A CHARACTERIZATION OF GROUP RINGS AS A SPECIAL CLASS OF HOPF ALGEBRAS

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By a group ring we mean in this paper a ring defined by a finite group G and an integral domain K :

$$A = KG ,$$

such that A contains G and is freely generated by G over K, so that

$$K\text{-rank of } A = \text{the order of } G .$$

The ring $A = KG$ has a co-multiplication

$$A \xrightarrow{\gamma} A \otimes_K A$$

defined by

$$\gamma\Big(\sum_{x \in G} \alpha_x x\Big) = \sum_{x \in G} \alpha_x (x \otimes x)$$

so that A is a Hopf algebra.

Let $B = \hat{A} = \mathrm{Hom}_K(A, K)$ be the dual K-module of A. Then the co-multiplication γ induces a multiplication $\hat{\gamma}$ in B. It is easy to verify that B, under $\hat{\gamma}$, is a commutative strongly semi-simple K-algebra in the following sense:

$$B = B_1 \oplus \ldots \oplus B_n , \qquad B_i \simeq K$$

as algebras over K, and each homomorphism

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$$\chi_i ; B \longrightarrow B_i \cong K .$$

is represented by an invertible element $x_i \in A$:

$$\chi_i(\hat{a}) = \hat{a}(x_i) .$$

The aim of this paper is to show, conversely, that a Hopf algebra whose co-multiplication is commutative and strongly semi-simple is, in fact, a group ring of a suitable finite group G.

Techniques of the proof are taken from those of the Tannaka duality theorem for compact groups[1], and, in fact, the above characterization can be seen as a dual formulation of this duality theorem.[*]

1. Hopf algebras[2]. Let K be a commutative ring with the identity 1. A K-module A is called a Hopf algebra if there are four K-linear operations

$$\mu : A \otimes_K A \longrightarrow A$$

$$\gamma : A \longrightarrow A \otimes_K A$$

$$\varepsilon : K \longrightarrow A$$

$$\delta : A \longrightarrow K$$

called multiplication, co-multiplication, augmentation and co-augmentation, respectively, such that following diagrams are all commutative:

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1) In particular J. L. Kelley, Duality for compact groups, Proc. N. A. S. 49 (1963) pp. 457-458.

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2) We follow the presentation of S. MacLane, Homology, 1963, pp. 197-198.

$$
\begin{array}{ccc}
A \otimes_K A \otimes_K A & \xrightarrow{1 \otimes \mu} & A \otimes_K A \\
{\scriptstyle \mu \otimes 1} \downarrow & & \downarrow {\scriptstyle \mu} \\
A \otimes_K A & \xrightarrow[\mu]{} & A
\end{array}
\quad , \quad
\begin{array}{ccccc}
K \otimes_K A & = & A & = & A \otimes_K K \\
{\scriptstyle \varepsilon \otimes 1} \downarrow & & \| & & \downarrow {\scriptstyle 1 \otimes \varepsilon} \\
A \otimes_K A & \xrightarrow[\mu]{} & A & \xleftarrow[\mu]{} & A \otimes_K A
\end{array}
\tag{1}
$$

$$
\begin{array}{ccc}
A & \xrightarrow{\gamma} & A \otimes_K A \\
{\scriptstyle \gamma} \downarrow & & \downarrow {\scriptstyle 1 \otimes \gamma} \\
A \otimes_K A & \xrightarrow[\gamma \otimes 1]{} & A \otimes_K A \otimes_K A
\end{array}
\quad , \quad
\begin{array}{ccccc}
A \otimes_K A & \xleftarrow{\gamma} & A & \xrightarrow{\gamma} & A \otimes_K A \\
{\scriptstyle \delta \otimes 1} \downarrow & & \| & & \downarrow {\scriptstyle 1 \otimes \delta} \\
K \otimes_K A & = & A & = & A \otimes_K K
\end{array}
\tag{2}
$$

$$
\begin{array}{ccc}
K & \xrightarrow{\varepsilon} & A \\
\| & & \downarrow {\scriptstyle \gamma} \\
K \otimes_K K & \xrightarrow[\varepsilon \otimes \varepsilon]{} & A \otimes_K A
\end{array}
\quad , \quad
\begin{array}{ccc}
A \otimes_K A & \xrightarrow{\delta \otimes \delta} & K \otimes_K K \\
{\scriptstyle \mu} \downarrow & & \| \\
A & \xrightarrow[\delta]{} & K
\end{array}
\tag{3}
$$

$$
\begin{array}{ccccc}
A \otimes_K A & \xrightarrow{\gamma \otimes \gamma} & A \otimes_K A \otimes_K A \otimes_K A & \xrightarrow{1 \otimes \tau \otimes 1} & A \otimes_K A \otimes_K A \otimes_K A \\
{\scriptstyle \mu} \downarrow & & & & \downarrow {\scriptstyle \mu \otimes \mu} \\
A & & \xrightarrow[\gamma]{\hspace{6em}} & & A \otimes_K A
\end{array}
\quad ,
\tag{4}
$$

where

$$\tau(a_1 \otimes a_2) = a_2 \otimes a_1 .$$

Diagrams (1) say that A is an algebra by the multiplication μ, with the identity:

