $\tau > 0$ such that if $x \in X$ and U is any open set with $x \in U$, then there are some point $y \in U$ and a natural number $k > 0$ such that $d\big(f^{kn}(x), f^{kn}(y)\big) > \tau$ for all $n = 1, 2, \ldots$.*

REMARK 5.12. In the statements of Theorem 5.2 and Lemma 5.3, we can not replace the assumption that f is continuum-wise expansive, by the assumption that f has sensitive dependence on initial conditions. Let $1: I^\infty \longrightarrow I^\infty$ be the identity map of the Hilbert cube I^∞ and let $f_2: I \longrightarrow I$ be as in Example 3.5. Put $g = 1 \times f_2: I^\infty \times I \longrightarrow I^\infty \times I$. Then the shift map $\tilde{g}$ of g is a homeomorphism and $\tilde{g}$ has sensitive dependence on initial conditions, but $\dim(I^\infty \times I, g) = \infty$. Let S^1 be the unit circle and let r_α denote the rotation of length $2\pi\alpha$ on S^1. Define a map $f: S^1 \times I \longrightarrow S^1 \times I$ by $f(x, t) = \big(r_{t\alpha}(x), t\big)$ for $x \in S^1$ and $t \in I$, where α is an irrational number. Then f is a homeomorphism and f has sensitive dependence on initial conditions. Note that $M = S^1 \times \{1\}$ is a minimal set of f and $\dim M = 1$.

6. **Continuum-wise expansive homeomorphisms and indecomposability.** There are several theorems concerning the existence of expansive homeomorphisms (see the references). In this section, we show that almost all results can be generalized to the case of continuum-wise expansive homeomorphisms.

By Proposition 2.3 and Proposition 5.1, we obtain the following theorems. The proofs are just the same as those of [10], [11] and [16], if we replace expansive homeomorphisms by continuum-wise expansive homeomorphisms.

THEOEREM 6.1 (*cf.* [16, (3.1)]). *If $f: X \longrightarrow X$ is a continuum-wise expansive homeomorphism of a compact metric space X and* $\dim X > 0$, *then there is a closed subset Z of X such that*

(1) each component of Z is nondegenerate,
(2) the space of components of Z is a Cantor set,
(3) the decomposition space of Z into components is upper and lower semicontinuous, and
(4) all components of Z are members of W^s or W^u.

THEOREM 6.2 (*cf.* [11]). *There are no Peano continua in the plane admitting continuum-wise expansive homeomorphisms.*

THEOREM 6.3 (*cf.* [10]). *Let X be a Peano continuum which contains a 1-dimensional* AR *neighborhood. Then there is no continuum-wise expansive homeomorphism on X.*

REMARK 6.4. In Example 4.4, we showed that there are many chainable continua admitting continuum-wise expansive homeomorphisms. Note that each chainable continuum can be embedded into the plan and it is acyclic.

Next, we will consider continuum-wise expansive homeomorphisms on continua which are not locally connected. We need the following notation. By a *refinement* of a finite collection $\mathcal{U}$ of subsets of a space X, we mean, as usual, any finite collection of subsets of X whose elements are contained in some elements of $\mathcal{U}$. Let $C_1, C_2, \ldots, C_m$ be a sequence of subsets of X. Then the sequence is said to be a *chain* and is denoted by $[C_1, C_2, \ldots, C_m]$, provided that $C_i \cap C_j \neq \emptyset$ if and only if $|i-j| \leq 1$ for each $1 \leq i$, $j \leq m$. A chain $[C_1, C_2, \ldots, C_m]$ is said to be an η-*chain* if $\operatorname{mesh}(\{C_1, C_2, \ldots, C_m\}) = \sup\{\operatorname{diam} C_i \mid 1 \leq i \leq m\} < \eta$. Let $[V_1, V_2, \ldots, V_m]$ be a chain such that $\mathcal{V} = \{V_1, V_2, \ldots, V_m\}$ is a refinement of a finite open cover $\mathcal{U}$ of X. Let U_1 and U_2 be elements of $\mathcal{U}$. Then the chain $[V_1, V_2, \ldots, V_m]$ is said to be *crooked between U_1 and U_2* if there are $1 \leq i(1) < i(2) < i(3) < i(4) \leq m$ such that $V_{i(1)} \subset U_1$, $V_{i(2)} \subset U_2$, $V_{i(3)} \subset U_1$ and $V_{i(4)} \subset U_2$. A chain $[V_1, V_2, \ldots, V_m]$ is said to be *chain from x to y* if $x \in V_1$ and $y \in V_m$.

