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THE SIMPLICIAL HELIX AND THE EQUATION $\tan n\theta = n \tan \theta$

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In Memoriam Robert Smith

ABSTRACT. Buckminster Fuller has coined the name *tetrahelix* for a column of regular tetrahedra, each sharing two faces with neighbours, one 'below' and one 'above' [A. H. Boerdijk, Philips Research Reports **7** (1952), p. 309]. Such a column could well be employed in architecture, because it is both strong and attractive. The $(n-1)$-dimensional analogue is based on a skew polygon such that every n consecutive vertices belong to a regular simplex. The generalized twist which shifts this polygon one step along itself is found to have the characteristic equation

$$(\lambda - 1)^2\{(n-1)\lambda^{n-2} + 2(n-2)\lambda^{n-3} + 3(n-3)\lambda^{n-4} + \ldots + (n-2)2\lambda + (n-1)\} = 0,$$

which can be derived from $\tan n\theta = n \tan \theta$ by setting $\lambda = \exp(2\theta i)$.

1. **Summary**. A sequence of regular tetrahedra $A_0A_1A_2A_3$, $A_1A_2A_3A_4$, $A_2A_3A_4A_5$, ..., each having one face in common with the next, forms a kind of twisted pillar which the late Buckminster Fuller called a *tetrahelix* ([1], 412; [4], 518–524). It can be shifted one step along itself by an isometry whose characteristic equation is $(\lambda-1)^2(3\lambda^2+4\lambda+3)=0$. This 'twist' is the product of a translation, given by the factor $(\lambda-1)^2$, and a rotation through an angle (about 131°49′) whose cosine is $-\frac{2}{3}$. The translation is through a distance $10^{-1/2}$ times the edge-length of the tetrahedra (see Boerdijk ([0], p. 309), whose skilful drawing of the tetrahelix was copied by J. D. Bernal and Remy Mosseri).

This is the case $n = 4$ of an arrangement in Euclidean $(n-1)$-space, where regular simplexes form a 'simplicial helix', shifted one step along itself by an isometry whose characteristic equation is

$$(\lambda-1)^2 \sum_{\nu=1}^{n-1} (n-\nu)\nu\lambda^{\nu-1} = 0.$$

The factor $(\lambda-1)^2$ still indicates a translation. When n is odd there is also a factor $\lambda + 1$, indicating a reflection. The 'nontrivial' part of this generalized twist is the

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product of $[n/2] - 1$ commutative rotations whose angles are the values of 2θ that satisfy the surprisingly elegant equation

$$\tan n\theta = n \tan \theta.$$

Similar results were obtained independently by Hurley [6].

2. **Regular skew polygons**. In spherical or Euclidean $(n - 1)$-space, a skew polygon $\ldots A_0A_1A_2 \ldots$ is said to be *regular* if its edges $A_\mu A_{\mu+1}$ are all congruent and its diagonals $A_\mu A_{\mu+\nu}$ are congruent for each $\nu = 2, 3, \ldots, n - 1$. Since the isometry S which relates the two congruent simplexes $A_0A_1 \ldots A_{n-1}$ and $A_1A_2 \ldots A_n$ shifts the polygon one step along itself, the diagonals $A_\mu A_{\mu+\nu}$ are congruent also for each $\nu \geq n$. If A_h coincides with A_0, the polygon is closed (a 'skew h-gon') and we naturally define $A_{h+\nu}$ to be an alternative name for A_ν.

