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Symmetric Genuine Spherical Whittaker Functions on $\overline{GSp_{2n}(F)}$

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Abstract. Let F be a p-adic field of odd residual characteristic. Let $\overline{GSp_{2n}(F)}$ and $\overline{Sp_{2n}(F)}$ be the metaplectic double covers of the general symplectic group and the symplectic group attached to the $2n$ dimensional symplectic space over F, respectively. Let σ be a genuine, possibly reducible, unramified principal series representation of $\overline{GSp_{2n}(F)}$. In these notes we give an explicit formula for a spanning set for the space of Spherical Whittaker functions attached to σ. For odd n, and generically for even n, this spanning set is a basis. The significant property of this set is that each of its elements is unchanged under the action of the Weyl group of $\overline{Sp_{2n}(F)}$. If n is odd, then each element in the set has an equivariant property that generalizes a uniqueness result proved by Gelbart, Howe, and Piatetski-Shapiro.Using this symmetric set, we construct a family of reducible genuine unramified principal series representations that have more then one generic constituent. This family contains all the reducible genuine unramified principal series representations induced from a unitary data and exists only for n even.

Introduction

Let F be a p-adic field and let $\mathbb{O}_F$ be its ring of integers. Let $\overline{Sp_{2n}(F)}$ be the metaplectic double cover of $Sp_{2n}(F)$ and let $\overline{GSp_{2n}(F)} = \overline{Sp_{2n}(F)} \rtimes F^*$ be the metaplectic double cover of $GSp_{2n}(F)$. Let $\overline{T_{2n}(F)}$ and $\overline{T'_{2n}(F)}$ be the inverse images inside $\overline{GSp_{2n}(F)}$ of $T_{2n}(F)$ and $T'_{2n}(F)$, the maximal tours of $Sp_{2n}(F)$ and $GSp_{2n}(F)$ respectively. While $\overline{T_{2n}(F)}$ is commutative, $\overline{T'_{2n}(F)}$ is not. The isomorphism class of a genuine, smooth, admissible, irreducible representation of $\overline{T'_{2n}(F)}$ is determined by its central character. Let $I(\omega)$ be a, possibly reducible, genuine principal series representation of $\overline{GSp_{2n}(F)}$ parabolically induced from a representation of $\overline{T'_{2n}(F)}$ whose central character is ω . Its Whittaker model is not unique. In fact

$$\dim\Big(\mathrm{Hom}_{\overline{GSp_{2n}(F)}}\big(I(\omega), \mathrm{Ind}^{\overline{GSp_{2n}(F)}}_{\overline{N_{2n}(F)}}\,\theta\big)\Big) = [F^* : {F^*}^2].$$

Here θ is a nondegenerate genuine character of $\overline{N_{2n}(F)}$, the inverse image of the maximal unipotent radical of $Sp_{2n}(F)$. We fix a natural basis, $B(\omega, \theta)$, for this space.

Suppose now that F has odd residual characteristic. In this case, $\overline{GSp_{2n}(F)}$ splits over $GSp_{2n}(\mathbb{O}_F)$. While for odd n the embedding of $GSp_{2n}(\mathbb{O}_F)$ inside $\overline{GSp_{2n}(F)}$ is essentially unique, there are two non conjugate embeddings if n is even. Let $K^{\eta}_{2n}(F)$ be an image of $GSp_{2n}(\mathbb{O}_F)$ inside $\overline{GSp_{2n}(F)}$. Assume that $I(\omega)$ is spherical, *i.e.,* contains a one dimensional invariant $K^{\eta}_{2n}(F)$ space. Note that this property is independent of the particular embedding of $GSp_{2n}(\mathbb{O}_F)$. In these notes we provide an explicit

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formulas for $W(\omega, \theta, \eta)$, the set of images of the spherical vector under $B(\omega, \theta)$, *i.e.,* a spanning set for the $K^{\eta}_{2n}(F)$ invariant subspace inside the subspace of $\mathrm{Ind}_{\overline{N_{2n}(F)}}^{\overline{GSp_{2n}(F)}} \theta$ generated by the images of $I(\omega)$; see Theorem 5.2 for the explicit formulas. For odd n, and generically for even n, $W(\omega, \theta, \eta)$ is a basis.

Let $\overline{T^{+}_{2n}(F)}$ be the centralizer of $\overline{T_{2n}(F)}$ inside $\overline{T'_{2n}(F)}$. Then $\overline{T^{+}_{2n}(F)}$ is a maximal Abelian subgroup of $\overline{T'_{2n}(F)}$. Let $E(\omega)$ be the set of extensions of ω to $\overline{T^{+}_{2n}(F)}$. The set $W(\omega, \theta, \eta)$ is naturally parameterized by $E(\omega)$. The important property of the spanning set constructed here is that it consists of symmetric functions, *i.e.,* each element is preserved under the action of the Weyl group of $\overline{Sp_{2n}(F)}$ on the characters of $\overline{T^{+}_{2n}(F)}$. Given the global functional equation satisfied by Eisenstein series (see [21, Theorem IV.1.10]), we hope that this property will be useful for global applications. In the cases where a Weyl element induces a nontrivial permutation on $E(\omega)$, $W(\omega, \theta, \eta)$ is linearly dependent and $I(\omega)$ is reducible. We give exact descriptions of those cases that occur only if n is even. As an application, we construct a family of reducible genuine unramified principal series representations of $\overline{GSp_{2n}(F)}$ which includes all the reducible genuine unramified principal series representations induced from unitary data. The representations in this family are a direct sum of two non-isomorphic θ-generic $\overline{GSp_{2n}(F)}$ irreducible submodules; each contains an invariant vector with respect to a different embedding of $GSp_{2n}(\mathbb{O}_F)$. The dimension of Whittaker functionals on each summand is 2.

It should be noted that $\overline{T^{+}_{2n}(F)}$ is not an analog to the Kazhdan–Patterson standard maximal Abelian subgroup, *i.e.,* it is not the centralizer of $\overline{T'_{2n}(F)} \cap K^{\eta}_{2n}(F)$ inside $\overline{T'_{2n}(F)}$. The fact that $\overline{T^{+}_{2n}(F)}$ is the centralizer of $\overline{T_{2n}(F)}$ inside $\overline{T'_{2n}(F)}$, rather than just being a maximal Abelian subgroup, is a crucial property. Furthermore, unlike the Kazhdan–Patterson standard maximal Abelian subgroup, $\overline{T^{+}_{2n}(F)}$ is a maximal Abelian subgroup of $\overline{T'_{2n}(F)}$ over 2-adic fields as well as over the field of real numbers. Its existence is a particular case of the relation between the representation theories of $\overline{GSp_{2n}(F)}$ and $\overline{Sp_{2n}(F)}$ given in [22]. This relation is a generalization of some of the results that appeared in [9] where the representation theory of $\overline{GL_2(F)}$ was related to the representation theory of $\overline{SL_2(F)}$. The other main ingredient of our argument in this paper is the work of Bump, Friedberg, and Hoffstein on the unique, up to normalization, Spherical Whittaker function on $\overline{Sp_{2n}(F)}$ attached to an unramified genuine representation [3]. As explained in the end of Section 3, it is the uniqueness of Whittaker's model for $\overline{Sp_{2n}(F)}$ which is ultimately responsible for our main result.

In a recent paper, Chinta and Offen [6] used the method of Casselman and Shalika [5], to construct a spanning set of Spherical Whittaker functions for covers of the general linear group. It is clear from their construction that a symmetric spanning set can be constructed by introducing Jacquet integrals whose functional equations are diagonal and then properly normalizing each integral. For one particular $\overline{T_{2n}(F)}$ orbit of Whittaker characters, one can extract the normalization factor from [3]. For our purpose, it is necessary to extend the result of [3] for the other orbits. As we show at the end of Section 3, for $n = 1$ our result coincides with the recipe given in [6, Section 7]. In fact, our work gives a natural explanation for the diagonalization given in [6] in the $n = 1$ case. It would be interesting to provide a similar explanation for the diagonalization in the higher rank cases studied in [6].

The paper is organized as follows. In Sections 1 and 2 we define some general notation and collect some information on $\overline{GSp_{2n}(F)}$. In Section 3 we construct the genuine principal series representations and their Whittaker maps. We also discuss in some details the Weyl group action and prove some irreducibility results. In Section 4 we give an extension of the results of [3]. This extension is used in Section 5 where we prove our main result. Finally, in Section 6 we apply the main result to construct the family of reducible genuine unramified principal series representations mentioned above.

1 General Notation

Let F be a p-adic field, let $\mathbb{O}_F$ be the ring of integers of F, let $\mathbb{P}_F$ be its maximal ideal, and let π be a generator of $\mathbb{P}_F$. Let q be the cardinality of the residue field $\mathbb{O}_F/\mathbb{P}_F$ and let $|\cdot|$ be the absolute value on F normalized in the usual way. For $a \in F^*$ we denote its order by $\mathrm{ord}(a)$. Thus, $\mathrm{ord}(a) = -log_q(|a|)$.

Let $(\cdot,\cdot)_F$ be the quadratic Hilbert symbol of F. It is a nondegenerate symmetric bilinear form on F^*/F^{*2}. For $a \in F^*$ we define η_a to be the quadratic character of F^* attached to a, that is,

$$\eta_a(b) = (a, b)_F.$$

Let ψ be a nontrivial additive character of F. We define its conductor to be the smallest integer such that ψ is trivial on $\mathbb{P}_F^n$. We say that ψ is unramified if its conductor is 0. For $a \in F^*$ define

$$\psi_a(x) = \psi(ax).$$

It is also a nontrivial additive character of F. For $a \in F^*$ let $\gamma_\psi(a) \in \mathbb{C}^1$ be the normalized Weil factor associated with the character of second degree of F given by $x \mapsto \psi_a(x^2)$ (see [23, Theorem 2, Section 14]). It is known that γ_ψ is a fourth root of unity and that $\gamma_\psi(F^{*2}) = 1$. Also,

$$\gamma_\psi(ab) = \gamma_\psi(a)\gamma_\psi(b)(a,b)_F, \quad \gamma_{\psi_b} = \eta_b \cdot \gamma_\psi, \quad \gamma_\psi^{-1} = \gamma_{\psi^{-1}}, \quad \gamma_\psi(F^{*2}) = 1. \tag{1.1}$$

For a diagonal matrix t inside $GL_n(F)$ we define

$$\gamma_\psi(t) = \gamma_\psi\big(\det(t)\big), \quad \eta_a(t) = \eta_a\big(\det(t)\big).$$

In Sections 2.3, 4, 5, and 6, we assume that F has odd residual characteristic. In this case, $(\mathbb{O}_F^*, \mathbb{O}_F^*)_F = 1$. Moreover, it follows from [20, Lemma 1.5] that if the conductor of ψ is even, then $\gamma_\psi(\mathbb{O}_F^*) = 1$. In the odd residual characteristic case we fix a set of representatives of F^*/F^{*2} of the form $\{1, u_0, \pi, \pi u_0\}$, where u_0 is a nonsquare element in $\mathbb{O}_F^*$, fixed once and for all. In this case, the nondegeneracy of the Hilbert symbol implies that

$$\eta_{u_0}(a) = (-1)^{\mathrm{ord}(a)}. \tag{1.2}$$

For p-adic fields of odd residual characteristic, there exist exactly two quadratic characters of $\mathbb{O}_F^*$. The nontrivial one is

$$u \mapsto \eta_\pi(u) = (u, \pi)_F = \begin{cases} 1 & u \in \mathbb{O}_F^{*2}, \\ -1 & u \notin \mathbb{O}_F^{*2}. \end{cases}$$

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Any of the two quadratic characters η of $\mathbb{O}_F^*$ extends uniquely to a quadratic character of $\mathbb{O}_F^* F^{*2}$. We shall continue to denote this character by η.