We need the following key lemma.

LEMMA 6.5 (*cf.* [16, (4.3)]). *Let $f: X \longrightarrow X$ be a continuum-wise expansive homeomorphism of a continuum X. Then there is $\delta_1 > 0$ such that for any $\rho > 0$, there is a natural number $N > 0$ and $\eta > 0$ such that if $\mathcal{U}$ is any finite open cover of X with* $\operatorname{mesh}(\mathcal{U}) < \eta$ *and $[U_1, U_2, \ldots, U_m]$ is a chain of $\mathcal{U}$ with $d(U_1, U_m) \geq \rho$, then one of the following conditions holds:*

(1) $d\big(f^N(U_1), f^N(U_r)\big) \geq \delta_1$ for some $1 < r \leq m$.
(2) $d\big(f^{-N}(U_1), f^{-N}(U_r)\big) \geq \delta_1$ for some $1 < r \leq m$.

PROOF. Let c, ε and δ be as in Proposition 2.2. Let $0 < \delta_1 < \delta/3$. We may assume that $\rho < \delta$. Choose a natural number N such that if $A \in C(X)$ and $\operatorname{diam} A \geq \rho$, then $c < \sup\{\operatorname{diam} f^n(A) \mid |n| \leq N\}$. We shall show that δ_1 and N satisfy the desired condition. Suppose, on the contrary, that there is a sequence $\eta_1 > \eta_2 > \cdots$ of positive numbers with $\lim_{i\to\infty} \eta_i = 0$ for each i and there is a finite open cover $\mathcal{U}_i$ of X with $\operatorname{mesh}(\mathcal{U}_i) < \eta_i$ such that for some chain $[U_{i,1}, U_{i,2}, \ldots, U_{i,m(i)}]$ of $\mathcal{U}_i$, $d(U_{i,1}, U_{i,m(i)}) \geq \rho$ and $d\big(f^N(U_{i,1}), f^N(U_{i,k})\big) < \delta_1$, $d\big(f^{-N}(U_{i,1}), f^{-N}(U_{i,k})\big) < \delta_1$ for each $i = 1, 2, \ldots,$ and $1 \leq k \leq m(i)$. For each i, choose $0 < n(i) \leq m(i)$ such that $d(U_{i,1}, U_{i,k}) \leq \rho$ for each $1 \leq k \leq n(i)$ and $d(U_{i,1}, U_{i,n(i)+1}) > \rho$. Set $A_i = \operatorname{Cl}(\bigcup_{k=1}^{n(i)} U_{i,k})$. Since $2^X = \{C \mid C$ is a nonempty closed subset of $X\}$ is a compact metric space with the Hausdorff metric, we may assume that $\lim_{i\to\infty} A_i = A$. Clearly, A is a subcontinuum of X with $\operatorname{diam} A = \rho$. By the definition of N, we see that $\operatorname{diam} f^N(A) \geq \delta$ or $\operatorname{diam} f^{-N}(A) \geq \delta$ (see Proposition 2.3). On the other hand, $\operatorname{diam} f^N(A) = \operatorname{diam} f^N(\lim_{i\to\infty} A_i) \leq 2\delta_1 < \delta$. Similarly, $\operatorname{diam} f^{-N}(A) < \delta$. This is a contradiction.

By using Lemma 6.5 and the same techniques as the proofs of [16, (4.4) and (4.5)], we can prove the following lemma.

LEMMA 6.6 [*cf.* [16, (4.6)]). *Suppose that X is a continuum and $f: X \to X$ is a continuum-wise expansive homeomorphism of X. Then there exists $\delta_1 > 0$ such that if $x, y \in X$, $x \neq y$, and $\mathcal{U}$ is any finite open cover of X, then there is a natural number $N > 0$ and $\eta > 0$ such that if $[V_1, V_2, \ldots, V_m]$ is an η-chain from x to y, then $[f^N(V_1), f^N(V_2), \ldots, f^N(V_m)]$ or $[f^{-N}(V_1), f^{-N}(V_2), \ldots, f^{-N}(V_m)]$ is a refinement of $\mathcal{U}$ and is crooked between U_s and U_t, where U_s, $U_t \in \mathcal{U}$ and $d(U_s, U_t) \geq \delta_1 - 2 \cdot \operatorname{mesh}(\mathcal{U})$.*

By using Lemma 6.6 and the same manner as in the proofs of [16, (4.1)] and [17, (3.1)], we can prove the following theorems, if we replace expansive homeomorphisms by continuum-wise expansive homeomorphisms. The proofs are just the same.