Let us call the regular skew polygon a *simplicial helix* if the simplex $A_0A_1 \ldots A_{n-1}$ is regular, so that the diagonal A_0A_ν is congruent to the edge A_0A_1 for all $\nu < n$. For instance, in spherical 3-space (that is, on a 3-sphere in Euclidean 4-space, so that $n = 4$), if $A_0A_1 = \pi/5$, the simplicial helix is a skew 30-gon: the Petrie polygon of the regular spherical honeycomb $\{3, 3, 5\}$ ([3], pp. 7, 29, 52). In this case the vertex A_{30} coincides with A_0, and $A_0A_1 = A_0A_2 = A_0A_3 = \pi/5$, $A_0A_4 = A_0A_5 = \pi/3$, $A_0A_6 = 2\pi/5$, $A_0A_7 = A_0A_8 = \pi/2$, $A_0A_9 = 3\pi/5$, $A_0A_{10} = A_0A_{11} = 2\pi/3$, $A_0A_{12} = A_0A_{13} = A_0A_{14} = 4\pi/5$, $A_0A_{15} = \pi$, $A_0A_{30-\nu} = A_0A_\nu$. These distances on the 3-sphere are $2 \arcsin(a/2\tau)$ in the notation of *Regular Polytopes* ([2], pp. 238, 298, Table V(iii)). The '5' in the Schläfli symbol $\{3, 3, 5\}$ means that each edge is surrounded by 5 tetrahedra, so that the dihedral angle of each tetrahedron is $2\pi/5$. S is a double rotation through angles $\xi_1 = \pi/15$ and $\xi_2 = 11\pi/15$. These angles were found by setting $p = q = 3$, $r = 5$ in the equation

$$X^4 - (c_p + c_q + c_r)X^2 + c_pc_r = 0 \tag{2.1}$$

where $\quad c_p = \cos^2\pi/p,\ c_q = \cos^2\pi/q,\ c_r = \cos^2\pi/r$

([2], p. 221). The roots of this equation are the values of

$$X = \pm \cos \tfrac{1}{2}\xi_\nu$$

for the Petrie polygon of the regular polytope $\{p, q, r\}$. More generally, for any 'spherical tetrahelix' (with $n = 4$) we can use the same equation

$$X^4 - (\tfrac{1}{2} + c_r)X^2 + \tfrac{1}{4}c_r = 0 \tag{2.2}$$

when the dihedral angle is $2\pi/r$, even if r is irrational.

In the limiting case when $c_r = \frac{2}{3}$, so that the equation becomes

$$(X^2 - 1)(X^2 - \tfrac{1}{6}) = 0,$$

the first factor yields $\xi_1 = 0$. This shows that the first rotation is reduced to a translation, S is an ordinary twist, and we have the Euclidean tetrahelix described at the beginning of §1.

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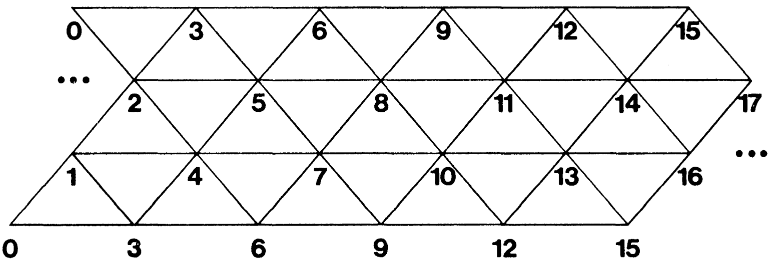

Each tetrahedron shares two faces with its neighbours. The remaining two faces ($A_\nu A_{\nu+1} A_{\nu+3}$ and $A_\nu A_{\nu+2} A_{\nu+3}$) belong to an infinite skew polyhedron consisting of equilateral triangles, six round each vertex. As Fuller remarks, a model can be constructed by copying the above Figure (in which 'A_ν' has been abbreviated to 'ν') on a sheet of carboard, and folding it along all the internal edges: gently up along the edges $A_\nu A_{\nu+1}$, gently down along $A_\nu A_{\nu+2}$, and sharply down along $A_\nu A_{\nu+3}$. Finally, the pairs of external edges A_0A_3, A_3A_6, A_6A_9, ... have to be glued together.

3. **Cartesian coordinates in 3 dimensions**. For a tetrahelix in Euclidean space, S is, as we have seen, a twist: the product of a translation through distance ξ_1 and a rotation through angle ξ_2. Taking the translation to be along the z-axis, and the circumscribed cylinder to have unit radius, we may give A_ν the coordinates

$$(\cos \nu\xi_2,\ \sin \nu\xi_2,\ \nu\xi_1) \qquad (\nu = \ldots -1, 0, 1, 2, \ldots).$$

Then

$$A_\mu A_{\mu+\nu}{}^2 = A_0 A_\nu{}^2 = (1 - \cos \nu\xi_2)^2 + \sin^2 \nu\xi_2 + (\nu\xi_1)^2$$
$$= 2 - 2\cos \nu\xi_2 + \nu^2\xi_1{}^2.$$