Let G be a group. We denote its center by $Z(G)$. For $h, g \in G$ we denote $g^{-1}hg$ by h^g. Let H be a subgroup of G. Let σ be a representation of H. If $g \in G$ normalizes H, we denote by ${}^g\sigma$ the representation of H defined by $h \mapsto \sigma(h^g)$.

2 Groups

2.1 Linear Groups

Let $GSp_{2n}(F)$ be the general symplectic group attached to the 2n dimensional symplectic space over F. We shall realize $GSp_{2n}(F)$ as the group

$$\{g \in GL_{2n}(F) \mid gJ_{2n}g^t = \lambda(g)J_{2n}\},$$

where $J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ and $\lambda(g) \in F^*$ is the similitude factor of g. The similitude map $g \mapsto \lambda(g)$ is a rational character of $GSp_{2n}(F)$. The kernel of the similitude map is the symplectic group $Sp_{2n}(F)$. Then F^* is embedded in $GSp_{2n}(F)$ via

$$\lambda \mapsto i(\lambda) = \begin{pmatrix} I_n & 0 \\ 0 & \lambda I_n \end{pmatrix}.$$

Using this embedding we define an action of F^* on $Sp_{2n}(F)$ by $(g, \lambda) \mapsto g^{i(\lambda)}$. Let $F^* \ltimes Sp_{2n}(F)$ be the semi-direct product corresponding to this action. For $g \in GSp_{2n}(F)$ define $g_1 = i\big(\lambda^{-1}(g)\big)g \in Sp_{2n}(F)$. The map $g \mapsto (\lambda(g), g_1)$ is an isomorphism between $GSp_{2n}(F)$ and $F^* \ltimes Sp_{2n}(F)$. We define $GSp^+_{2n}(F)$ to be the subgroup of $GSp_{2n}(F)$ that consists of elements whose similitude factor lies in F^{*2}. $GSp^+_{2n}(F)$ is a normal subgroup of $GSp_{2n}(F)$ that contains $Sp_{2n}(F)$. Clearly,

$$[GSp_{2n}(F) : GSp^+_{2n}(F)] = [F^* : F^{*2}] < \infty.$$

For any H' of $GSp_{2n}(F)$ we denote by H^+ and H its intersection with $GSp^+_{2n}(F)$ and $Sp_{2n}(F)$ respectively.

Let $N_{GL_n}(F)$ be the group of upper triangular unipotent matrices in $GL_n(F)$ and let $N_{2n}(F)$ be the following maximal unipotent subgroup of $Sp_{2n}(F)$:

$$\left\{ \begin{pmatrix} z & b \\ 0 & {}^t z^{-1} \end{pmatrix} \;\middle|\; z \in N_{GL_n}(F), b \in Mat_{n\times n}(F),\ b^t = z^{-1}bz^t \right\}.$$

We normalize the Haar measure on $N_{2n}(F)$ in the usual way. Let ψ be a nontrivial character of F. A character θ of $N_{2n}(F)$ is called nondegenerate if

$$\theta(z) = \psi\big(a_n z_{(n,2n)} + \Sigma_{i=1}^{n-1} a_i z_{(i,i+1)}\big),$$

where $a_1, a_2, \ldots, a_n$ are elements of F^*. We say that θ is unramified if ψ_{a_i} is unramified for any $1 \leq 1 \leq n$. We say that θ agree with ψ on the long root if $a_n = 1$.

Let $T_{GL_n}(F)$ be the group of diagonal elements inside $GL_n(F)$ and let

$$T'_{2n}(F) = \{[t, y] = \operatorname{diag}(t, yt^{-1}) \mid y \in F^*,\ t \in T_{GL_n}(F)\}$$

be the subgroup of diagonal matrices inside $GSp_{2n}(F)$. Note that $\lambda([t, y]) = y$. Denote $T'_{2n}(F) \ltimes N_{2n}(F)$ by $B'_{2n}(F)$. This is a Borel subgroup of $GSp_{2n}(F)$. Let δ be the modular function on $B'_{2n}(F)$.

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2.2 Rao's Cocycle and Metaplectic Groups

Let $\overline{Sp_{2n}(F)}$ be the unique nontrivial two-fold cover of $Sp_{2n}(F)$. The action of F^* on $Sp_{2n}(F)$ lifts uniquely to an action of F^* on $\overline{Sp_{2n}(F)}$; see [14, p. 36]. Using this lift we define

$$\overline{GSp_{2n}(F)} \simeq F^* \ltimes \overline{Sp_{2n}(F)},$$

the unique two-fold cover of $GSp_{2n}(F)$ that contains $\overline{Sp_{2n}(F)}$.

We shall realize $\overline{GSp_{2n}(F)}$ as the set $\overline{GSp_{2n}(F)} = GSp_{2n}(F) \times \{\pm 1\}$ equipped with the multiplication law $(g_1, \epsilon_1)(g_2, \epsilon_2) = \big(g_1 g_2, \epsilon_1 \epsilon_2 \widetilde{c}(g_1, g_2)\big)$. Here,

$$\widetilde{c}\colon GSp_{2n}(F) \times GSp_{2n}(F) \to \{\pm 1\}$$

is the cocycle constructed in [19, Section 2B] and studied in [22, Section 2]. Note that $\widetilde{c}$ is an extension of Rao's cocycle

$$c\colon Sp_{2n}(F) \times Sp_{2n}(F) \to \{\pm 1\}$$

constructed in [16]. Hence, the inverse image of $Sp_{2n}(F)$ inside $\overline{GSp_{2n}(F)}$ is a realization of $\overline{Sp_{2n}(F)}$. For any subset H of $GSp_{2n}(F)$ we denote by $\overline{H}$ its inverse image inside $\overline{GSp_{2n}(F)}$. Let H be a subgroup of $GSp_{2n}(F)$. A representation π of $\overline{H}$ is called *genuine* if it does not factor through the projection map on H. Thus, a representation of $\overline{H}$ with a central charter is genuine if and only $\pi(I_{2n}, -1) = -\operatorname{Id}$. For $(g, \epsilon) \in \overline{GSp_{2n}(F)}$ we define $\lambda(g, \epsilon) = \lambda(g)$.

Lemma 2.1 *The following hold:*

(i) $\widetilde{c}([t, y], [t', y']) = (\det(t), y' \det(t'))_F$.

(ii) $\widetilde{c}(i(F^*), GSp_{2n}(F)) = 1$.

(iii) $(g, \epsilon)^{(aI_{2n}, \epsilon')} = (g, \epsilon(\lambda(g), a^n)_F)$ *and* $(aI_{2n}, \epsilon)^{(g, \epsilon')} = (a, \epsilon(\lambda(g), a^n)_F)$.

(iv) $Z(\overline{GSp_{2n}(F)}) = \begin{cases} \overline{F^* I_{2n}} & n \text{ is even}, \\ \overline{F^{*2} I_{2n}} & n \text{ is odd}. \end{cases}$

(v) $Z(\overline{GSp^+_{2n}(F)}) = \overline{F^* I_{2n}}$.

(vi) $Z(\overline{T'_{2n}(F)}) = \{([t, y], \epsilon) \mid y, \det(t) \in F^{*2}\}$.

(vii) $\overline{T_{2n}(F)}$ *is an abelian group,* $\overline{T^+_{2n}(F)}$ *is the centralizer of* $\overline{T_{2n}(F)}$ *inside* $\overline{T'_{2n}(F)}$ *and it is a maximal Abelian subgroup of* $\overline{T'_{2n}(F)}$. *Furthermore,*

$$\overline{T^+_{2n}(F)} = \overline{T_{2n}(F)} Z\big(\overline{GSp^+_{2n}(F)}\big) \quad \text{and} \quad Z\big(\overline{GSp^+_{2n}(F)}\big) \cap \overline{T_{2n}(F)} = \overline{\pm I_{2n}}. \tag{2.1}$$

(viii) $N_{2n}(F)$ *embeds into* $\overline{Sp_{2n}(F)}$ *via the trivial section.* $\overline{T'_{2n}(F)}$ *normalizes* $(N_{2n}(F), 1)$.

Proof (i)–(iii) follow immediately from the cocycle formulas given in [19, pp. 456 and 460]. Since $Z(\overline{H}) \subseteq \overline{Z(H)}$ for any subgroup H of $GSp_{2n}(F)$, (iv) and (v) follow from (iii). (vi) and (vii) follow from (i). We now prove (viii). Since $N_{2n}(F) \subseteq Sp_{2n}(F)$, the fact that $N_{2n}(F)$ is embedded inside $\overline{GSp_{2n}(F)}$ via the trivial section is a property of Rao cocycle; see [16, Corollary 5.5]. This corollary also implies that $\overline{T_{2n}(F)}$ normalizes $(N_{2n}(F), 1)$. Thus, to finish the proof of (viii) one only need to show that $\overline{i(F^*)}$ normalizes $(N_{2n}, 1)$. This fact can be verified directly from the cocycle formula. All the assertions given in this lemma are proved in more details in [22, Section 2]. ■

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From the last part of this lemma it follows that if θ is a character of $N_{2n}(F)$, then $(z,\epsilon)\mapsto \epsilon\theta(z)$ defines a genuine character of $\overline{N_{2n}(F)}$. We shall continue to denote this character by θ.

2.3 Splittings of Maximal Compact Subgroups

In this subsection we assume that F is a p-adic field of odd residual characteristic. In this case, $\overline{Sp_{2n}(F)}$ splits over $Sp_{2n}(\mathbb{O}_F)$ and $\overline{GSp_{2n}(F)}$ splits over $GSp_{2n}(\mathbb{O}_F)$; see [13, Theorem 2]. Furthermore, there exists a unique map

$$\iota_{2n}\colon Sp_{2n}(\mathbb{O}_F)\to\{\pm1\},$$

such that

$$k\mapsto \kappa_{2n}(k)=\big(k,\iota_{2n}(k)\big)$$

is an embedding of $Sp_{2n}(\mathbb{O}_F)$ into $\overline{Sp_{2n}(F)}$. It is known that ι_{2n} is identically 1 on $B_{2n}(F)\cap Sp_{2n}(\mathbb{O}_F)$; see [21, section 1.4]. We shall denote the image of $Sp_{2n}(\mathbb{O}_F)$ under this embedding by $K_{2n}(F)$. The splitting of $\overline{GSp_{2n}(F)}$ over $GSp_{2n}(\mathbb{O}_F)$ is not unique. Let η be one of the two quadratic characters of $\mathbb{O}_F^*$. Define

$$\iota_{2n}^{\eta}\colon GSp_{2n}(\mathbb{O}_F)\to\{\pm1\}$$

to be the following extension of ι_{2n}:

$$\iota_{2n}^{\eta}(k)=\eta\big(\lambda(k)\big)\,\iota_{2n}\big(k_{1}\big)\,. \tag{2.2}$$

Note that these two maps are identically 1 on $T_{2n}^{+}(F)\cap GSp_{2n}(\mathbb{O}_F)$.

