A mechanism for solving equations of the n^{th} degree.

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§1.

The proposed mechanism explained here is based on the geometrical properties of Fig. 1.

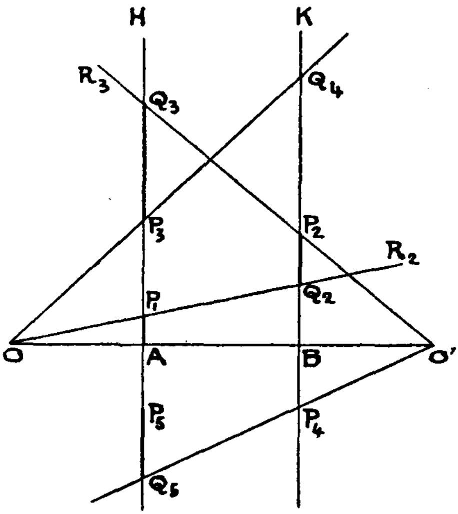

Fig. 1.

OABO′ is a straight line having $OA = BO' = 1$, and $OB = AO' = x$ and AH, BK are lines perpendicular to AB.

We take $AP_1 = a$ on AH, and draw OP_1Q_2 to meet BK in Q_2.

Then take $Q_2P_2 = b$ and draw $O'P_2Q_3$ to meet AH in Q_3

„ „ $Q_3P_3 = c$ „ „ OP_3Q_4 „ „ BK „ Q_4

„ „ $Q_4P_4 = d$ „ „ $O'P_4Q_5$ „ „ AH „ Q_5

„ „ $Q_5P_5 = e$, and so on.

(The figure corresponds to the case when c and d are negative.)

It follows at once that

$$BQ_2 = ax,\ BP_2 = ax + b,\ AQ_3 = ax^2 + bx,\ AP_3 = ax^2 + bx + c,$$

$$BQ_4 = ax^3 + bx^2 + cx,\ BP_4 = ax^3 + bx^2 + cx + d,\ AQ_5 = ax^4 + bx^3 + cx^2 + dx,$$

$$AP_5 = ax^4 + bx^3 + cx^2 + dx + e.$$

And by continuing in this manner we can get

$$BP_{2n} = ax^{2n-1} + bx^{2n-2} + cx^{2n-3} + \ldots + kx + l$$

$$AP_{2n+1} = ax^{2n} + bx^{2n-1} + cx^{2n-2} + \ldots + kx^2 + lx + m.$$

If now a machine is constructed in which two guides AH and BK have sliding pieces P_3Q_3, P_5Q_5; P_2Q_2, P_4Q_4, etc., which can move along the guides, and if on these sliding pieces there are pivots at P_2Q_2, etc., whose axes are perpendicular to the plane of the figure, and which have guide holes in their projecting parts, through which rods $OP_1Q_2R_2$, $O'P_2Q_3R_3$, etc., can pass, the rods being pivoted at O and O′, and if the distance P_2Q_2, P_3Q_3, etc., between the pivots can be adjusted so as to represent the quantities b, c, etc., and if further, the construction of the machine permits $OB = x$ to be varied at will, without disturbing the geometrical conditions laid down, then we have a mechanism which will give real roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ by changing the value of x until P_5 coincides with A.

In order, however, that the various parts should not interfere with one another's motion, it would be necessary to substitute for each of the guides AH, BA, a set of parallel guides, lying in planes perpendicular to that of Fig. 1, the number of parallel guides being such that none would have more than one sliding piece on it.

The manner in which this might be carried out is indicated diagrammatically in Fig. 2, where A_1, A_2, A_3; B_1, B_2, B_3; O_1, O_2, O_3; and O_1', O_2', O_3' are sets of points lying in lines which are perpendicular to the plane of $O_1A_1B_1O_1'H_1K_1$.

It is not proposed here to give a completed mechanical design, but merely to indicate its nature and essential features. Details might be modified, *e.g.* instead of pivots with guide holes, it might be better to have pins clamped to the sliding pieces, which would move in slots in the hinged rods.

We may suppose the lines L_2R_2, L_3R_3, etc., in Fig. 2 to represent the centre lines of the slots of the rods which are hinged, so as to rotate about the lines $O_1O_2O_3$, $O_1'O_2'O_3'$, into which slots enter the pins projecting from the sliding pieces P_2Q_2, P_3Q_3, etc.