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$$e = \varepsilon . 1 \in A$$

$$\mu(a \otimes e) = \mu(e \otimes a) = a .$$

Diagrams (2) say that A is a co-algebra by the co-multiplication γ, with the co-identity δ.

Diagrams (3) say that the co-multiplication γ operates on the identity e as $\gamma \cdot e = e \otimes e$, and the multiplication μ operates on the co-identity as $\delta \cdot \mu = \delta \otimes \delta$.

Finally, diagrams (4) say that the multiplication μ is a homomorphism of the co-algebra (A, γ), and the co-multiplication is a homomorphism of the algebra (A, μ).

2. Strong semi-simplicity. Suppose K is an integral domain[3], and A is a finitely generated free K-module. Then

$$B = \hat{A} = \mathrm{Hom}_K(A, K)$$

is also a finitely generated K-module and the co-multiplication

$$\gamma : A \longrightarrow A \otimes_K A$$

induces a multiplication $\hat{\gamma}$ on B:

$$\hat{\gamma}(\hat{a}_1 \otimes \hat{a}_2)(a) = (\hat{a}_1 \otimes \hat{a}_2)(\gamma a), \quad \hat{a}_1, \hat{a}_2 \in B, \qquad a \in A.$$

Further, the co-identity δ defines a map $\hat{\delta} : K \longrightarrow B$

$$\hat{\delta} \cdot \alpha(a) = \alpha \cdot \delta a \in K$$

and $\hat{\delta} \cdot 1 \in B$ is the identity of B.

Suppose B is an absolutely semi-simple commutative K-algebra under $\hat{\gamma}$:

$$B \simeq B_1 \oplus \ldots \oplus B_n , \qquad B_i \simeq K .$$

Then the following conditions are equivalent.

3) Always commutative with the identity 1 .

S_1) For all $\hat{a} \in B = \mathrm{Hom}_K(A, K)$, $\hat{a} \neq 0$, there exists $x \in A$ such that x is μ-invertible,

$$\gamma x = x \otimes x, \qquad \text{and} \qquad \hat{a}(x) \neq 0.$$

S_2) Each $\hat{\gamma}$-homomorphism $\chi_i : B \to B_i \xrightarrow{\sim} K$ is representable by a μ-invertible element $x_i \in A : \chi_i(b) = b(x_i)$.

Proof of $S_1) \Rightarrow S_2)$. If $\gamma x = x \otimes x$, then the map $B \ni b \to \chi(b) = b(x)$ is a $\hat{\gamma}$-homomorphism. In fact,

$$\chi(\hat{\gamma}(b_1 \otimes b_2)) = \hat{\gamma}(b_1 \otimes b_2)(x) = (b_1 \otimes b_2)(\gamma x) =$$

$$(b_1 \otimes b_2)(x \otimes x) = b_1(x) \cdot b_2(x) = \chi(b_1)\, \chi(b_2).$$

Let

$$\hat{\delta} \cdot 1 = e_1 + \ldots + e_n$$

be the decomposition of the identity $\hat{\delta} \cdot 1$ of B into idempotents according to the decomposition

$$B \simeq B_1 \oplus \ldots \oplus B_n, \qquad B_i \xrightarrow{\sim} K.$$

Then for any $\hat{\gamma}$-homomorphism $\chi : B \to K$,

$$1 = \chi(\hat{\delta} \cdot 1) = \chi(e_1) + \ldots + \chi(e_n)$$

$$\chi(\hat{\gamma}(e_i \otimes e_i)) = \chi(e_i)^2 = \chi(e_i)$$

$$\chi(\hat{\gamma}(e_i \otimes e_j)) = \chi(e_i)\, \chi(e_j) = 0, \qquad i \neq j.$$

So there exists one and only one i such that

$$\chi(e_i) = 1, \qquad \chi(e_j) = 0, \qquad j \neq i;$$

i.e., χ coincides with

$$\chi_i : B \to B_i \cong K .$$

Now, by S_1), $e_i \neq 0$ implies existence of a μ-invertible x, with $\gamma x = x \otimes x$, such that $e_i(x) \neq 0$; i.e., the $\hat{\gamma}$-homomorphism χ determined by x has the property:

$$\chi(e_i) = e_i(x) \neq 0 .$$

Since $\chi(e_i)^2 = \chi(e_i^2) = \chi(e_i)$, $\chi(e_i) \neq 0$ implies $\chi(e_i) = 1$ so that $\chi = \chi_i$. In other words, χ_i is represented by a μ-invertible element $x \in A$.