Lemma 2.2 *There are exactly two maps*

$$\iota'_{2n}\colon GSp_{2n}(\mathbb{O}_F)\to\{\pm1\},$$

such that $k\mapsto (k,\iota'_{2n}(k))$ is an embedding of $GSp_{2n}(\mathbb{O}_F)$ inside $\overline{GSp_{2n}(F)}$. These are the two maps defined in (2.2). These two embeddings are conjugates via an element of $\overline{GSp_{2n}(F)}$ if and only if n is odd.

Proof The restriction of ι'_{2n} to $Sp_{2n}(\mathbb{O}_F)$ must agree with ι_{2n}. Thus

$$\iota'_{2n}(k)=\iota'_{2n}\big(i(\lambda(k)\big)\,\iota_{2n}(k_{1})\widetilde{c}\big(i(\lambda(k),k_{1}\big)\,.$$

Lemma 2.1(ii) now implies that the restriction of ι'_{2n} to $i(\mathbb{O}_F^*)$ must be a quadratic character and that

$$\iota'_{2n}(k)=\iota'_{2n}\big(i(\lambda(k)\big)\,\iota_{2n}(k_{1}).$$

This shows that ι'_{2n} must be one of the two maps given in (2.2). On the other hand, if ι'_{2n} is a splitting map, then so is $\iota'_{2n}\cdot(\eta_\pi\circ\lambda)$.

Let ι'_{2n} be any of the two splittings given in (2.2). Suppose that n is odd. By Lemma 2.1(iii),

$$\big(k,\iota'_{2n}(k)\big)^{(\pi I_{2n},1)}=\big(k,\iota'_{2n}(k)\eta_\pi(\lambda(k))\big)\,.$$

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Hence, if n is odd, the two embeddings constructed here are conjugates. It remains to show that if n is even, then these two are not conjugates. From the Cartan decomposition it follows that the normalizer of $GSp_{2n}(\mathbb{O}_F)$ inside $\overline{GSp_{2n}(F)}$ is $Z(GSp_{2n}(F))GSp_{2n}(\mathbb{O}_F)$. Since the inner conjugation map of $\overline{GSp_{2n}(F)}$, $\sigma \mapsto \sigma^{(g,\epsilon)}$, is independent of ϵ, it follows that if the two embeddings mentioned above are conjugates then a conjugating element must lie in $\overline{Z(GSp_{2n}(F))}$. However, if n is even, then by Lemma 2.1, $\overline{Z(GSp_{2n}(F))} = Z(\overline{GSp_{2n}(F)})$. ■

From this point we assume that $GSp_{2n}(\mathbb{O}_F)$ is embedded in $\overline{GSp_{2n}(F)}$ via

$$k \mapsto \left(k, \iota_{2n}^{\eta}(k)\right),$$

where η is one of the two characters above. We shall denote the image of $GSp_{2n}(\mathbb{O}_F)$ under this embedding by $K_{2n}^{\eta}(F)$.

3 Representations

3.1 Genuine Principal Series Representations

Lemma 3.1 *Let χ be a character of the diagonal subgroup inside $GL_n(F)$, and let ξ be a character of F^*.*

(i) *The map*

$$\left([t,1],\epsilon\right) \mapsto \chi_{\psi}\left([t,1],\epsilon\right) = \epsilon\chi(t)\gamma_{\psi}(t) \tag{3.1}$$

defines a genuine character of $\overline{T_{2n}(F)}$, and all genuine characters of $\overline{T_{2n}(F)}$ have this form. Also,

$$\chi_{\psi_a} = (\chi \cdot \eta_a)_{\psi}. \tag{3.2}$$

(ii) *The map $(aI_{2n},\epsilon) \mapsto \xi_{\psi}(aI_{2n},\epsilon) = \epsilon\xi(a)\gamma_{\psi}(a^n)$ defines a genuine character of $Z(\overline{GSp_{2n}^{+}(F)})$, and all genuine characters of $Z(\overline{GSp_{2n}^{+}(F)})$ have this form.*

(iii) *Assume in addition that $\xi(-1) = \chi(-I_n)$, or equivalently that ξ_{ψ} and χ_{ψ} agree on $Z(\overline{GSp_{2n}^{+}(F)}) \cap \overline{T_{2n}(F)}$. Then we can define a genuine character of $\overline{T_{2n}^{+}(F)}$ by*

$$h = zg \mapsto (\chi \boxtimes \xi)_{\psi}(zg) = \chi_{\psi}(g)\xi_{\psi}(z).$$

Here, $g \in \overline{T_{2n}(F)}$ and $z \in Z(\overline{GSp_{2n}^{+}(F)})$. All the genuine characters of $\overline{T_{2n}^{+}(F)}$ have this form.

(iv) *$(\chi \boxtimes \xi)_{\psi}$ and $(\chi' \boxtimes \xi')_{\psi}$ have the same restriction to $Z(\overline{T_{2n}'(F)})$ if and only if they are conjugates via an element of $\overline{T_{2n}'(F)}$ in which case*

$$(\chi' \boxtimes \xi')_{\psi} = (\chi \cdot \eta_c \boxtimes \xi \cdot \eta_c^n)_{\psi} = {}^{(i(c),1)}\left((\chi \boxtimes \xi)_{\psi}\right) \tag{3.3}$$

for some $c \in F^*$.

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Proof The fact that (3.1) defines a genuine character of $\overline{T_{2n}(F)}$ follows from Lemma 2.1(i), and (1.1). Since the product of any two genuine characters of $\overline{T_{2n}(F)}$ is a nongenuine character of $\overline{T_{2n}(F)}$ and since any nongenuine character of $\overline{T_{2n}(F)}$ may be viewed as a character $T_{2n}(F)$, the first two assertions in (i) follow. Equation (3.2) also follows now from (1.1). (ii) is proved by similar arguments. (iii) follows from (i), (ii), and Lemma 2.1(vii). The equality between the middle and right wings of (3.3) follows from Lemma 2.1(i). The fact the any two genuine characters of $\overline{T^{+}_{2n}(F)}$ which satisfy the relation given in (3.3) have the same restriction to $Z(\overline{T'_{2n}(F)})$ follows from Lemma 2.1(vi) and (vii). Note now that $[\overline{T^{+}_{2n}(F)}:Z(\overline{T'_{2n}(F)})] = [F^*:F^{*2}]$ and that

$$\{(i(c),1) \mid c \in F^*/F^{*2}\}$$

is a set of representatives for $\overline{T'_{2n}(F)}/\overline{T^{+}_{2n}(F)}$. The proof of (iv) is finished once we note that $c \notin F^{*2}$ implies that

$$(\chi \boxtimes \xi)_\psi \neq (\chi \cdot \eta_c \boxtimes \xi \cdot \eta_c^n)_\psi. \qquad \blacksquare$$

Let ω be a genuine character of $Z(\overline{T'_{2n}(F)})$ we shall denote by $E(\omega)$ the set of its extensions to $\overline{T^{+}_{2n}(F)}$. The group $\overline{T'_{2n}(F)}$ is not Abelian. Its smooth admissible irreducible representations have the following form.

Lemma 3.2 *The isomorphism class of a smooth admissible genuine irreducible representation of $\overline{T'_{2n}(F)}$ is determined by its central character. Any smooth admissible genuine irreducible representation of $\overline{T'_{2n}(F)}$ is $[F^*:F^{*2}]$ dimensional. It can be realized as an induction from a genuine character of $\overline{T^{+}_{2n}(F)}$. If φ and φ' are two genuine characters of $\overline{T^{+}_{2n}(F)}$, then the corresponding inductions on $\overline{T'_{2n}(F)}$ are isomorphic if and only if $\varphi, \varphi' \in E(\omega)$, where ω is a genuine character of $Z(\overline{T'_{2n}(F)})$.*

Proof This is a variation of the Stone–von Neumann Theorem; see [13, Theorem 3]. The crucial facts here are that $\overline{T'_{2n}(F)}$ is a two step nilpotent group and that $\overline{T^{+}_{2n}(F)}$ is a maximal Abelian subgroup. $\blacksquare$

We shall denote by $\tau(\omega)$ the isomorphism class of irreducible admissible genuine representation of $\overline{T'_{2n}(F)}$ whose central character is ω. We denote its realization mentioned in the last theorem by

$$i\big((\chi \boxtimes \xi)_\psi\big) = \operatorname{Ind}^{\overline{T'_{2n}(F)}}_{\overline{T^{+}_{2n}(F)}} (\chi \boxtimes \xi)_\psi.$$

Due to Lemma 2.1(viii) we shall regard, as in the linear case, each character (resp., irreducible smooth admissible representation) of $\overline{T_{2n}(F)}$ (resp., of $\overline{T'_{2n}(F)}$) as a character (resp., representation) of $\overline{B_{2n}(F)}$ (resp., of $\overline{B'_{2n}(F)}$) by extending it trivially on $(N_{2n}(F),1)$. We shall always assume that a character (resp., representation) of $\overline{B_{2n}(F)}$ (resp., $\overline{B'_{2n}(F)}$) has this form. Genuine principal series representation of $\overline{Sp_{2n}(F)}$ (resp., $\overline{GSp_{2n}(F)}$) is a representation parabolically induced from a genuine character (resp., irreducible smooth admissible genuine representation) of $\overline{B_{2n}(F)}$ (resp., $\overline{B'_{2n}(F)}$). We shall assume that the induction is normalized. We let

$$I(\chi_\psi) = \operatorname{Ind}^{\overline{Sp_{2n}(F)}}_{\overline{B_{2n}(F)}} \chi_\psi.$$

For a genuine character ω of $Z(\overline{T'_{2n}(F)})$ we denote by $I(\omega)$ the isomorphism class of genuine principal series representation of $\overline{GSp_{2n}(F)}$ induced from $\tau(\omega)$. By Lemma 3.2, $I(\omega)$ can be realized as

$$I\big((\chi \boxtimes \xi)_\psi\big) = \mathrm{Ind}_{\overline{B^+_{2n}(F)}}^{\overline{GSp_{2n}(F)}} (\chi \boxtimes \xi)_\psi,$$

where $(\chi \boxtimes \xi)_\psi$ is any of the $[F^* : F^{*2}]$ elements of $E(\omega)$.