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It is not necessary that OA and BO′ should remain fixed when AB varies; all that is needed is that OB : OA should be variable while OA remains equal to BO′.

It would be somewhat difficult to design a practical machine which could without adjustment at once determine both positive and negative roots, and also roots both greater and less than 1. It might be found advisable to restrict it to positive roots lying between 0 and 1, so that the order of the points OABO′ would always be the same. The other real roots could be got by finding the roots between 0 and 1 of the three related equations:

$$a + bx + cx^2 + dx^3 + ex^4 = 0$$
$$a - bx + cx^2 - dx^3 + ex^4 = 0$$
$$ax^4 - bx^3 + cx^2 - dx + e = 0$$

More generally, if the machine were designed to determine the roots lying in any finite interval, say 1 to 2 or 1 to 10, the other real roots could be found by suitably transforming the equation.

As for imaginary roots, the finding of these can be reduced to the determination of the real roots of a related equation of degree $n(n-1)/2$, n being the degree of the equation to be solved.

§2.

A modification of the construction in Fig. 1, which would apply to the solution of simultaneous equations of the first degree, will now be explained, taking as an example the equations:

$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$
$$a_3x + b_3y + c_3z + d_3 = 0.$$

This is shown in Fig. 3. Here $X_1'X_1Y_1'Y_1Z_1'Z_1O_1$ is a straight line, and X_1X, $X_1'X'$, etc., are straight lines perpendicular to it.

We have $X_1'X_1 = Y_1'Y_1 = Z_1'Z_1 = 1$ and $X_1O_1 = x$, $Y_1O_1 = y$, $Z_1O_1 = z$.

On $X_1'X'$ we lay off $X_1'A_1 = a_1$ and draw A_1X_1 to meet $Y_1'Y'$ in P_1, Y_1Y in y_1, and O_1O in F_1.

On P_1Y' we lay off $P_1B_1 = b_1$ and draw B_1Y_1 to meet $Z_1'Z'$ in Q_1, Z_1Z in z_1, and O_1O in G_1.

On Q_1Z' we lay off $Q_1C_1 = c_1$ and draw C_1z_1 to meet O_1O in R_1.

On R_1O we lay off $R_1D_1 = d_1$.