Since the edges A_0A_ν all have the same length, for $\nu = 1, 2, 3$, we find

$$2 - 2\cos \xi_2 + \xi_1{}^2 = 2 - 2\cos 2\xi_2 + 4\xi_1{}^2 = 2 - 2\cos 3\xi_2 + 9\xi_1{}^2;$$

so

$$3(\cos 3\xi_2 - \cos \xi_2) = 12\xi_1{}^2 = 8(\cos 2\xi_2 - \cos \xi_2)$$

or, in terms of $x = \cos \xi_2$,

$$3(4x^3 - 3x - x) = 8(2x^2 - 1 - x),$$
$$3x^3 - 4x^2 - x + 2 = 0,$$
$$(x - 1)^2(3x + 2) = 0.$$

Discarding the superfluous root $x = 1$, we deduce that the angle of rotation ξ_2 is given by

$$\cos \xi_2 = -\tfrac{2}{3}, \qquad \xi_2 \approx 131°49',$$

and the translation-distance, given by

$$3\xi_1^2 = 2(2x^2 - x - 1) = 10/9,$$

is $\xi_1 = (10/27)^{1/2}$. Also the edge-length 2ϕ is given by

$$(2\phi)^2 = A_0A_1^2 = 2 - 2\cos\xi_2 + \xi_1^2 = 100/27.$$

Thus the ratio of translation distance to edge-length is

$$\xi_1/2\phi = \sqrt{10}/10 = 1/\sqrt{10}. \tag{3.1}$$

The corresponding ratio in spherical 3-space can be found by solving the equation 2.2 and observing that the edge-length 2ϕ of the spherical honeycomb $\{3, 3, r\}$ is given by the continued fraction

$$\sin^2\phi = 1 - c_p/1 - c_q/1 - c_r$$

([3], p. 35), where $c_p = c_q = \cos^2\pi/3 = \frac{1}{4}$ and $c_r = \cos^2\pi/r$; thus

$$\sin^2\phi = 1 - \tfrac{1}{4}/1 - \tfrac{1}{4}/1 - c_r = (2 - 3c_r)/(3 - 4c_r).$$

Since $\cos^2\frac{1}{2}\xi_1$ is the greater root of 2.2, regarded as a quadratic equation in X^2, we have

$$\begin{aligned}\sin^2\tfrac{1}{2}\xi_1 &= 1 - \tfrac{1}{2}(\tfrac{1}{2} + c_r + \sqrt{\tfrac{1}{4} + c_r^2})\\ &= \tfrac{1}{4}(3 - 2c_r - \sqrt{1 + 4c_r^2})\\ &= (2 - 3c_r)/(3 - 2c_r + \sqrt{1 + 4c_r^2}).\end{aligned}$$

Thus

$$\sin^2\tfrac{1}{2}\xi_1/\sin^2\phi = (3 - 4c_r)/(3 - 2c_r + \sqrt{1 + 4c_r^2}).$$

The Euclidean case arises when c_r tends to $\frac{2}{3}$, so that both ξ_1 and ϕ tend to zero and

$$\lim\,(\xi_1/2\phi)^2 = \lim\,(\sin^2\tfrac{1}{2}\xi_1/\sin^2\phi) = \tfrac{1}{3}/\tfrac{10}{3} = \tfrac{1}{10},$$

in agreement with 3.1.

Although the regular honeycomb $\{3, 3, 5.104\ldots\}$ exists only in a statistical sense ([1], p. 411), there is no need to be surprised about the success of this procedure. In fact, the background for the basic equation 2.1 depends only on the dihedral angles π/p, π/q, π/r of the characteristic orthoscheme for $\{p, q, r\}$.