3.2 Weyl Group Action

Let W_{2n} be the Weyl group of $Sp_{2n}(F)$. We choose representatives of W_{2n} inside $Sp_{2n}(\mathbb{O}_F)$ and take their images inside $K_{2n}(F)$. We shall continue to denote the outcome by W_{2n}. By [21, (1.18), Section 1.3],

$$(3.4) \qquad (t,1)^w = (t^w, 1)$$

for any $t \in T_{2n}(F)$ and $w \in W_{2n}$. In fact, (2.1) implies that (3.4) holds for all $t \in T^+_{2n}(F)$. Thus, ${}^w(\chi_\psi) = \big({}^w\chi\big)_\psi$ and

$$(3.5) \qquad {}^w(\chi \boxtimes \xi)_\psi = ({}^w\chi \boxtimes \xi)_\psi$$

For any $w \in W_{2n}$. Here ${}^w\chi$ is defined via a similar action of W_{2n} on $T_{2n}(F)$ and its characters. Note that W_{2n} also acts on the set of genuine characters of $Z(\overline{T'_{2n}(F)})$ and hence acts on isomorphism classes of genuine irreducible admissible representations of $\overline{T_{2n}(F)}$. Clearly, $E({}^w\omega) = {}^wE(\omega)$ for any $w \in W_{2n}$. Given ω, a genuine character of $Z(\overline{T'_{2n}(F)})$, fix $(\chi \boxtimes \xi)_\psi \in E(\omega)$ and define $R(\omega)$ to be the the following subgroup of F^*/F^{*2}:

$$R(\omega) = \left\{ c \in F^*/F^{*2} \mid \exists w \in W_{2n}.\ {}^{(i(c),1)}(\chi \boxtimes \xi)_\psi = {}^w(\chi \boxtimes \xi)_\psi \right\}.$$

Clearly, $R(\omega)$ is well defined, *i.e.,* does not depend on the particular choice of $(\chi \boxtimes \xi)_\psi \in E(\omega)$. So $R(\omega)$ is nontrivial if and only if there exists $w \in W_{2n}$ that fixes ω and induces a nontrivial permutation on $E(\omega)$.

Lemma 3.3 *Let ω be a genuine character of $Z(\overline{T'_{2n}(F)})$.*

(i) *If n is odd and ${}^w\omega = \omega$ for some $w \in W_{2n}$, then w preserves the elements of $E(\omega)$ pointwise. In particular, $R(\omega)$ is trivial for any ω.*

(ii) *If n is even, then there exists a genuine character ω_0 of $Z(\overline{T'_{2n}(F)})$ such that $R(\omega_0)$ is nontrivial. Furthermore, up to conjugation by $w \in W$, an element of $E(\omega_0)$ has the form $(\chi_0 \boxtimes \xi)_\psi$, where*

$$(3.6) \qquad \chi_0[\mathrm{diag}(t_n, t_{n-1}, \dots, t_1), 1] = \chi_1(t_1 t_2)\chi_2(t_3, t_4) \cdots \chi_{\frac{n}{2}}(t_{n-1} t_n)\eta_c(t_2 t_4 \cdots t_n).$$

Here $\chi_1, \dots, \chi_{\frac{n}{2}}$ are characters of F^* and c is a nonsquare element in F^*.

Proof The first assertion follows from a comparison of (3.3) and (3.5). If n is odd, then

$$(\chi' \boxtimes \xi')_\psi = {}^{(i(c),1)}\big((\chi \boxtimes \xi)_\psi\big)$$

implies that $c \in F^{*2}$. Next, if n is even, define

$$w_{\circ} = \operatorname{diag}(w', w', \dots, w') \in W_{2n}, \tag{3.7}$$

where $w' = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. By a straightforward computation

$${}^{w_0}\big(\chi_{\circ} \boxtimes \xi\big)_\psi = (\chi_{\circ}\eta_c \boxtimes \xi)_\psi.$$

Last, W_{2n} is generated by simple reflections. Denote these reflections by $w_{i,i+1}$ where $1 \le i \le n-1$ and by w_n where $w_{i,i+1}$ acts on $[\operatorname{diag}(t_n, t_{n-1}, \dots, t_1), 1])$ by switching t_i and t_{i+1}, and where w_n acts on $[\operatorname{diag}(t_n, t_{n-1}, \dots, t_1), 1]$ by inverting t_1. Thus, if χ' defined by

$$\operatorname{diag}(t_n, t_{n-1}, \dots, t_1) = \prod_{i=1}^{n} \chi_i'(t_i)$$

is not Weyl conjugated to $\chi_{\circ}$, then there exists $1 \le i \le n$ such that $\chi_i' \ne \chi_j^{\pm 1}\eta_c$ for any $1 \le j \le n$ and $\eta_c \ne 1$. This shows that χ_ψ' is not Weyl conjugated to any of its quadratic twists. ∎

Theorem 3.4 *Let ω be a genuine character of $Z(\overline{T_{2n}'(F)})$. Fix $(\chi \boxtimes \xi)_\psi \in E(\omega)$.*

(i) *$I(\omega)$ is irreducible if and only if $R(\omega)$ is trivial and $I(\chi_\psi)$ is an irreducible $\overline{Sp_{2n}(F)}$ module.*
(ii) *If $I(\chi_\psi)$ is an irreducible $\overline{Sp_{2n}(F)}$ module and $R(\omega)$ has two elements, then $I(\omega)$ breaks into a direct sum of two $\overline{GSp_{2n}(F)}$ irreducible modules.*
(iii) *If the restriction of ω to $Z(\overline{T_{2n}'(F)}) \cap \overline{Sp_{2n}(F)}$ is unitary and $R(\omega)$ has two elements, then $I(\omega)$ breaks into a direct sum of two $\overline{GSp_{2n}(F)}$ irreducible modules.*

Proof Using induction by stages we can realize $I(\omega)$ as

$$\operatorname{Ind}_{\overline{GSp_{2n}^{+}(F)}}^{\overline{GSp_{2n}(F)}} I'\big((\chi \boxtimes \xi)_\psi\big),$$

where

$$I'\big((\chi \boxtimes \xi)_\psi\big) = \operatorname{Ind}_{\overline{B_{2n}^{+}(F)}}^{\overline{GSp_{2n}^{+}(F)}} (\chi \boxtimes \xi)_\psi. \tag{3.8}$$

Since $\overline{GSp_{2n}^{+}(F)}$ is a normal subgroup of finite index inside $\overline{GSp_{2n}(F)}$, it follows from Clifford theory that $I(\omega)$ is irreducible if and only if $I'((\chi \boxtimes \xi)_\psi)$ is irreducible and is not isomorphic to any of its conjugations by elements of $\overline{GSp_{2n}(F)}$ that lie outside $\overline{GSp_{2n}^{+}(F)}$. Note that since $\overline{GSp_{2n}^{+}(F)} = \overline{Sp_{2n}(F)}Z(\overline{GSp_{2n}^{+}(F)})$, it follows that $I'((\chi \boxtimes \xi)_\psi)$ is an irreducible $\overline{GSp_{2n}^{+}(F)}$ module if and only if $I(\chi_\psi)$ is an irreducible

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$\overline{Sp_{2n}(F)}$ module. If C is a set of representatives of F^*/F^{*2}, then $(i(C),1)$ is a set of representatives of $\overline{GSp_{2n}(F)}/\overline{GSp^+_{2n}(F)}$. For $c \in F^*$ we have

$$^{(i(c),1)}I'\big((\chi \boxtimes \xi)_\psi\big) \simeq I'\big(^{(i(c),1)}(\chi \boxtimes \xi)_\psi\big).$$

Suppose now that $I'((\chi \boxtimes \xi_a)_\psi)$ is irreducible. If n is odd, then by Lemma 3.3(i), $R(\omega)$ is trivial and for $c \notin F^{*2}$

$$I'\big((\chi \boxtimes \xi)_\psi\big) \not\simeq {}^{(i(c),1)}I'\big((\chi \boxtimes \xi)_\psi\big),$$

since by Lemma 2.1(iii), these two representations have different central characters. This proves (i) for odd n. Suppose now that n is even. Then, by (3.3), $I'((\chi \boxtimes \xi)_\psi)$ and $I'(^{(i(c),1)}(\chi \boxtimes \xi)_\psi)$ have the same central character. Hence,

$$I'\big((\chi \boxtimes \xi)_\psi\big) \simeq {}^{(i(c),1)}I'\big((\chi \boxtimes \xi)_\psi\big)$$

if and only if

$$I\big(\chi_\psi\big) \simeq I\big((\chi\eta_c)_\psi\big). \tag{3.9}$$

Since we assume that $I(\chi_\psi)$ is irreducible, it follows, similarly to linear groups, that (3.9) holds if and only if $(\chi\eta_c)_\psi = (^w\chi)_\psi$ for some $w \in W_{2n}$. This finishes the proof of the first assertion.

We now prove the second assertion. Since we assume that $I(\chi_\psi)$ is irreducible, then, by Clifford theory, the assertion will follow once we show that $A(\omega)$, the commuting algebra of

$$\mathrm{Ind}^{\overline{GSp_{2n}(F)}}_{\overline{GSp^+_{2n}(F)}} I'\big((\chi \boxtimes \xi)_\psi\big)$$

is 2 dimensional. By Frobenius reciprocity

$$\begin{aligned} A(\omega) &\simeq \mathrm{Hom}_{\overline{GSp^+_{2n}(F)}}\Big(\mathrm{Ind}^{\overline{GSp_{2n}(F)}}_{\overline{GSp^+_{2n}(F)}} I'\big((\chi \boxtimes \xi)_\psi\big), I'\big((\chi \boxtimes \xi)_\psi\big)\Big) \\ &\simeq \bigoplus_{y \in F^*/F^{*2}} \mathrm{Hom}_{\overline{GSp^+_{2n}(F)}}\Big(\mathrm{Ind}^{\overline{GSp^+_{2n}(F)}}_{\overline{B^+_{2n}(F)}}\big((\chi\eta_y \boxtimes \xi)_\psi\big), \mathrm{Ind}^{\overline{GSp^+_{2n}(F)}}_{\overline{B^+_{2n}(F)}}(\chi \boxtimes \xi)_\psi\Big) \\ &\simeq \bigoplus_{y \in Y} \mathrm{Hom}_{\overline{Sp_{2n}(F)}}\Big(I\big(\chi\eta_y)_\psi\big), I\big(\chi_\psi\big)\Big). \end{aligned}$$

By an argument we already used and by the definition of $R(\omega)$, we are finished. The third assertion follows from the second and from the fact that, regardless of the parity of n, any genuine principal series of $\overline{Sp_{2n}(F)}$ induced from a unitary character is irreducible; see [10] or [21, Theorem 5.1]. ■

Note We focus here on the case where $R(\omega)$ has two elements, since, as we shall see in Section 6, if ω is unramified then $R(\omega)$ has at most two elements. However it is quite possible, for ramified ω, that $R(\omega)$ will have 4 elements, which is the upper bound if F is a p-adic field of odd residual characteristic. Indeed, if in (3.6) we assume that $\chi_1^2 = \chi_2^2 = \cdots = \chi_{\frac{n}{2}}^2 = \eta_d$, where d is a non square element in F^* such that $dc^{-1} \notin F^{*2}$, then it is easy to see that since $^{J_{2n}}(\chi_0 \boxtimes \xi)_\psi = (\chi_0\eta_d \boxtimes \xi)_\psi$, it follows that $R(\omega)$ has 4 elements.