Then $OF_1 = a_1x$, $F_1G_1 = b_1y$, $G_1R_1 = c_1z$, and $R_1D_1 = d$,

$$\therefore \quad OD_1 = ax_1 + by_1 + cz + d_1.$$

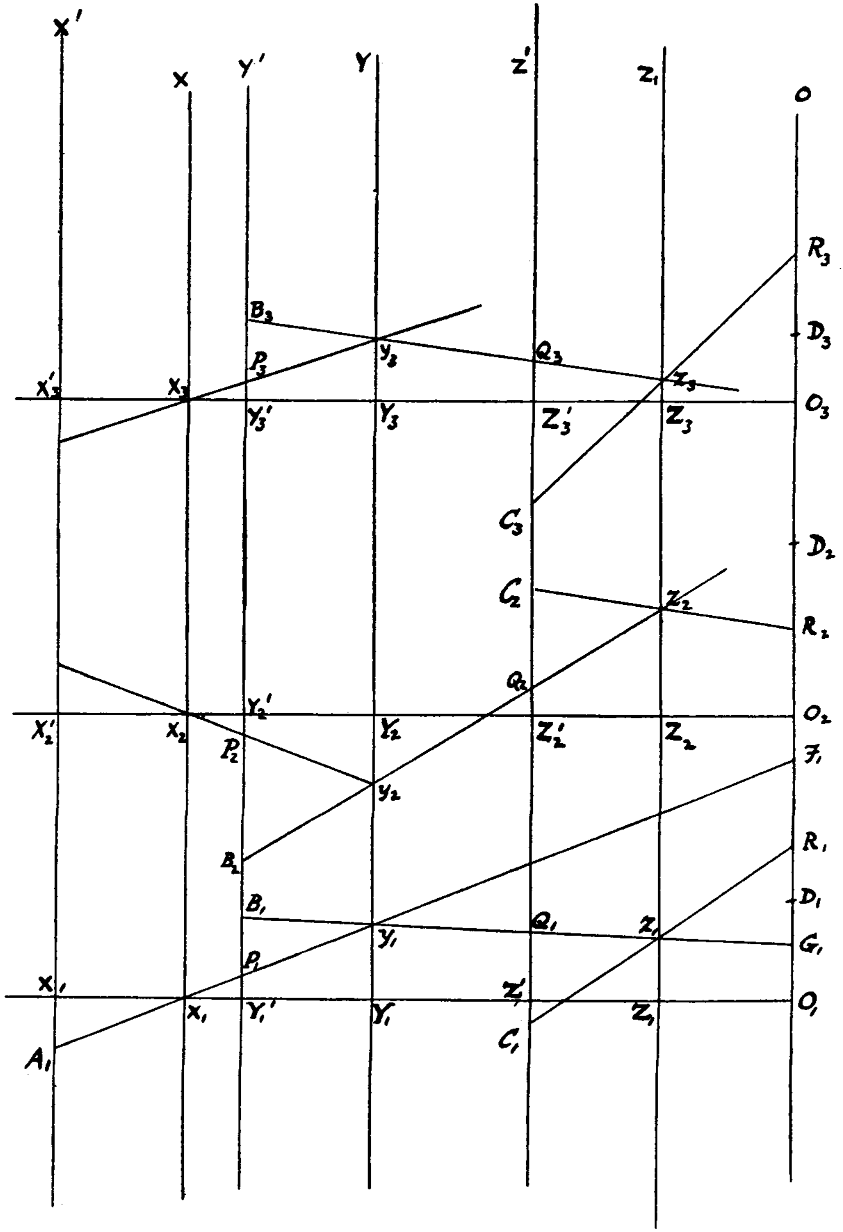

Fig. 3.

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The figure indicates also similar constructions for

$$a_2x + b_2y + c_2z + d_2 \quad \text{and} \quad a_3x + b_3y + c_3z + d_3,$$

showing that these are represented by O_2D_2 and O_3D_3 respectively.

If now x, y and z be successively or simultaneously varied so that D_1, D_2, D_3 coincide with O_1, O_2, O_3 respectively, the values which x, y, z then have will be the solution of the given equations.

To make a practical working machine, the same kind of development would be required as was indicated in the mechanism for solving one equation of the n^{th} degree.

§3. *Simultaneous Equations of Degree higher than the First.*

A further extension of the preceding constructions, which will now be obvious, would enable us to design a mechanism which would solve simultaneous equations of degree higher than the first. Probably a workable machine could be constructed only for the simpler cases.

It may be remarked that in the case of simultaneous equations we take a separate axis XO for each equation, all the axes being parallel to one another, and either in the same plane as OXX′ or in a plane perpendicular to that, the only essential being that the guide-lines belonging to any one unknown should be parallel, and in a plane perpendicular to OX.

On another occasion this subject may be further developed: for the present I merely indicate in Fig. 4 the nature of a mechanism for constructing $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$. Fig. 4, like Fig. 2, shows how the mechanism may be arranged to permit the various rods to move without interfering with one another.

Here X_1X_1' and Y_1Y_1' are of unit length and $O_1X_1 = x$, $O_1Y_1 = y$. $X_1'A_1 = a$, $P_2H_2 = 2h$, $Q_3G_3 = 2g$, $Y_8'B_8 = b$, $S_7F_7 = 2f$, $R_5C_5 = c$.

The linkage $G_3O_3aaaaX_4'G_4$ serves to keep $G_4X_4' = O_3G_3$, and the linkage $F_7O_7\beta\beta\beta\beta Y_6'F_6$ to keep $F_6Y_6' = O_7F_7$. These conditions could probably be maintained by a simpler mechanism, the former, *e.g.*, by having pins on the sliders at G_4 and G_3 entering the slot of a rod pivoted half-way between X′ and O. The linkage $T_5\gamma\gamma\gamma C_5D_5$ serves to keep D_5 half-way between T_5 and C_5, and we have $2O_5D_5 = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$.

In this figure the values of h, g and c are *negative*.