4. **The case $n = 5$.** Similarly, a simplicial helix in spherical 4-space, composed of 4-simplexes of dihedral angle $2\pi/s$, can be investigated by setting $p = q = r = 3$ in the analogue of 2.1 for the 5-dimensional polytope $\{p, q, r, s\}$, namely

$$X^4 - (c_p + c_q + c_r + c_s)X^2 + (c_pc_r + c_pc_s + c_qc_s) = 0 \tag{4.1}$$

([2], pp. 135, 220). Since $\cos^2\frac{1}{2}\xi_1$ is the greater root of this equation

$$X^4 - (\tfrac{3}{4} + c_s)X^2 + \tfrac{1}{2}(\tfrac{1}{8} + c_s) = 0,$$

regarded as a quadratic in X^2, the amount of the 'spherical translation' ξ_1 is given by

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$$\begin{aligned}\sin^2\tfrac{1}{2}\xi_1 &= 1 - \tfrac{1}{2}(\tfrac{3}{4} + c_s + \tfrac{1}{4}\sqrt{5 - 8c_s + 16c_s^2})\\ &= \tfrac{1}{8}(5 - 4c_s - \sqrt{5 - 8c_s + 16c_s^2})\\ &= (5 - 8c_s)/2(5 - 4c_s + \sqrt{5 - 8c_s + 16c_s^2}).\end{aligned}$$

On the other hand, the edge-length 2ϕ of the spherical honeycomb $\{3, 3, 3, s\}$ is given by

$$\begin{aligned}\sin^2\phi &= 1 - c_p/1 - c_q/1 - c_r/1 - c_s\\ &= 1 - \frac{1}{4}\Big/1 - \frac{1}{4}\Big/1 - \frac{1}{4}\Big/1 - c_s\\ &= 1 - \frac{1}{4}\Big/\frac{2 - 3c_s}{3 - 4c_s} = \frac{5 - 8c_s}{4(2 - 3c_s)}\end{aligned}$$

Thus

$$\sin^2\tfrac{1}{2}\xi_1/\sin^2\phi = 2(2 - 3c_s)/(5 - 4c_s + \sqrt{5 - 8c_s + 16c_s^2}).$$

The simplicial helix in Euclidean 4-space arises when both ξ_1 and ϕ tend to zero, that is, when c_s tends to $\frac{5}{8}$. Thus the ratio of the amount of translation to the edge-length of the simplexes is the square root of

$$\lim\,(\xi_1/2\phi)^2 = \lim(\sin^2\tfrac{1}{2}\xi_1/\sin^2\phi) = \tfrac{1}{4}/5 = 1/20:$$

the ratio itself (analogous to $\frac{1}{2}$ when $n = 3$ and $1/\sqrt{10}$ when $n = 4$) is now $1/2\sqrt{5}$. Hurley [6] finds the general expression to be

$$\binom{n+1}{3}^{-1/2}.$$

5. **The simplicial helix in spherical ($n - 1$)-space**. For the extension to higher spaces we shall find it convenient to use the Chebyshev polynomial $U_{n-1}(X) = \sin n\theta/\sin\theta$, where $X = \cos\theta$, and to express this polynomial as a determinant having $n - 1$ rows:

$$\frac{\sin n\theta}{\sin\theta} = \begin{vmatrix} 2X & 1 & 0 & \dots & 0 & 0\\ 1 & 2X & 1 & \dots & 0 & 0\\ & \dots & & \dots & & \\ 0 & 0 & 0 & \dots & 2X & 1\\ 0 & 0 & 0 & \dots & 1 & 2X \end{vmatrix} = 2^{n-1}\begin{vmatrix} X & \frac{1}{2} & 0 & \dots & 0 & 0\\ \frac{1}{2} & X & \frac{1}{2} & \dots & 0 & 0\\ & \dots & & \dots & & \\ 0 & 0 & 0 & \dots & X & \frac{1}{2}\\ 0 & 0 & 0 & \dots & \frac{1}{2} & X \end{vmatrix}.$$

In fact, the analogue of 2.1 and 4.1 for the Petrie polygon of the general regular polytope $\{p, q, \dots, v, w\}$ ([2], p. 220) is

$$\begin{vmatrix} X & \sqrt{c_p} & 0 & 0 & \dots & 0 & 0 & 0 \\ \sqrt{c_p} & X & \sqrt{c_q} & 0 & \dots & 0 & 0 & 0 \\ 0 & \sqrt{c_q} & X & \sqrt{c_r} & \dots & 0 & 0 & 0 \\ & \dots & & & \dots & & \dots & \\ 0 & 0 & 0 & 0 & \dots & \sqrt{c_v} & X & \sqrt{c_w} \\ 0 & 0 & 0 & 0 & \dots & 0 & \sqrt{c_w} & X \end{vmatrix} = 0.$$