3.3 Whittaker Functionals

Let θ be a nondegenerate character of $N_{2n}(F)$ and let σ be a genuine representation of $\overline{Sp_{2n}(F)}$ (resp., $\overline{GSp_{2n}(F)}$). A θ-Whittaker model for σ is a nonzero image of σ inside $\mathrm{Ind}_{N_{2n}(F)}^{\overline{Sp_{2n}(F)}} \theta$ (resp., $\mathrm{Ind}_{N_{2n}(F)}^{\overline{GSp_{2n}(F)}} \theta$) under an $\overline{Sp_{2n}(F)}$ (resp., $\overline{GSp_{2n}(F)}$) map. Then σ is called θ-generic if it has a θ-Whittaker model. Note that if θ and θ' are conjugates via an element of $\overline{T_{2n}(F)}$ (resp., $\overline{T'_{2n}(F)}$), then σ is θ-generic if and only if it is θ'-generic. Note that there is only one orbit of nondegenerate characters of $N_{2n}(F)$ under the action of $\overline{T'_{2n}(F)}$, while there are $[F^*:F^{*2}]$ orbits of nondegenerate characters of $N_{2n}(F)$ under the action of $\overline{T_{2n}(F)}$. If c varies over a set of representatives of F^*/F^{*2}, then $\theta_c = {}^{(i(c),1)}\theta$ varies over all $T_{2n}(F)$ orbits of a nondegenerate genuine characters of $N_{2n}(F)$. Note that we can choose a set of representatives F^*/F^{*2} that consists of elements whose order is 0 or 1. Any genuine principal series representation σ is θ-generic. Its θ-Whittaker model is unique; see [3] or [19].

Lemma 3.5 *For $f \in I(\chi_\psi)$ and $g \in \overline{Sp_{2n}(F)}$ the Jacquet integral*

$$W_f(g) = \int_{N_{2n}(F)} f\big((J_{2n}u,1)g\big)\,\theta^{-1}(u,1)\,du$$

converges absolutely for a dominant character and has an analytic continuation to the full set of genuine characters. The map $f \mapsto W_f$ is the unique (up to a scalar) map from $I(\chi_\psi)$ to its θ-Whittaker model.

Proof This is well known and follows from [4, Theorem 5.2]. See, for example, [1, Chapter 4] or [13, Theorem 6.3]. ■

Whittaker models for $\overline{GSp_{2n}(F)}$ are not unique. For principal series representations we have the following lemma.

Lemma 3.6 *Fix C, a set of $[F^*:F^{*2}]$ representatives of $\overline{T'_{2n}(F)}/\overline{T^{+}_{2n}(F)}$. For $f \in I(\chi \boxtimes \xi)_\psi)$, $g \in \overline{GSp_{2n}(F)}$, and $c \in C$, the Jacquet integral*

$$\widetilde{W}^c_f(g) = \int_{N_{2n}(F)} f\big(c(J_{2n}u,1)g\big)\,\theta^{-1}(u,1)\,du$$

converges absolutely for a dominant character and has an analytic continuation to the full set of genuine characters. The set of maps $\{f \mapsto \widetilde{W}^c_f \mid c \in C\}$ forms a basis for

$$\mathrm{Hom}_{\overline{GSp_{2n}(F)}}\Big(I\big((\chi\boxtimes\xi)_\psi\big)\,,\mathrm{Ind}_{N_{2n}(F)}^{\overline{GSp_{2n}(F)}}\theta\Big)\,.$$

Proof This is proved exactly as in [11, Lemmas 1.3.1 and 1.3.2]. The idea here is that $\{f \mapsto f(c) \mid c \in C\}$ is a basis of the space of functionals on $i((\chi \boxtimes \xi)_\psi)$ and that the Jacquet integral is an isomorphism from this space to

$$\mathrm{Hom}_{\overline{GSp_{2n}(F)}}(I(\chi\boxtimes\xi)_\psi), \mathrm{Ind}_{N_{2n}(F)}^{\overline{GSp_{2n}(F)}}\theta).$$

This theorem can also be deduced from the arguments in [22, Section 5]. ■

Corollary 3.7 *Keep the notation of Lemma 3.6 and assume that n is odd. Denote by ω' the central character of $I((\chi\boxtimes\xi)_\psi)$. ω' is the restriction of $(\chi\boxtimes\xi)_\psi$ to $Z(\overline{GSp_{2n}(F)}) = \overline{F^{*2}I_{2n}}$. Let Ω be the set of the $[F^*:F^{*2}]$ extensions of ω' to $Z(\overline{GSp^+_{2n}(F)}) = \overline{F^*I_{2n}}$. For each $\mu\in\Omega$, exactly one of the maps $f\mapsto \widetilde{W}^c_f$ is an element in the one dimensional space*

$$\mathrm{Hom}_{\overline{GSp_{2n}(F)}}\Big(I\big((\chi\boxtimes\xi)_\psi\big), \mathrm{Ind}^{\overline{GSp_{2n}(F)}}_{Z(\overline{GSp_{2n}(F)})\times\overline{N_{2n}(F)}}\mu\times\theta\Big).$$

Proof Let W be the space of θ Whittaker functionals on $I((\chi\boxtimes\xi)_\psi)$, *i.e.,* the set of functionals on $I((\chi\boxtimes\xi)_\psi)$ such that $\lambda(\rho(n)f)=\theta(n)\lambda(f)$ for any $n\in\overline{N_{2n}(F)}$. Here ρ stands for right translations. By Frobenius reciprocity,

$$W\simeq \mathrm{Hom}_{\overline{GSp_{2n}(F)}}\Big(I((\chi\boxtimes\xi)_\psi), \mathrm{Ind}^{\overline{GSp_{2n}(F)}}_{\overline{N_{2n}(F)}}\theta\Big),$$

and $\{f\mapsto \widetilde{W}^c_f(I_{2n},1)\mid c\in C\}$ forms a basis for W. Clearly $Z(\overline{GSp^+_{2n}(F)})$ acts on W. By examining its action on this basis it follows that W decomposes over $Z(\overline{GSp^+_{2n}(F)})$ with multiplicity 1 to $[F^*:F^{*2}]$ one-dimensional spaces and that Ω is the set of $Z(\overline{GSp^+_{2n}(F)})$ eigenvalues. The lemma follows now from Frobenius reciprocity. ■

Lemma 3.8 *Let C be a set of representatives of F^*/F^{*2}. For $f\in I((\chi\boxtimes\xi)_\psi)$, $g\in\overline{GSp_{2n}(F)}$, and $c\in C$, define $W^c_f(g)$ to be the analytic continuation of*

$$\int_{N_{2n}(F)} f\big((J_{2n}u,1)(i(c),1)g\big)\,\theta_c^{-1}(u,1)\,du.$$

The set of maps $\{f\mapsto W^c_f\mid c\in C\}$ forms a basis for

$$\mathrm{Hom}_{\overline{GSp_{2n}(F)}}\Big(I\big((\chi\boxtimes\xi)_\psi\big), \mathrm{Ind}^{\overline{GSp_{2n}(F)}}_{\overline{N_{2n}(F)}}\theta\Big).$$

Proof As a set of representatives of $\overline{T'_{2n}(F)}/\overline{T^+_{2n}(F)}$ we choose

$$\Big\{\big(\mathrm{diag}(cI_{2n},I_{2n}),1\big)\mid c\in C\Big\}.$$

We note that up to $(I_{2n},\pm1)$

$$(\mathrm{diag}(cI_{2n},I_{2n}),1)(J_{2n}u,1)=(J_{2n}u,1)(i(c),1).$$

The lemma follows now from Lemma 3.6 by an integration change of variable: $u\mapsto u^{(i(c),1)}$. ■

Remark We shall now outline the uniqueness result that is responsible for our main result to be proved in Section 5.2. For $c\in F^*/F^{*2}$, let $I^c((\chi\boxtimes\xi)_\psi)\subseteq I((\chi\boxtimes\xi)_\psi)$ be the subspace of functions whose support is contained in the open set of elements

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whose similitude character lies in cF^{*2}. Obviously, $I^c((\chi \boxtimes \xi)_\psi)$ is a $\overline{GSp^+_{2n}(F)}$-invariant space and

$$I\big((\chi \boxtimes \xi)_\psi\big) = \bigoplus_{c \in F^*/F^{*2}} I^c\big((\chi \boxtimes \xi)_\psi\big).$$

The crucial fact is that, as $\overline{Sp_{2n}(F)}$ modules,

$$I^c\big((\chi \boxtimes \xi)_\psi\big) \simeq I\big((\chi\eta_c)_\psi\big).$$

It is now clear that any Whittaker functional on $I((\chi \boxtimes \xi)_\psi)$ that vanish on

$$I_c\big((\chi \boxtimes \xi)_\psi\big) = \bigoplus_{c \neq d \in F^*/F^{*2}} I^d\big((\chi \boxtimes \xi)_\psi\big)$$

defines a Whittaker functional on $I((\chi\eta_c)_\psi)$. It was proved in [19] that the space of Whittaker functionals on any representation of $\overline{Sp_{2n}(F)}$ parabolically induced from a smooth, admissible, irreducible, genuine, generic representation is one-dimensional. Consequently, the space of Whittaker functionals on $I((\chi \boxtimes \xi)_\psi)$ that vanish on $I_c((\chi \boxtimes \xi)_\psi)$ is one dimensional. A non zero element in this space is given by

$$f \mapsto \lambda_{(\chi \boxtimes \xi)_\psi, c, \theta}(f) = W_f^c(I_{2n}, 1).$$

Clearly, the set

$$\{\lambda_{(\chi \boxtimes \xi)_\psi, c, \theta} \mid c \in F^*/F^{*2}\}$$

is a basis for the space of Whittaker functionals on $I((\chi \boxtimes \xi)_\psi)$. For a Weyl element $w \in K_{2n}(F)$ let

$$A_w \colon I\big((\chi \boxtimes \xi)_\psi\big) \to I\big(({}^w\chi \boxtimes \xi)_\psi\big)$$

be the standard intertwining operator defined by the meromorphic continuation of

$$\big(A_w(f)\big)(g) = \int_{N_{2n}(F) \cap (w N^-_{2n}(F) w^{-1})} f(wng)\, dn.$$

Here $N^-_{2n}(F)$ is the unipotent radical opposite to $N_{2n}(F)$. Since A_w maps $I^c((\chi \boxtimes \xi)_\psi)$ into $I^c(({}^w\chi \boxtimes \xi)_\psi)$, it follows that $\lambda_{(\chi \boxtimes \xi)_\psi, c, \theta}$ and $\lambda_{({}^w\chi \boxtimes \xi)_\psi, c, \theta} \circ A_w$ are proportional. Consider the $n = 1$ case for example. In this case there is only one nontrivial Weyl element. For this element, by comparing the results of [20] and an unpublished result of J. Sweet, one sees that the proportion factor is

$$C(\chi, \psi, c) = \gamma_F(\psi^{-1}) \frac{\gamma(\chi^2, 0, \psi_2)}{\gamma(\chi \cdot \eta_c, \frac{1}{2}, \psi)}, \tag{3.10}$$

where $\gamma_F(\psi^{-1})$ is the nonnormalized Weil index attached to ψ^{-1} and $\gamma(\cdot, \cdot, \cdot)$ is the Tate gamma factor. These factors should be thought of as an analog to Shahidi Local coefficients [18] defined in a context where uniqueness of Whittaker models fails.

