TRANSVERSAL THEORY AND MATROIDS

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1. Introduction. In this paper I use techniques developed by Mirsky and Perfect (**5**) to generalize the extremely close relationship between transversal theory and the theory of matroids or independence structures. I extend in two directions a fundamental theorem of Rado (**8**) and use the techniques of Mirsky and Perfect to obtain easy proofs of known and unknown results about systems of representatives with repetition.

2. Basic concepts. In this section I review the results used subsequently. Throughout the paper, S will denote a finite set and $\mathbf{A}$ will denote the collection of subsets of S, $\{A_i, i \in I\}$, where I is a finite index set. $|K|$ will denote the cardinality of a set K and I use the notation

$$A(J) = \bigcup_{i \in J} A_i.$$

A subset $Y = \{Y_1, \ldots, Y_t\}$ of distinct element of S is called a *partial transversal* of $\mathbf{A}$ if there exists an injection $g \colon Y \to I$ such that

$$Y_i \in A_{g(Y_i)} \qquad (1 \leqq i \leqq t).$$

The *length* of such a partial transversal is t and a *transversal* of $\mathbf{A}$ is a partial transversal of length $|I|$.

A matroid $\mathbf{M}$ on S is defined as follows. $\mathbf{M}$ is a collection of subsets of S called *independent* sets which satisfies:

(1) $\emptyset \in \mathbf{M}$;

(2) $X \in \mathbf{M}$, $Y \subset X \Rightarrow Y \in \mathbf{M}$;

(3) If $Z \subset S$, then all maximal independent subsets of Z have the same cardinality, called the *rank* of Z and denoted by $r(Z)$.

The *rank of the matroid* is defined to be $r(S)$. We will denote a matroid by $(S, \mathbf{M})$, or when there is no possibility of confusion by $\mathbf{M}$.

Matroids were first introduced by Whitney (**11**) and many equivalent sets of axioms exist; we refer to (**9** or **11**). A *base* of a matroid $\mathbf{M}$ on S is any maximal independent subset of S. A matroid may be defined in terms of its bases as follows: A collection $\mathbf{G}$ of subsets of S is the set of bases of some matroid on S if:

(4) Each member of $\mathbf{G}$ has the same cardinality;

(5) If B_1 and B_2 are members of $\mathbf{G}$, and $e \in B_1$, there exists $f \in B_2$ such that

$$B_3 = B_1 - e + f \in \mathbf{G}.$$

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A fundamental relationship between transversal theory and matroids is given by the following theorem, proved independently by Mirsky and Perfect (**5**), and Edmonds and Fulkerson (**1**).

THEOREM 1. *If* $\mathbf{A}$ *is any collection of subsets of* S, *and* $T(\mathbf{A})$ *denotes the collection of partial transversals of* $\mathbf{A}$, *then* $T(\mathbf{A})$ *is the collection of independent sets of a matroid on* S, *which we call the transversal matroid of* $\mathbf{A}$.

We will give a generalization of this and also of the following result.

RADO'S THEOREM. *If* $(S, \mathbf{M})$ *is a matroid with rank function* r *and* $\mathbf{A}$ *is any collection of subsets of* S, *then* $\mathbf{A}$ *has a transversal which is an independent set in* $\mathbf{M}$ *if and only if*

$$r(A(J)) \geqq |J|$$

for all $J \subset I$.

An extension of this, first noted by Perfect (**7**), is the following.

COROLLARY. *Under the same conditions,* $\mathbf{A}$ *has a partial transversal with rank at least* t *in* $(S, \mathbf{M})$ *if and only if for all* $J \subset I$, *we have:*

$$r(A(J)) \geqq t + |J| - |I|.$$

3. p-transversals. For convenience, let $n = |I|$ and let $\mathbf{p} = (p_1, \ldots, p_n)$ denote any n-vector with non-negative integer coordinates p_i. If $\mathbf{A}$ is any family of subsets of S, we say that a subset X of S is a $\mathbf{p}$-*transversal* of A if and only if we can write

$$X = X_1 \cup \ldots \cup X_n,$$

where

$$X_i \cap X_j = \emptyset \quad (i \neq j), \qquad |X_i| = p_i \quad (i \in I), \qquad X_i \subset A_i.$$

Thus a $(1, 1, \ldots, 1)$-transversal is a transversal in the usual sense. Using an idea of Halmos and Vaughan (**3**), we prove the following result.

THEOREM 3. *For fixed* $\mathbf{p}$, *the collection of* $\mathbf{p}$-*transversals of* $\mathbf{A}$ *form the bases of a matroid on* S.