The angles ξ_ν $(\nu = 1, 2, \dots, [\frac{1}{2}n])$ of the component rotations of S are the values of 2θ for which

$$X = \cos\theta.$$

To investigate the simplicial helix formed by regular simplexes with dihedral angle $2\pi/w$ in spherical $(n-1)$-space, we set $p = q = \dots = v = 3$, so that the equation for X becomes

$$0 = 2^{n-1}\begin{vmatrix} X & \frac{1}{2} & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{1}{2} & X & \frac{1}{2} & 0 & \dots & 0 & 0 & 0 \\ 0 & \frac{1}{2} & X & \frac{1}{2} & \dots & 0 & 0 & 0 \\ & \dots & & & \dots & & \dots & \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{2} & X & \sqrt{c_w} \\ 0 & 0 & 0 & 0 & \dots & 0 & \sqrt{c_w} & X \end{vmatrix}$$

$$= X\frac{\sin n\theta}{\sin\theta} - 2c_w\frac{\sin(n-1)\theta}{\sin\theta} = \frac{\cos\theta\sin n\theta - 2c_w\sin(n-1)\theta}{\sin\theta}.$$

This multiplied by $\sin\theta$ is

$$\cos\theta\sin n\theta - 2\cos^2\frac{\pi}{w}(\cos\theta\sin n\theta - \sin\theta\cos n\theta)$$

$$= -\cos\frac{2\pi}{w}\cos\theta\sin n\theta + \left(1 + \cos\frac{2\pi}{w}\right)\sin\theta\cos n\theta$$

$$= -\cos\theta\cos n\theta\left\{\cos\frac{2\pi}{w}\tan n\theta - \left(1 + \cos\frac{2\pi}{w}\right)\tan\theta\right\}.$$

Thus our equation is equivalent to

(5.1) $$\tan n\theta = (1 + \sec 2\pi/w)\tan\theta.$$

When $w = 3$, we are considering the Petrie polygon of the regular simplex α_n, and the equation is $\tan n\theta = -\tan\theta$ or $n\theta = \nu\pi - \theta$. So

$$\xi_\nu = 2\theta = 2\nu\pi/(n+1); \qquad \nu = 1, 2, \dots, [(n+1)/2].$$

When $w = 4$, we are considering the Petrie polygon of the cross polytope (or 'octahedron analogue') β_n, and the equation is $\tan n\theta = \infty$ or $n\theta = (2\nu - 1)\pi/2$. So

$$\xi_\nu = 2\theta = (2\nu - 1)\pi/n; \qquad \nu = 1, 2, \ldots, [(n + 1)/2].$$

6. **The simplicial helix in Euclidean $(n - 1)$-space**. By regarding Euclidean space as a limiting case of spherical space, we can obtain the appropriate value of $2\pi/w$ by making θ tend to 0 in 5.1:

$$n = 1 + \sec 2\pi/w, \quad 2\pi/w = \operatorname{arc\,sec}(n - 1),$$

in agreement with the known dihedral angle $\pi - 2\psi = \operatorname{arc\,sec} n$ for the regular simplex α_n in Euclidean n-space ([2], p. 295). The equation for θ is now simply

$$\tan n\theta = n \tan \theta. \tag{6.1}$$

Setting $\lambda = e^{\xi i} = e^{2\theta i}$, so that

$$\tan \theta = \frac{\lambda^{1/2} - \lambda^{-1/2}}{i(\lambda^{1/2} + \lambda^{-1/2})} = \frac{\lambda - 1}{i(\lambda + 1)}$$

we obtain the characteristic equation for the 'twist' S in the form

$$\frac{\lambda^n - 1}{\lambda^n + 1} = n\frac{\lambda - 1}{\lambda + 1}$$

or

$$(n - 1)(\lambda^{n+1} - 1) - (n + 1)(\lambda^n - \lambda) = 0$$

or

$$(\lambda - 1)\{(n - 1)(\lambda^n + \ldots + \lambda + 1)$$
$$- (n + 1)(\lambda^{n-1} + \ldots + \lambda^2 + \lambda)\} = 0$$

or

$$(\lambda - 1)\{(n - 1)\lambda^n - 2(\lambda^{n-1} + \ldots + \lambda^2 + \lambda) + (n - 1)\} = 0$$