Note.—Dr P. Pinkerton has pointed out to me that a geometrical construction for a polynomial, very similar in principle to that of Fig. 1, is given by Lagrange in his *Leçons Élémentaires sur les Mathématiques* (1795). I find the construction referred to is near the end of the 4th Leçon, and is followed by the suggestion that an instrument might be constructed on that model which would approximately solve equations of all degrees, and could be made to draw the curve of an equation. I also find that this suggestion of Lagrange was anticipated by Rowning, who in a memoir dated 24th March 1768, and published in the *Phil. Trans.* for 1770 (vol. LX.), gives a drawing of an equation-solving machine of his own design, founded on the identical geometrical construction afterwards given by Lagrange. He refers to a paper by J. A. Segner, published in the *Novi Commentarii Acad. Sc. Imp. Petrop.*, tom VII., *pro annis* 1758–9, where the geometrical construction is given; but remarks, "This is a method I myself fell into ten or twelve years ago." It is, I think, somewhat remarkable that Rowning's machine has been so long neglected, as its merits seem much superior to those of many other machines that have been proposed.

The construction given in Fig. 1 of the present paper differs from that of Segner and Lagrange in having lines passing through fixed points instead of parallels.

The developments given here in §2 and §3 have not, so far as I know, been anticipated.

I hope at some future meeting of the Society to be able to exhibit some specimens of the equation-solving machines here described.

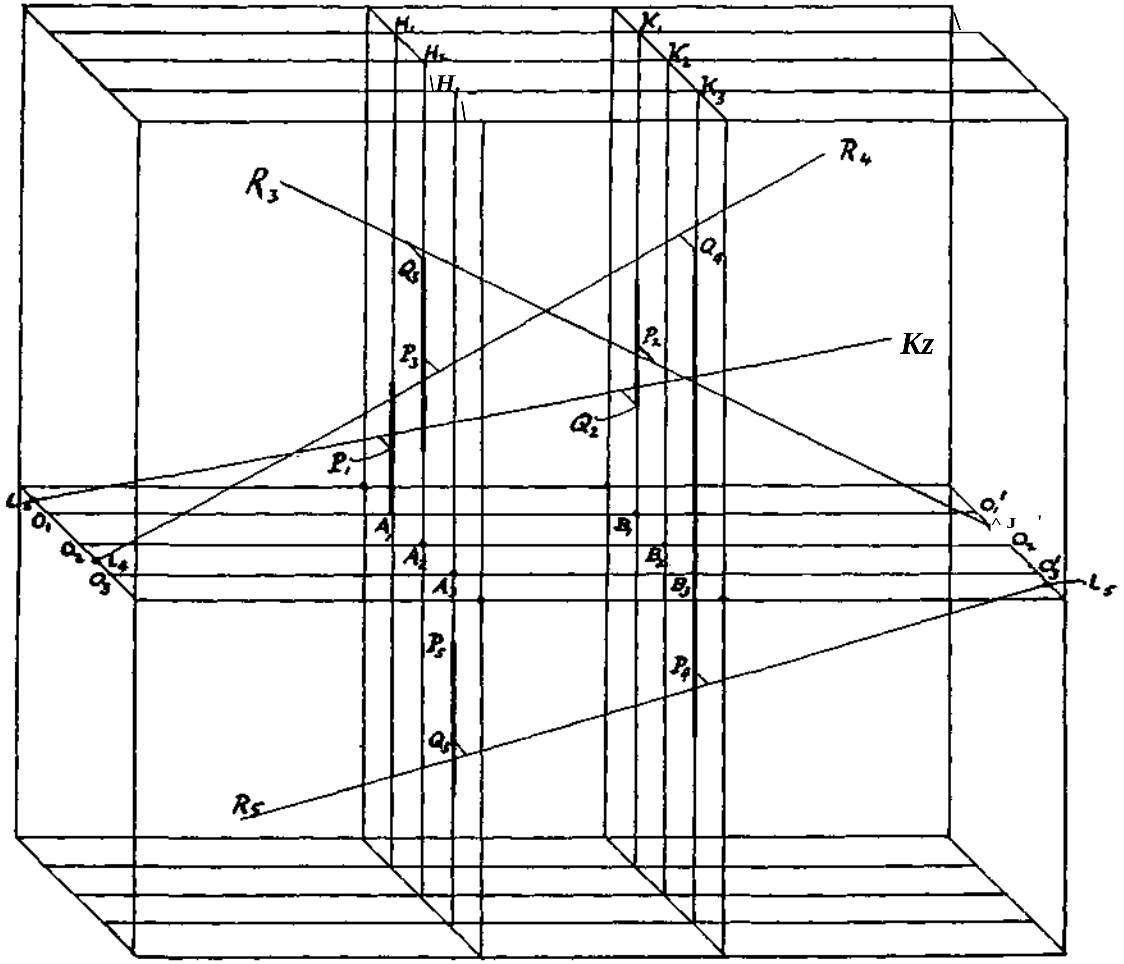

Fig. 2.