Proof. Let $\mathbf{A}^*$ be the collection of subsets of S consisting of p_1 copies of A_1, p_2 copies of $A_2, \ldots, p_n$ copies of A_n. It is easy to see that X is a $\mathbf{p}$-transversal of $\mathbf{A}$ if and only if X is a transversal of $\mathbf{A}^*$. Hence, by Theorem 1, the $\mathbf{p}$-transversals of $\mathbf{A}$ are the bases of a matroid on S.

Similarly, we may extend Rado's theorem as follows.

THEOREM 4. *If* $(S, \mathbf{M})$ *is a matroid and* $\mathbf{A}$ *is any collection of subsets of* S, *then* $\mathbf{A}$ *has a* $\mathbf{p}$-*transversal which is independent in* $\mathbf{M}$ *if and only if for all* $J \subset I$, *we have*

$$r(A(J)) \geqq \sum_{i \in J} p_i.$$

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Proof. Let $\mathbf{A}^*$ be the collection of subsets of S defined in the proof of Theorem 3. Then $\mathbf{A}$ has an independent $\mathbf{p}$-transversal if and only if $\mathbf{A}^*$ has an independent transversal, and the result follows from Rado's theorem.

4. k-transversals. If $Z = \{z_1, z_2, \ldots, z_n\}$ is any collection of not necessarily distinct elements of S and $e \in S$, we denote the number of times that e occurs in Z by $f(e; Z)$.

If $\mathbf{A}$ is any collection of subsets of S and k is any non-negative integer, then we say that the subset $X = \{x_1, \ldots, x_\alpha\}$ of distinct elements of S is a *k-transversal* of $\mathbf{A}$ if there exists a set $Y = \{Y_i, i \in I\}$ of not necessarily distinct elements such that $Y_i \in X$ and

$$Y_i \in A_i \quad (i \in I), \qquad 1 \leqq f(x_i; Y) \leqq k \qquad (1 \leqq i \leqq \alpha).$$

Thus, a 1-transversal is a transversal in the usual sense. k-transversals are more usually described as systems of representatives with repetition; see (**2** or **6**).

In general, the collection of k-transversals of a family $\mathbf{A}$ of subsets of S does not form the collection of bases of a matroid. However, we generalize Rado's theorem to obtain the following result.

THEOREM 5. *If $(S, \mathbf{M})$ is a matroid with rank function r, then $\mathbf{A}$ has a k-transversal with rank not less than t if and only if*

$$k|A(J)| \geqq |J|, \tag{6}$$

$$r(A(J)) \geqq |J| + t - |I|, \tag{7}$$

for all subsets $J \subset I$.

Before proving the theorem we need a preliminary result in matroid theory. For notational reasons let

$$S = \{e(1), e(2), \ldots, e(m)\}.$$

Define S^k to be the set of km elements,

$$S^k = \{e(1, 1), \ldots, e(1, k), e(2, 1), \ldots, e(2, k), \ldots, e(m, 1), \ldots, e(m, k)\}$$

and let $g: S^k \to S$ be the natural map

$$g(e(i, j)) = e(i).$$

If now $(S, \mathbf{M})$ is a matroid, let $\mathbf{G}$ be the collection of those subsets of S^k which satisfy the requirement that $X \in \mathbf{G}$ if and only if

(a) $|X| = r(S)$,

(b) $g(X)$ is a base of $(S, \mathbf{M})$.

Then it is easy to prove the following theorem.

THEOREM 6. *$\mathbf{G}$ is the set of bases of a matroid on S^k and we denote this matroid by $(S^k, \mathbf{M}^k)$.*

Proof. From (a), no member of $\mathbf{G}$ properly contains another. Let X and Y be two members of $\mathbf{G}$, say

$$X = e(i_1, j_1), \ldots, e(i_r, j_r) \quad \text{and} \quad Y = e(k_1, l_1), \ldots, e(k_r, l_r).$$

Let $B_1 = g(X)$, $B_2 = g(Y)$, so that B_1 and B_2 are bases of $(S, \mathbf{M})$. Consider

$$X' = X - e(i_t, j_t) \qquad (1 \leqq t \leqq r).$$

Then

$$g(X') = B_1 - e(i_t).$$

However, since B_1 and B_2 are bases of $(S, \mathbf{M})$, there exists u such that

$$B_3 = B_1 - e(i_t) + e(k_u)$$

is also a base of $(S, \mathbf{M})$. Hence

$$X - e(i_t, j_t) + e(k_u, l_u)$$

is a member of $\mathbf{G}$, and therefore $\mathbf{G}$ satisfies (4) and (5), which proves Theorem 6.