<u>Proof of</u> $S_2) \Rightarrow S_1)$. If $\hat{a} = \sum_i \alpha_i e_i \neq 0$, then there is an i such that $\alpha_i \neq 0$. Now, let x_i be a μ-invertible element in A such that

$$\chi_i(b) = b(x_i) .$$

Then

$$\hat{a}(x_i) = \alpha_i e_i(x_i) = \alpha_i \neq 0 .$$

We can show also that

$$\gamma x_i = x_i \otimes x_i .$$

In fact,

$$(e_j \otimes e_k)(\gamma x_i) = \hat{\gamma}(e_j \otimes e_k)(x_i) = \begin{cases} 1 & j=k=i \\ 0 & \text{all the other} \end{cases}$$

$$(e_j \otimes e_k)(x_i \otimes x_i) = e_j(x_i) e_k(x_i) = \begin{cases} 1 & j=i=k \\ 0 & \text{all the other} \end{cases}$$

i.e.,

$$(e_j \otimes e_k)(\gamma x_i) = (e_j \otimes e_k)(x_i \otimes x_i)$$

and $e_j \otimes e_k$ from a K-free basis of $B \otimes_K B$, so that

$$\gamma x_i = x_i \otimes x_i .$$

3. Main theorem. Let K be an integral domain, and A a finitely generated K-algebra. Then A is the group ring of a finite group G over K, if and only if, A has a co-multiplication, so that it is a Hopf algebra (§ 1), and its dual algebra $\hat{A} = B$ is commutative and strongly semi-simple (§ 2).

Proof of the necessity. Let G be a finite group and A the group ring over K. Then

$$A \ni \sum_{x \in G} \alpha_x x \rightsquigarrow \sum_{x \in G} \alpha_x (x \otimes x) \in A \otimes_K A$$

is a co-multiplication. Let $e \in G$ be the identity; then

$$\varepsilon : K \longrightarrow A$$

is defined by

$$\varepsilon \cdot 1 = e \in G \subset A .$$

Let $d = \sum_{x \in G} x \in A$; then $\delta : A \to K$ is defined by

$$a \cdot d = (\delta a) \cdot d .$$

Consider the dual algebra $\hat{A} = B$. One sees easily that

$$\hat{A} = \mathrm{Hom}_K(A, K) \cong C(G, K) ,$$

where $C(G, K)$ is the set of all K-valued functions over G, by the mapping

$$\hat{A} \ni \hat{a} \rightsquigarrow \hat{a}(x) \in C(G, K) .$$

Consider the dual multiplication $\hat{\gamma}$ on $\hat{A}$:

$$\hat{\gamma}(\hat{a}_1 \otimes \hat{a}_2)(x) = \hat{a}_1 \otimes \hat{a}_2(\gamma x) = \hat{a}_1 \otimes \hat{a}_2(x \otimes x) = \hat{a}_1(x)\hat{a}_2(x) .$$

i.e., the multiplication induced on $C(G, K)$ by $\hat{\gamma}$, under the above isomorphism, is pointwise multiplication:

$$(f \cdot g)(x) = f(x)g(x) , \quad x \in G, \quad f, g \in C(G, K) ,$$

so that

$$B = \hat{A} \cong C(G, K) \simeq B_1 \oplus \ldots \oplus B_n , \quad B_i \simeq K$$

is commutative and absolutely semi-simple. Moreover, each

$$\chi_i : B \to B_i \cong K$$

is given by the homomorphism:

$$C(G, K) \ni f \rightsquigarrow \chi_i(f) = f(x_i) , \qquad x_i \in G$$

i.e., represented by a μ-invertible element $x_i \in A$.

<u>Proof of the sufficiency</u>. Now A is a Hopf algebra whose dual algebra is commutative and strongly semi-simple. Let

$$G = \{ x \in A \mid x \text{ is } \mu\text{-invertible and } \gamma x = x \otimes x \} .$$

We are going to show that G is a finite group and A is the group ring of G over K. We divide the proof into several steps.