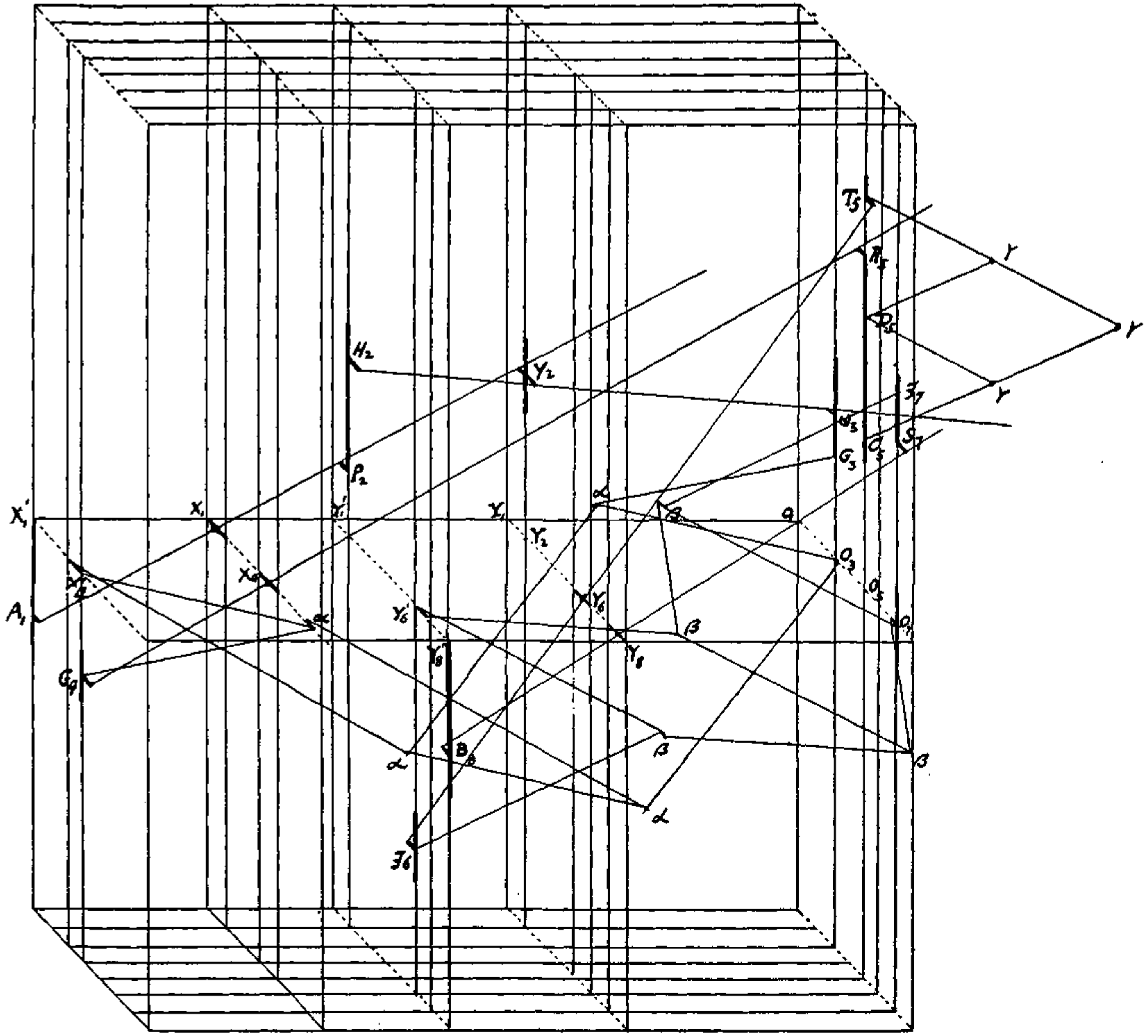

Fig. 4.

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