Proof of Theorem 5. Define the collection $\mathbf{A}^k = \{A_i^k, i \in I\}$ of subsets of S^k by

$$e(i, j) \in A_u^k \Leftrightarrow e(i) \in A_u \qquad (u \in I).$$

Let r be the rank function of the matroid $(S, \mathbf{M})$ and r_k the rank function of $(S^k, \mathbf{M}^k)$. Suppose that $\mathbf{A}$ has a k-transversal X such that $r(X) \geqq t$. Then $\mathbf{A}^k$ has a transversal Y such that $g(Y) = X$ and $r_k(Y) \geqq t$. However, from the extension of Rado's theorem, this implies that for any $J \subset I$ we have:

$$r_k[A^k(J)] \geqq |J| + t - |I|.$$

Hence

$$r[A(J)] \geqq |J| + t - |I|.$$

The necessity of the condition (6) follows from a theorem of Rado (**2**, Theorem 3.2).

Conversely, if (7) holds, then for any $J \subset I$ we have:

$$r_k[A^k(J)] \geqq |J| + t - |I|.$$

Hence $\mathbf{A}^k$ has a partial transversal Y of rank at least t in $(S^k, \mathbf{M}^k)$. However by (6), and (**2**, 3.2), $\mathbf{A}^k$ has a transversal and since any partial transversal of a family of sets may be augmented to a transversal (if it exists) (**5**), there exists $Z \subset S^k$ such that

(8) $$Z \text{ is a transversal of } \mathbf{A}^k,$$

(9) $$Y \subset Z,$$

and therefore

(10) $$r_k(Z) \geqq t.$$

Hence $g(Z)$ is a k-transversal of $\mathbf{A}$ and $r[g(Z)] \geqq t$, which proves the theorem.

5. Some applications. I now show how Theorems 4 and 5 may be used to give simple proofs of various theorems about **p**-transversals and k-transversals. The basic technique is to construct a "useful" matroid on S, find its rank function, and then apply the above theorems.

Example 1. Let **M** be the matroid of S in which every subset is independent. Then for any $X \subset S$ we have:

$$r(X) = |X|.$$

Hence using this matroid in Theorems 4 and 5, we obtain the following results.

THEOREM 7. *A has a* **p***-transversal if and only if for all $J \subset I$ we have:*

$$|A(J)| \geqq \sum_{i \in J} p_i.$$

THEOREM 8. *A has a k-transversal of cardinality at least t if and only if for all $J \subset I$ we have:*

$$k|A(J)| \geqq |J|, \tag{11}$$

$$|A(J)| \geqq |J| + t - |I|. \tag{12}$$

Example 2. Let U be any subset of S and define $(S, \mathbf{M})$ to be a matroid having the single base U. Verifying that this is a matroid is trivial, and if $X \subset S$, the rank of X is

$$r(X) = |X \cap U|.$$

Now the collection **A** of subsets of S has a **p**-transversal which contains U if and only if **A** has a **p**-transversal whose rank in **M** is not less than $|U|$. Hence, from Theorem 4 we obtain the following result.

THEOREM 9. **A** *has a* **p***-transversal which contains the subset $U \subset S$ if and only if for all $J \subset I$, we have:*

$$|A(J) \cap U| \geqq \sum_{i \in J} p_i.$$

For k-transversals we obtain in the same way.

THEOREM 10. **A** *has a k-transversal which contains the subset $U \subset S$ if and only if for all $J \subset I$, we have:*

$$k|A(J)| \geqq |J| \quad \textit{and} \quad |A(J) \cap U| \geqq |J| + |U| - |I|.$$

Theorems 9 and 10 are generalizations of a well-known theorem of Hoffman and Kuhn (**4**).

Example 3. Let $\{E_i, i \in D\}$ be a partition of S into disjoint subsets, and let $\{a_i, i \in D\}$ be given non-negative integers. Define $(S, \mathbf{M})$ to be the matroid having as bases those subsets Z of S which satisfy

$$|Z \cap E_i| = a_i \qquad (i \in D).$$

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It is easy to verify that this is a matroid and the rank of X in $(S, \mathbf{M})$ is given by

$$r(X) = |X| - \sum_{D} (|X \cap E_i| - a_i)^+,$$

where $(x)^+$ denotes $\max\{x, 0\}$. Now $\mathbf{A}$ has a $\mathbf{p}$-transversal X which satisfies

$$|X \cap E_i| \leqq a_i \qquad (i \in D)$$

if and only if $\mathbf{A}$ has a $\mathbf{p}$-transversal which is an independent set in $(S, \mathbf{M})$. Hence by Theorem 4 we obtain the following result.