Thus, we have shown that the functional equations satisfied by the base constructed here are diagonal. For an arbitrary choice of a basis $B((\chi \boxtimes \xi)_\psi)$ of the space of Whittaker functionals on $I((\chi \boxtimes \xi)_\psi)$ one can expect that the composition of an element of $B((\chi \boxtimes \xi)_\psi)$ with the intertwining operator will be a linear combination of elements of $B(({}^w\chi \boxtimes \xi)_\psi)$ in which more then one summand appear; see [6], for example.

Returning to the $n = 1$ case, it follows from (3.10) that if $c_1 c_2^{-1} \notin F^{*2}$, then $C(\chi, \psi, c_1) \neq C(\chi, \psi, c_2)$. This shows that the diagonalization above is unique up to permutations and normalization. This explains why our result in this case coincides with the result in [6, Section 7]. In the $\overline{GSp_{2n}(F)}$ case, it can be proved that the local coefficients attached to different elements of F^*/F^{*2} are distinct; *i.e.,* formulas similar to (3.10) can be given for all n.

4 An Extension of a Result of Bump, Friedberg, and Hoffstein

From this point until the end of the paper, we assume that the residual characteristic of F is odd. We fix ψ, an unramified character of F, and θ, a nondegenerate unramified character of $N_{2n}(F)$ that agrees with ψ^{-1} on the long root.

A genuine character of $\overline{T_{2n}(F)}$ is called unramified if it is trivial on $K_{2n}(F) \cap \overline{T_{2n}(F)}$. Fix $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{C}^{*n}$ and define χ_α to be a character of $T_{2n}(F)$ by

$$\chi_\alpha\big(\operatorname{diag}(a_n, a_{n-1}, \dots, a_1, a_n^{-1}, a_{n-1}^{-1}, \dots, a_1^{-1}\big) = \prod_{i=1}^{n} \alpha_i^{\operatorname{ord}(a_i)}.$$

Since ψ has an even conductor, $\gamma_\psi(\mathbb{O}_F^*) = 1$. Thus, any unramified genuine character of $\overline{T_{2n}(F)}$ can be written as $(\chi_\alpha)_\psi$ for some $\alpha \in \mathbb{C}^{*n}$. Note that the action of W_{2n} maps an unramified character of $\overline{T_{2n}(F)}$ to an unramified character. Denote

$${}^w(\chi_\alpha)_\psi = (\chi_{{}^w\alpha})_\psi.$$

This action induces an action of W_{2n} on the polynomial ring

$$R = \mathbb{C}[\alpha_1, \alpha_2, \dots, \alpha_n, \alpha_1^{-1}, \alpha_2^{-1}, \dots, \alpha_n^{-1}].$$

Let A be the following linear map on R:

$$A(p) = \sum_{w \in W_{2n}} (-1)^{\operatorname{length}(w)}\, {}^w p.$$

Define

$$\Delta(\alpha) = A\Big(\prod_{i=1}^{n} \alpha_i^i\Big).$$

If $(\chi_\alpha)_\psi$ is a genuine unramified character of $\overline{T_{2n}(F)}$, then $I((\chi_\alpha)_\psi)$ contains a one dimensional $K_{2n}(F)$ fixed subspace. We denote by $f^0_{(\chi_\alpha)_\psi}$ the normalized Spherical function of $I((\chi_\alpha)_\psi)$, *i.e.,* the unique $K_{2n}(F)$ invariant element inside $I((\chi_\alpha)_\psi)$ such that

$$f^0_{(\chi_\alpha)_\psi}(I_{2n}, 1) = 1.$$

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An image of $f^0_{(\chi_\alpha)_\psi}$ inside the θ_c-Whittaker model of $I((\chi_\alpha)_\psi)$, which is unique up to a scalar, is called a Spherical θ_c-Whittaker function attached to $I((\chi_\alpha)_\psi)$. A Spherical θ_c Whittaker function of $W^0_{\alpha,\psi,c,\theta}$ is called symmetric if

$$W^0_{\alpha,\psi,c,\theta} = W^0_{{}^w\alpha,\psi,c,\theta} \tag{4.1}$$

for any $w \in W_{2n}$. Following [3], we shall now give an explicit formulas for these elements, where the order of c is either 0 or 1. Define

$$D(\alpha, c) = \prod_{i<j}\big((1-q^{-1}\alpha_j\alpha_i^{-1})(1-q^{-1}\alpha_j\alpha_i)\big)\begin{cases}\prod_{i=1}^n(1+\eta_\pi(-c)q^{-\frac{1}{2}}\alpha_i) & |c|=1,\\ \prod_{i=1}^n(1-q^{-1}\alpha_i^2) & |c|=q^{-1},\end{cases}$$

and define

$$W^0_{\alpha,\psi,c,\theta}(g) = D(\alpha,c)^{-1}\int_{N_{2n}(F)} f^0_{\chi_{\alpha,\psi}}\big((J_{2n}n,1)g\big)\,\theta_c^{-1}(n)\,dn.$$

Lemma 4.1 *For $t = \operatorname{diag}(a_n, a_{n-1}, \dots, a_1) \in T_{GL_n}(F)$, let $k_i = \operatorname{ord}(a_i)$. Then for $c \in \mathbb{O}_F^*$ we have*

$$\begin{aligned} W^0_{\alpha,\psi,c,\theta}\big([t,1],1\big) &= \Delta^{-1}(\alpha)\delta(t)^{\frac{1}{2}}\gamma_\psi^{-1}(t) \\ &\times\begin{cases}A\big(\prod_{i=1}^n(1-\eta_\pi(-c)q^{\frac{-1}{2}}\alpha_i^{-1})\alpha_i^{k_i+i}\big) & 0\le k_1\le k_2\le\dots\le k_n,\\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{4.2}$$

For c such that $|c| = q^{-1}$ we have

$$\begin{aligned} W^0_{\alpha,\psi,c,\theta}(t,1) &= \\ &\Delta^{-1}(\alpha)\delta(t)^{\frac{1}{2}}\gamma_\psi^{-1}(t)\begin{cases}A\big(\prod_{i=1}^n(\alpha_i^{k_i+i}\big) & 0\le k_1\le k_2\le\dots\le k_n,\\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{4.3}$$

In particular, for any $c \in F^*$ whose order is either 0 or 1, $W^0_{\alpha,\psi,c,\theta}$ is a symmetric Spherical θ_c-Whittaker function attached to $I((\chi_\alpha)_\psi)$.

Proof We first prove (4.2). For $c = 1$ this is exactly [3, Theorems 1.1 and 1.2]. Note that we denote by θ what those authors denote by θ^{-1}, see [3, p. 384]. However, this difference is irrelevant, since we use θ^{-1} rather then θ in the Jacquet integral. Given that the Whittaker character θ_c is unramified, the only relation of it with ψ used in the proof of those Theorems is that ψ_c, which is the restriction of θ_c to the long simple root, satisfies $\gamma_\psi = \gamma_{\psi_c}$. Recall that $\gamma_{\psi_c} = \gamma_\psi\eta_c$. This explains why the result for any $c \in \mathbb{O}_F^{*2}$ follows from [3, Theorems 1.1 and 1.2]. For a $c \in \mathbb{O}_F^*$ that is not a square, one can still use the results in [3]: From (1.2) and (3.2) it follows that

$$(\chi_\alpha)_\psi = (\chi_{(c,\pi)\alpha})_{\psi_c} = (\chi_{-\alpha})_{\psi_c}, \quad \gamma_{\psi_c}(t) = \gamma_\psi(t)(-1)^{k_1+k_2+\dots k_n}.$$

Also,

$$\Delta(\alpha) = (-1)^{(1+2+\cdots+n)}\Delta(-\alpha). \tag{4.4}$$

This implies that by changing θ to θ_c, γ_ψ to γ_{ψ_c} and α_i to $-\alpha_i$ in [3, p. 384–386], one obtains the result above.

We shall now explain how to modify the proofs in [3] to prove (4.3). We start by establishing the absolute convergence of the Jacquet integral in some cone in $\mathbb{C}^n$, its holomorphic extension to $\mathbb{C}^n$, and the functional equation satisfied by $W^0_{\alpha,\psi,c,\theta}$, namely (4.1). The only modification one needs in the proof of [3, Theorem 1.1] is to the $\overline{SL_2(F)}$ computation. By a direct computation, similar to one given in [3, p. 387], one proves that if $Re(\alpha) > 0$, then for $t = u\pi^{k_1}$, where $u \in \mathbb{O}_F^*$,

$$\int_F f^0_{(\chi_{\alpha_1})_\psi}\left(\left(\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}, 1\right)\left(\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}, 1\right)\right)\right)\psi_\pi^{-1}(x)dx =$$
$$(1-q^{-1}\alpha_1^2)\delta(t)^{\frac{1}{2}}\gamma_\psi^{-1}(t)\begin{cases}\frac{(a_1^{k_1+1}-a_1^{-k_1-1})}{\alpha_1-\alpha_1^{-1}} & 0 \le k_1\\ 0 & \text{otherwise.}\end{cases}$$

Since the restriction of θ_c to the short simple roots are unramified the rest of the argument follows word for word the proof of [3, Theorem 1.1]. Next, the fact that $W^0_{\alpha,\psi,c,\theta}(t,1)$ vanishes unless $0 \le k_1 \le k_2 \le \cdots \le k_n$ is proved by the same simple argument that is given in [3, p. 392]. Indeed, this argument works for any Whittaker character under the assumption that its restriction to any short simple root subgroup is nontrivial on $\mathbb{P}_F^{-1}$ and that its restriction to the long simple root subgroup is nontrivial on $\mathbb{P}_F^{-2}$. The main remaining ingredient in the proof is an expansion of $f^0_{\chi_{\alpha,\psi}}$ as a linear combination of Iwahori fixed vectors. This step is independent of the Whittaker character. Last, since θ_c is trivial on $N_{2n}(\mathbb{O}_F)$, it follows that

$$\int_{N_{2n}(F)} \phi_J(Ju,1)\theta_c^{-1}(u)\,du = 1,$$

where ϕ_J is the Iwahori fixed vector defined in [3, p. 392]. This gives an exact analog to [3, (3.4), p. 395]. From this formula and from the functional equation established already, the explicit formula (4.3) follows exactly as in [3, pp. 395–396]. ■

Note that as in [3], the normalizing factors $D(\alpha, c)$ are the values of the Jacquet integrals at the identity.