I) Let $B = B_1 \oplus \ldots \oplus B_n$, $B_i \simeq K$, and let $\# G$ denote the number of elements in G. Then $\# G = n$.

Proof. By § 2, there are exactly n K-algebra homomorphisms $\chi_i : B \to K$, and each $x \in G$ determines such a homomorphism:

$$\chi(\hat{a}) = \hat{a}(x) ,$$

so $\# G \leq n$. But B is strongly semi-simple, so each $\chi_i : B \to B_i \simeq K$ is represented by a μ-invertible element x_i, which by § 2, satisfies

$$\gamma x_i = x_i \otimes x_i$$

i.e., $x_i \in G$. Hence $\# G = n$.

II) $A = Kx_1 + \ldots + Kx_n , \quad x_i \in G$

as K-modules.

Proof. By hypothesis, $B = Ke_1 + \ldots + Ke_n$ is a free K-module of rank n, and, by I)

$$\chi_i(e_j) = e_j(x_i) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \qquad 1 \leq i, \ j \leq n$$

where χ_i is the $\hat{\gamma}$-homomorphism of B in K, determined by x_i. Hence the K-dual module

$$\hat{B} = \mathrm{Hom}_K(B, K) = K\chi_1 + \ldots + K\chi_n$$

is a free K-module of rank n. Consider the K-homomorphism $\Phi : A \to B$ defined by

$$\Phi\left(\sum_{i=1}^{n} \alpha_i x_i\right) = \sum_{i=1}^{n} \alpha_i \chi_i , \qquad \alpha_i \in K .$$

Φ is well defined, because $\sum_i \alpha_i x_i = 0$ implies

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$$e_j\left(\sum_{i=1}^{n} \alpha_i x_i\right) = \alpha_j = 0 \quad \text{for all } j = 1, 2, \ldots, n .$$

By definition, Φ is onto. But Φ is also injective, in fact,

$$\sum_{i=1}^{n} \alpha_i \chi_i = 0 \quad \text{implies} \quad \sum_i \alpha_i \chi_i(e_j) = \alpha_j = 0$$

for all $j = 1, 2, \ldots, n$. Hence $\Phi : A \xrightarrow{\sim} \hat{B}$ is an isomorphism of K-modules and

$$A = Kx_1 + \ldots + Kx_n .$$

III) The identity $e = \varepsilon . 1$ of A is in G and it is also the identity of G.

<u>Proof</u>. By definition e is μ-invertible, and

$$\gamma e = \gamma \cdot \varepsilon \cdot 1 = (\varepsilon \otimes \varepsilon)(1 \otimes 1) = \varepsilon 1 \otimes \varepsilon 1 = e \otimes e$$

by the diagrams (3) of Hopf algebras (§1). So $e \in G$. By definition, for all $x \in G \subset A$,

$$\mu(e \otimes x) = \mu(x \otimes e) = x .$$

IV) $x \in G$, $y \in G$ imply $x \cdot y = \mu(x \otimes y) \in G$.

<u>Proof</u>. x invertible and y invertible imply $x \cdot y$ invertible. By diagrams of §1

$$(x \cdot y) = \gamma\mu(x \otimes y) = (\mu \otimes \mu)(1 \otimes \tau \otimes 1)(\gamma \otimes \gamma)(x \otimes y)$$

$$= (\mu \otimes \mu)(1 \otimes \tau \otimes 1)(\gamma x \otimes \gamma y)$$

$$= (\mu \otimes \mu)(1 \otimes \tau \otimes 1)(x \otimes x \otimes y \otimes y)$$

$$= (\mu \otimes \mu)(x \otimes y \otimes x \otimes y) = \mu(x \otimes y) \otimes \mu(x \otimes y)$$

$$= x \cdot y \otimes x \cdot y ,$$

so that $x \cdot y \in G$.

V) $x \in G$ implies $x^{-1} \in G$.

<u>Proof</u>. Let $G = \{x_1 = e, x_2, \ldots, x_n\}$. By IV), $x \in G$ and $x_i \in G$ imply $x \cdot x_i \in G$, so there is $j = j(i)$ such that $x \cdot x_i = x_j$. But $x \in G$ is by definition invertible, so $x_i \neq x_j$ implies $x \cdot x_i \neq x \cdot x_j$. Hence there exists $x_i \in G$ such that $x \cdot x_i = e$ and $x^{-1} = x_i \in G$.

This finishes the proof of sufficiency. In fact, by III), IV), V), G is a finite group in A under μ-multiplication, and by II) A is generated freely by G over K.

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