THEOREM 11. *If $\{E_i, i \in D\}$ is any partition of S into disjoint subsets, then $\mathbf{A}$ has a p-transversal X such that*

$$|X \cap E_i| \leqq a_i,$$

where $\{a_i, i \in D\}$ are prescribed integers if and only if for all subsets J of I we have:

$$|A(J)| - \sum_{i \in D} (E_i \cap A(J) - a_i)^+ \geqq \sum_{i \in J} p_i.$$

This is a generalization of a theorem in (**4**) for ordinary transversals.

Using Theorem 3 we use a technique used by Mirsky and Perfect (**5**) for finding conditions for two families of sets to have a common transversal, to obtain a generalization of a well-known result of Ford and Fulkerson (**2**).

THEOREM 12. *Let $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ be n-vectors such that*

$$\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = N.$$

Then if $\mathbf{A}$ and $\mathbf{B}$ are any two families of n subsets of S, $\mathbf{A}$ has a $\mathbf{p}$-transversal which is also a $\mathbf{q}$-transversal of $\mathbf{B}$ if and only if for any subsets J, K of $(1, \ldots, n)$, we have:

$$|A(J) \cap B(K)| \geqq \sum_{J} p_i + \sum_{K} q_i - N.$$

Proof. By Theorem 3, the set of $\mathbf{p}$-transversals of $\mathbf{A}$ form the bases of a matroid on S, we denote this by $[S, \mathbf{M}(A)]$. Hence, if r_A denotes the rank function of $\mathbf{M}(A)$, $\mathbf{B}$ has a $\mathbf{q}$-transversal which is also a $\mathbf{p}$-transversal of $\mathbf{A}$ if and only if X is a base of $\mathbf{M}(A)$. The condition for this is that for any $K \subset \{1, \ldots, n\}$, we have:

$$r_A(B(K)) \geqq \sum_{K} q_i. \tag{13}$$

Now if X is any subset of S, $r_A(X) \geqq t$ if and only if the collection of subsets consisting of p_1 copies of $A_1 \cap X$, p_2 copies of $A_2 \cap X$, $\ldots$, p_n copies of $A_n \cap X$, has a partial transversal of length at least t. A simple deduction from the defect version of Hall's theorem (**2**, Theorem 3.4) shows that necessary and sufficient conditions for this are that for any subset $J \subset \{1, \ldots, n\}$, we have:

$$|X \cap A(J)| \geqq \sum_{i \in J} p_i - N + t. \tag{14}$$

Hence

$$r_A(B(K)) \geqq \sum_{i \in K} q_i$$

if and only if

$$B(K) \cap A(J) \geqq \sum_{i \in J} p_i + \sum_{i \in K} q_i - N$$

for any subsets J, K of $\{1, \ldots, n\}$, which proves the theorem.

By a similar method we may prove the following result.

THEOREM 13. *If* $\mathbf{A} = \{A_1, \ldots, A_n\}$ *and* $\mathbf{B} = \{B_1, \ldots, B_m\}$ *are two collections of subsets of* S, *then if* $\mathbf{p} = (p_1, \ldots, p_n)$ *is any* n-*vector, there exists a subset* X *of* S *which is a* $\mathbf{p}$-*transversal of* $\mathbf{A}$ *and a* k-*transversal of* $\mathbf{B}$ *if and only if for any* $J \subset \{1, \ldots, n\}$ *and* $K \subset \{1, \ldots, m\}$, *we have:*

$$k|B(K)| \geqq |K|, \tag{15}$$

$$|A(J) \cap B(K)| \geqq \sum_{i \in J} p_i + |K| - m. \tag{16}$$

Proof. Let $[S, \mathbf{M}(A)]$ be the $\mathbf{p}$-transversal matroid of A, as in the proof of Theorem 12. Then $\mathbf{B}$ has a k-transversal X which is also a $\mathbf{p}$-transversal of A if and only if $\mathbf{B}$ has a k-transversal which is of rank $\sum p_i$ in $[S, \mathbf{M}(A)]$. By Theorem 5, the necessary and sufficient conditions for this are that (1) holds and also

$$r_A[B(K)] \geqq |K| + \sum_{i=1}^{n} p_i - m \qquad (K \subset (1, \ldots, m)).$$

By (14), this is equivalent to

$$\begin{aligned}|B(K) \cap A(J)| &\geqq \sum_{i \in J} p_i - \sum_{i=1}^{n} p_i + |K| + \sum_{i=1}^{n} p_i - m \\ &= \sum_{i \in J} p_i + |K| - m\end{aligned}$$

for any $J \subset \{1, \ldots, n\}$, $K \subset \{1, \ldots, m\}$.

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