5 Unramified Principal Series Representation of $\overline{GSp_{2n}(F)}$.

5.1 A Standard Model

Let ω be a genuine character of $Z(\overline{T'_{2n}(F)})$. We call ω, $\tau(\omega)$, and $I(\omega)$ unramified if the restriction of ω to $Z(\overline{T'_{2n}(F)}) \cap K^\eta_{2n}(F)$ is trivial. Note that by Lemma 2.2, this definition does not depend on η. From Lemma 3.1(iv) it follows that ω is unramified

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if η is any of the two quadratic characters of $\mathbb{O}_F^*$. exactly two of the four elements of $E(\omega)$ are trivial on $\overline{T^+_{2n}(F)} \cap K^\eta_{2n}(F)$. We call these two extensions, standard extensions. If $(\chi \boxtimes \xi)_\psi$ is one of them, we call $i((\chi \boxtimes \xi)_\psi)$ and $I((\chi \boxtimes \xi)_\psi)$ standard models for $\tau(\omega)$ and $I(\omega)$ respectively. Given that ψ has an even conductor, a genuine character $(\chi \boxtimes \xi)_\psi$ of $\overline{T^+_{2n}(F)}$ has a trivial restriction to $\overline{T^+_{2n}(F)} \cap K^\eta_{2n}(F)$ if and only if it has the form

$$(\chi \boxtimes \xi)_\psi = (\alpha \boxtimes \beta)^+_\psi = (\chi_\alpha \boxtimes \xi_\beta)_\psi,$$

where $\beta \in \mathbb{C}^*$ and ξ_β is the unramified character of F^* defined by $x \mapsto \beta^{\operatorname{ord}(x)}$. Note that the restriction of $(\chi_\alpha \boxtimes \xi_\beta)_\psi$ to $\overline{T_{2n}(F)}$ is $(\chi_\alpha)_\psi$. Also note that the two standard extensions $(\chi_\alpha \boxtimes \xi_\beta)_\psi$ and $(\chi_{\alpha'} \boxtimes \xi_{\beta'})_\psi$ of ω satisfy

$$(\chi_{\alpha'} \boxtimes \xi_{\beta'})_\psi = {}^{(i(u_0,1))}(\chi_\alpha \boxtimes \xi_\beta)_\psi = (\chi_{-\alpha} \boxtimes \xi_{(-1)^n\beta})_\psi.$$

Thus, if $(\alpha \boxtimes \beta)^+_\psi$ is one of the two standard extensions of an unramified genuine character ω of $Z(\overline{T'_{2n}(F)})$ to $\overline{T^+_{2n}(F)}$, then the two nonstandard extensions are

$$(\alpha \boxtimes \beta)^-_\psi = {}^{((\pi),1)}\big((\alpha \boxtimes \beta)^+_\psi\big) \quad \text{and} \quad (-\alpha \boxtimes (-1)^n\beta)^-_\psi.$$

Remark Let ω be a genuine unramified character of $Z(\overline{T'_{2n}(F)})$. Let $E'(\omega)$ be the set of restrictions of the elements of $E(\omega)$ to $\overline{T_{2n}(F)}$. Note that $E'(\omega)$ has four elements. From [22, Section 3.3] it follows that the restriction of $I(\omega)$ to $\overline{Sp_{2n}(F)}$ is isomorphic to

$$\bigoplus_{\chi \in E'(\omega)} \operatorname{Ind}^{\overline{Sp_{2n}(F)}}_{\overline{B_{2n}(F)}} \chi.$$

Exactly two of the four summands here have a $K_{2n}(F)$ invariant element. These are the two representations induced from restrictions of standard extensions of ω. The other two summands have a $K_{2n}(F)^{(i(\pi),1)}$ invariant element. See [7, section 2.6].

Lemma 5.1 *Let ω be a genuine character of $Z(\overline{T'_{2n}(F)})$. $\tau(\omega)$ has a nontrivial $K^\eta_{2n}(F)$-invariant subspace if and only if ω unramified. In this case the dimension of the invariant subspace is* 1. *In particular, if $\eta_1 \neq \eta_2$ are the two quadratic characters of $\mathbb{O}_F^*$, then $\tau(\omega)$ contains a fixed vector under the action of $K^{\eta_1}_{2n}(F)$ if and only if it contains a fixed vector under the action of $K^{\eta_2}_{2n}(F)$.*

Proof Clearly, the existence of a nontrivial $K^\eta_{2n}(F)$ invariant subspace implies that ω is unramified. Conversely, assume that ω is unramified and that $i((\chi \boxtimes \xi)_\psi)$ is a standard model for $\tau(\omega)$. As a set of representatives of

$$\overline{T^+_{2n}(F)} \backslash \overline{T'_{2n}(F)} \;/\; \overline{T'_{2n}(F)} \cap K^\eta_{2n}(F)$$

we can take $\{(I_{2n},1),\,(i(\pi),1)\}$. Note that if $f \in i((\chi \boxtimes \xi)_\psi)$ is invariant under $\overline{T'_{2n}(F)} \cap K^\eta_{2n}(F)$, it follows from Lemma 2.1(i) that for any $([u,1],1) \in \overline{T^+_{2n}(F)} \cap K^\eta_{2n}(F)$ such that $\det(u) \notin F^{*2}$,

$$f\big(i(\pi),1\big) = f\big((i(\pi),1)([u,1],1)\big) = f\big(([u,1],-1)(i(\pi),1)\big) = -f(i(\pi),1).$$

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Thus, $(i(\pi), 1)$ cannot support a $\overline{T'_{2n}(F)} \cap K^{\eta}_{2n}(F)$ invariant function. On the other hand,

$$f(t) = \begin{cases} (\chi \boxtimes \xi)_{\psi}(t^{+}) & \text{if } t = t^{+}u, \text{ where } t^{+} \in \overline{T^{+}_{2n}(F)},\, u \in \overline{T'_{2n}(F)} \cap K^{\eta}_{2n}(F), \\ 0 & \text{otherwise,} \end{cases}$$

defines a non zero $\overline{T'_{2n}(F)} \cap K^{\eta}_{2n}(F)$ invariant function inside $i((\chi \boxtimes \xi)_{\psi})$. ■

From this lemma and from the Iwasawa decomposition it follows that an unramified genuine principal series representation $I(\omega)$ of $\overline{GSp_{2n}(F)}$ contains a one-dimensional $K^{\eta}_{2n}(F)$ invariant subspace. If $I((\chi \boxtimes \xi)_{\psi})$ is a standard model, then

$$(5.1) \quad f^{\eta}_{(\chi\boxtimes\xi)_{\psi}}(g) = \begin{cases} \delta^{\frac{1}{2}}(b)(\chi \boxtimes \xi)_{\psi}(b) & \text{if } g = bk, \text{ where } b \in \overline{B^{+}_{2n}(F)},\, k \in K^{\eta}_{2n}(F), \\ 0 & \text{otherwise} \end{cases}$$

is the unique $K^{\eta}_{2n}(F)$ invariant function in $I((\chi \boxtimes \xi)_{\psi})$ such that $f^{\eta}_{(\chi\boxtimes\xi)_{\psi}}(I_{2n}, 1) = 1$.

5.2 Main Result

Assume that ψ and θ are as in Section 4. For $\alpha \in \mathbb{C}^{*n}$, $\beta \in \mathbb{C}^{*}$ define two functions

$$k^{\eta\,\pm}_{\alpha,\beta} \colon \overline{T'_{2n}(F)} \to \mathbb{C}$$

by

$$k^{\eta\,+}_{\alpha,\beta}(h) = \epsilon\eta\big(\lambda(h)\big)\,\gamma_{\psi}^{-1}(b^{n})\beta^{l}\Delta^{-1}(\alpha)\delta(t)^{\frac{1}{2}}\gamma_{\psi}^{-1}(t) \times \begin{cases} A\big(\prod_{i=1}^{n}(1-\eta_{\pi}(-1)q^{\frac{-1}{2}}\alpha_i^{-1})\alpha_i^{k_i+i}\big) & 0 \le k_1 \le k_2 \le \cdots \le k_n,\ m=0 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$k^{\eta\,-}_{\alpha,\beta}(h) = \epsilon\eta\big(\pi\lambda(h)\big)\,\gamma_{\psi}^{-1}(b^{n})\beta^{l}\Delta^{-1}(\alpha)\delta(t)^{\frac{1}{2}}\gamma_{\psi}^{-1}(t) \times \begin{cases} A\big(\prod_{i=1}^{n}\alpha_i^{k_i+i}\big) & 0 \le k_1 \le k_2 \le \cdots \le k_n,\ m=1 \\ 0 & \text{otherwise.} \end{cases}$$

Here

$$(5.2) \qquad h = \big(i(\pi^{-m}), 1\big)\,\big(bI_{2n}, 1\big)\,\big([t, 1], 1\big)\,\big(i(u), \epsilon\big),$$

where $m \in \{0, 1\}$, $b \in F^{*}$, $u \in \mathbb{O}_F^{*}$, $t = \operatorname{diag}(a_n, a_{n-1}, \dots, a_1) \in T_{GL_n}(F)$ and where we denote $\operatorname{ord}(a_i) = k_i$, $\operatorname{ord}(b) = l$ (note that $\lambda(h) = b^{2}u\pi^{m}$). We now extend these functions to

$$k^{\eta\,\pm}_{\alpha,\beta,\theta} \colon \overline{GSp_{2n}(F)} \to \mathbb{C}$$

by setting
$$k^{\eta\,\pm}_{\alpha,\beta,\theta}\big((n,1)hu\big) = \theta(n)k^{\eta}_{\alpha,\beta}{}^{\pm}(h),$$
where $n \in N_{2n}(F)$, $u \in K^{\eta}_{2n}(F)$. From Theorem 5.2 it will follow that these extensions are well defined. Observe that this definition implies that these are symmetric functions; *i.e.*,

$$k^{\eta\,\pm}_{\alpha,\beta,\theta} = k^{\eta\,\pm}_{w_{\alpha,\beta,\theta}} \tag{5.3}$$

for any $w \in W_{2n}$.

Theorem 5.2 *Let $I(\omega)$ be a genuine unramified principal series representation of $\overline{GSp_{2n}(F)}$. Suppose*

$$E(\omega) = \{(\alpha \boxtimes \beta)^{\pm}_{\psi},\ (-\alpha \boxtimes (-1)^n\beta)^{\pm}_{\psi}\}.$$

Then the set

$$\{k^{\eta\,\pm}_{\alpha,\beta,\theta},\ k^{\eta\,\pm}_{-\alpha,(-1)^n\beta,\theta}\} \tag{5.4}$$

forms a symmetric spanning set for the space of genuine Spherical θ-Whittaker functions attached to $I(\omega)$.