or

$$(\lambda - 1)^2 \sum_{\nu=0}^{n-1} (2\nu - n + 1)\lambda^\nu = 0$$

or

$$(\lambda - 1)^3 \sum_{\nu=1}^{n-1} (n - \nu)\nu\lambda^{\nu-1} = 0.$$

Since S is an isometry in Euclidean $(n - 1)$-space, the characteristic equation should have degree n. In fact, the third factor $\lambda - 1$ has no geometric significance, and we should more properly write

$$(\lambda - 1)^2 \sum_{\nu=1}^{n-1} (n - \nu)\nu\lambda^{\nu-1} = 0. \tag{6.2}$$

When n is odd, there is also a factor $\lambda + 1$ (because S then has a reflection as one

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of its components), and the characteristic equation becomes

$$(6.3) \qquad (\lambda - 1)^2(\lambda + 1) \sum_{\nu=1}^{n-2} \left[\frac{n-\nu}{2}\right]\left[\frac{\nu+1}{2}\right]\lambda^{\nu-1} = 0.$$

For instance, when $n = 3$ we have simply $(\lambda - 1)^2(\lambda + 1) = 0$, and S is a glide-reflection.

When $n = 4$, 6.2 becomes

$$(\lambda - 1)^2(3\lambda^2 + 4\lambda + 3) = 0.$$

and the angle of the twist is ξ where $\cos \xi = \frac{1}{2}(\lambda + \lambda^{-1}) = -\frac{2}{3}$, so that $\xi = 2\mu + \frac{1}{2}\pi \approx 131°49'$ ([1], p. 412), remarkably close to the value $11\pi/15 = 132°$ which occurs in the Petrie polygon for the 600-cell $\{3,3,5\}$ ([2], p. 221).

When $n = 5$, 6.3 becomes

$$(\lambda - 1)^2(\lambda + 1)(2\lambda^2 + \lambda + 2) = 0$$

and S is a 'gliding twist' whose angle ξ is given by $\cos \xi = -\frac{1}{4}$, so that $\xi = \pi - 2\eta \approx 104°29'$.

The coefficients in the non-trivial part of 6.3, with $n = 3, 5, 7, \ldots$, form an amusing variant of Pascal's triangle of binomial coefficients:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 2 & 1 & 2 & & & & & \\
 & & & & 3 & 2 & 4 & 2 & 3 & & & & \\
 & & & 4 & 3 & 6 & 4 & 6 & 3 & 4 & & & \\
 & & 5 & 4 & 8 & 6 & 9 & 6 & 8 & 4 & 5 & & \\
 & 6 & 5 & 10 & 8 & 12 & 9 & 12 & 8 & 10 & 5 & 6 & \\
7 & 6 & 12 & 10 & 15 & 12 & 16 & 12 & 15 & 10 & 12 & 6 & 7 \\
 & & & & \cdots & & & & \cdots & & & &
\end{array}$$

7. **The equation for $t = \tan \theta$.** Since

$$\tan n\theta = \sum_{\nu=0}^{[(n-1)/2]} (-1)^\nu \binom{n}{2\nu+1} \tan^{2\nu+1}\theta \Big/ \sum_{\nu=0}^{[n/2]} (-1)^\nu \binom{n}{2\nu} \tan^{2\nu}\theta$$

([5], p. 53) and

$$n\binom{n}{2\nu} - \binom{n}{2\nu+1} = 2\nu\binom{n+1}{2\nu+1} = 2\nu\binom{n+1}{n-2\nu},$$

we can express 6.1 as an equation for $t = \tan \theta$:

$$t^3 \sum_{\nu=1}^{[n/2]} (-1)^\nu \nu \binom{n+1}{n-2\nu} t^{2\nu-2} = 0.$$

For instance, when $n = 6$, the non-zero roots are given by

$$-\binom{7}{4} + 2\binom{7}{2}t^2 - 3\binom{7}{0}t^4 = 0 \quad \text{or} \quad 3t^4 - 42t^2 + 35 = 0.$$

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Thus $t^2 = 7 \pm 4\sqrt{7/3}$ and

$$\cos 2\theta = (1 - t^2)/(1 + t^2) = (-4 \pm \sqrt{21})/10$$
$$\approx 0.06 \text{ or } -0.86.$$

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