THEOREM 6.7. *If a tree-like continuum admits a continuum-wise expansive homeomorphism, then it contains an indecomposable subcontinuum.*

Note that there are many indecomposable tree-like continua admitting continuum-wise expansive homeomorphisms (see Example 3.5, Corollary 3.7, Example 4.4 and Theorem 4.9).

COROLLARY 6.8 (*cf.* [14]). *There are no continuum-wise expansive homeomorphisms on dendroids (= arcwise connected tree-like continua).*

THEOREM 6.9. *Suppose that $\mathbb{F}$ is a finite collection of graphs and a continuum X is $\mathbb{F}$-like. If X admits a continuum-wise expansive homeomorphism, then X contains an indecomposable (nondegenerate) subcontinuum.*

COROLLARY 6.10. *Let $\mathbb{F}$ be a finite collection of graphs. If a continuum X is homeomorphic to an inverse limit of an inverse sequence $\{G_n, f_{n+1}\}$ such that each G_n is an element of $\mathbb{F}$, and X admits a continuum-wise expansive homeomorphism, then X contains an indecomposable subcontinuum. In particular, if $f: G \to G$ is an onto map of a graph G and the shift map $\tilde{f}$ of f is a continuum-wise expansive homeomorphism, then the inverse limit (G, f) of f has an indecomposable subcontinuum.*

EXAMPLE 6 11 In the statements of Theorem 6 2, Theorem 6 7 and Corollary 6 8, we can not replace the assumption that f is continuum-wise expansive, by the assumption that f has sensitive dependence on initial conditions In fact, there is a homeomorphism f $S^1 \times I \longrightarrow S^1 \times I$ such that f has sensitive dependence on initial conditions (see Corollary 5 11) Note that $S^1 \times I$ is a Peano continuum in the plane Let $D = \{0, 1\}$ and let $C = \prod_{\infty < n < \infty} D_n$, where $D_n = D$ Let σ $C \longrightarrow C$ be the shift map of C, *i e* , $\sigma\big((a_n)_n\big) = (a_{n \; 1})_n$ Let X be the cone of C, *i e* , $X = (C \times I)/(C \times \{0\})$, which is obtained from $C \times I$ by shrinking $C \times \{0\}$ to a point X is called the *Cantor fan* Define a map f $X \longrightarrow X$ by $f([a, t]) = [\sigma(a), \sqrt{t}]$ for $a \in C$ and $t \in I$ Then f is a homeomorphism and f has sensitive dependence on initial conditions, but X is a dendroid Hence X is a hereditarily decomposable tree-like continuum Note that $h(f) = \log 2 > 0$

The following questions remain open

QUESTION 1 If f $X \longrightarrow X$ is a homeomorphism of a continuum X and the topological entropy $h(f)$ is positive, is X not Suslinian? (Note that there is a hereditarily decomposable continuum X admitting a homeomorphism f of X such that $h(f) > 0$ (see Example 6 11))

QUESTION 2 If f $X \longrightarrow X$ is a continuum-wise expansive homeomorphism of a continuum, then does X contain an indecomposable subcontinuum?

QUESTION 3 What kinds of indecomposable continua admit continuum-wise expansive homeomorphisms? What kinds of plane continua admit continuum-wise expansive homeomorphisms?

QUESTION 4 Does the Menger's universal curve admit a continuum-wise expansive homeomorphism?

QUESTION 5 Is it true that if f $X \longrightarrow X$ is a continuum-wise expansive homeomorphism of a continuum X, then there is a subset D of X such that f is expansive on D and every nondegenerate subcontinuum of X has at least two points of D?

QUESTION 6 Is it true that if an onto map f $X \longrightarrow X$ of a continuum X has sensitive dependence on initial conditions, then $h(f)$ is positive?

QUESTION 7 Suppose that a continuum X admits a homeomorphism f which has sensitive dependence on initial conditions Is X not Suslinian? (Note that there is a hereditarily decomposable tree-like continuum admitting a homeomorphism which has sensitive dependence on the initial conditions of Example 6 11)

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