Proof Denote
$$C = \{1, u_0, \pi, \pi u_0\}.$$
As explained in Section 1, C is a set of representatives of F^*/F^{*2}. For $c \in C$ define

$$D^{\eta}(\alpha, c) = \eta(c\pi^{-\operatorname{ord}(c)})D(\alpha, c)$$

and define

$$W^{\eta}_{(\alpha\boxtimes\beta)_{\psi},c,\theta}(g) = D^{\eta}(\alpha, c)^{-1} \int_{N_{2n}(F)} f^{\eta}_{(\chi_\alpha\boxtimes\xi_\beta)_{\psi}}\big((J_{2n}n, 1)(i(c), 1)g\big)\,\theta_c^{-1}(n)\,dn.$$

By Lemma 3.8, $\{W^{\eta}_{(\alpha\boxtimes\beta)_{\psi},c,\theta} \mid c \in C\}$ is a spanning set for the space of genuine Spherical θ-Whittaker functions attached to $I(\omega)$. Thus, the proof of Theorem 5.2 amounts to showing the following:

$$W^{\eta}_{(\alpha\boxtimes\beta)_{\psi},1,\theta}(h) = k^{\eta\,+}_{\alpha,\beta,\theta}(h), \tag{5.5}$$

$$W^{\eta}_{(\alpha\boxtimes\beta)_{\psi},\pi,\theta}(h) = k^{\eta\,-}_{\alpha,\beta,\theta}(h), \tag{5.6}$$

$$W^{\eta}_{(\alpha\boxtimes\beta)_{\psi},u_0,\theta} = W^{\eta}_{(\chi_{-\alpha}\boxtimes\xi_{(-1)^n\beta})_{\psi},1\theta} \tag{5.7}$$

and

$$W^{\eta}_{(\alpha\boxtimes\beta)_{\psi},\pi u_0,\theta} = W^{\eta}_{(\chi_{-\alpha}\boxtimes\xi_{(-1)^n\beta})_{\psi},\pi,\theta}. \tag{5.8}$$

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Here h is the $\overline{T'_{2n}(F)}$ element defined in (5.2).

By (5.1), $f^{\eta}_{(\chi_\alpha \boxtimes \xi_\beta)_\psi}(g)$ vanishes if $\lambda(g) \notin \mathbb{O}_F^* F^{*2}$. This implies that $W^{\eta}_{(\alpha\boxtimes\beta)_\psi,y,\theta}(h)$ vanish if $m \neq \operatorname{ord}(c)$. We compute $W^{\eta}_{(\alpha\boxtimes\beta)_\psi,1,\theta}(h)$. Recall that $Z(\overline{GSp^{+}_{2n}(F)}) = \overline{F^* I_{2n}}$ and that

$$f^{\eta}_{(\chi_\alpha \boxtimes \xi_\beta)_\psi}\big(g(i(k),\epsilon)\big) = \epsilon\eta(k) f^{\eta}_{(\chi_\alpha \boxtimes \xi_\beta)_\psi}(g).$$

Thus, if $m = 0$, then

$$W^{\eta}_{(\alpha\boxtimes\beta)_\psi,1,\theta}(h) = \epsilon\eta(u)\gamma_\psi^{-1}(b^n)\beta^l W^{\eta}_{(\alpha\boxtimes\beta)_\psi,1,\theta}\big(t,1\big).$$

Since the restriction of $W^{\eta}_{(\alpha\boxtimes\beta)_\psi,1,\theta}$ to $\overline{Sp_{2n}(F)}$ is $W^{0}_{\alpha,\psi,1,\theta}$, (5.5) now follows from (4.2). To compute $W^{\eta}_{(\alpha\boxtimes\beta)_\psi,u_0,\theta}(h)$ we note that

$$\big(i(u_0),1\big)\, h \big(i(u_0),1\big)^{-1} = \big(I_{2n}, (u_0, b^n a_1, a_2, \dots, a_n)_F\big)\, h$$

and that

$$(u_0, b^n a_1, a_2, \dots, a_n)_F = (-1)^{(ln+k_1+k_2+\cdots+k_n)}.$$

Thus, arguing as before, if $m = 0$,

$$W^{\eta}_{(\alpha\boxtimes\beta)_\psi,u_0,\theta}(h) = (-1)^{(ln+k_1+k_2+\cdots+k_n)}\eta(u)\gamma_\psi^{-1}(b^n)\beta^l\Delta^{-1}(\alpha)\delta(t)^{\frac12}\gamma_\psi^{-1}(t)$$
$$\times \begin{cases} A\big(\prod_{i=1}^{n}(1-\eta_\pi(-u_0)q^{\frac{-1}{2}}\alpha_i^{-1})\alpha_i^{k_i+i}\big) & 0 \le k_1 \le k_2 \le \cdots \le k_n, \\ 0 & \text{otherwise.} \end{cases}$$

Now, (5.7) follows from (1.2) and (4.4), and (5.6) and (5.8) are proved by similar arguments that ultimately utilize (4.3). ■

Note that from the proof above it follows that for $c \in C$,

$$D^{\eta}(\alpha,c) = \int_{N_{2n}(F)} f^{\eta}_{(\chi_\alpha \boxtimes \xi_\beta)_\psi}\big((J_{2n}n,1)(i(c\pi^{-\operatorname{ord}(c)}),1)g\big)\,\theta_c^{-1}(n)\,dn.$$

Theorem 5.3 *If n is odd, then the spanning set constructed in Theorem 5.2 has an additional property. Let ω' be the central character of $I(\omega)$. Then ω' is the restriction of ω to $Z(\overline{GSp_{2n}(F)})$. Define Ω to be the set of four extensions of ω' to $Z(\overline{GSp^{+}_{2n}(F)})$. For each μ in Ω exactly one of the functions in (5.4) has the property*

$$f(tg) = \mu(t) f(g)$$

for any $t \in Z(\overline{GSp^{+}_{2n}(F)})$, $g \in \overline{GSp_{2n}(F)}$.

Proof This is verified at once by direct computation using the cocycle properties given in Lemma 2.1. In fact, it follows from Corollary 3.7. ■

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6 Reducible Unramified Principal Representations

Fix ψ, an unramified character of F. Throughout this section, let ω be an unramified genuine character of $Z(\overline{T'_{2n}(F)})$ and let $I(\omega)$ be an unramified principal series representation of $\overline{GSp_{2n}(F)}$ with $I((\chi_\alpha \boxtimes \xi_\beta)_\psi)$ and $I((\chi_{-\alpha} \boxtimes \xi_{(-1)^n\beta})_\psi)$ as its two standard models.

Lemma 6.1 *We have that $R(\omega) > 1$ if and only if n is even and, up to conjugation by elements of W_{2n}, α equals*

$$\alpha_0 = (\alpha_1, -\alpha_1, \alpha_2, -\alpha_2, \dots, \alpha_{\frac{n}{2}}, -\alpha_{\frac{n}{2}}). \tag{6.1}$$

In this case $R(\omega) = 2$.

Proof Since W_{2n} maps a standard extension of an unramified genuine charter of $Z(\overline{T'_{2n}(F)})$, to another standard extension of an unramified genuine charter of $Z(\overline{T'_{2n}(F)})$ it follows that $R(\omega) \leq 2$. The rest of this lemma follows from Lemma 3.3. ∎

Lemma 6.2 *If $R(\omega) > 1$, then the dimension of the space of Whittaker functionals on the space generated by the $K^\eta_{2n}(F)$-invariant subspace inside $I(\omega)$ is 2.*

Proof By Lemma 6.1, $R(\omega) > 1$ implies that n is even and that ${}^w\alpha = -\alpha$ for some $w \in W_{2n}$. Thus, by (5.3)

$$k^{\eta\,+}_{\alpha,\beta} = k^{\eta\,+}_{-\alpha,\beta}, \quad k^{\eta\,-}_{\alpha,\beta} = k^{\eta\,-}_{-\alpha,\beta}.$$

On the other hand $k^{\eta\,+}_{\alpha,\beta}$ and $k^{\eta\,-}_{\alpha,\beta}$ are linearly independent, since these two functions have disjoint supports. Hence, the lemma follows from the symmetry property of the spanning set constructed in Theorem 5.2. ∎

From this point we assume that n is even. For a character χ of $T_{2n}(F)$ and

$$\mathbf{s} = (s_1, s_2, \dots, s_n) \in \mathbb{C}^{*n},$$

let $\chi^{\mathbf{s}}$ be the character of $T_{2n}(F)$ given by

$$t = \big[\operatorname{diag}(t_n, t_{n-1}, \dots, t_1), 1\big] \mapsto \chi(t) \prod_{i=1}^{n} |t_i|^{s_i}.$$

Let w_0 be the W_{2n} element defined in (3.7). For $s = (s_1, s_2, \dots, s_n) \in \mathbb{C}^{*n}$ define a section $f_{\mathbf{s}} \in I((\chi^{\mathbf{s}})_\psi)$ and define a complex function $A_{\chi,\mathbf{s}}(f_{\mathbf{s}}) = A_{\mathbf{s}}(f_{\mathbf{s}})$ on $\overline{Sp_{2n}(F)}$ by

$$\big(A_{\mathbf{s}}(f_{\mathbf{s}})\big)(g) = \int_{N_{2n}(F)\cap w_0 N^-_{2n}(F) w_0^{-1}} f_{\mathbf{s}}\big((w_0 n, 1)g\big)\, dn. \tag{6.2}$$

Here $N^-_{2n}(F) = J_{2n} N_{2n}(F) J^{-1}_{2n}$ is the unipotent radical of $\overline{Sp_{2n}(F)}$ opposite to $N_{2n}(F)$.

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Lemma 6.3

(i) *The integral defined in (6.2) converges in some cone inside $\mathbb{C}^{*n}$ and has a meromorphic extension to $\mathbb{C}^{*n}$. Away from its poles it defines an $\overline{Sp_{2n}(F)}$ intertwining map*

$$A_s\colon I\big((\chi^{\mathbf{s}})_\psi\big) \to I\big({}^{w_0}(\chi^{\mathbf{s}})_\psi\big).$$

(ii) *Let χ_0 be as in (3.6). Then $A_{\chi_0,s}$ is analytic at $s=0$, i.e., at χ_0.*

(iii) *Let $\chi_0=\chi_{\alpha_0}$, where α_0 is as in (6.1), then*

$$A_{\chi_{\alpha_0},s}\big(f^0_{(\chi_{\alpha_0})_\psi}\big) = \Big(\frac{L(\eta_{u_0},0)}{L(\eta_{u_0},1)}\Big)^{\frac{n}{2}}\big(f^0_{(\chi_{-\alpha_0})_\psi}\big).$$

Proof (i) is well known. It is proved by the same standard arguments used for linear groups. To prove (ii) we can assume the g in (6.2) is $(I_{2n},1)$. Also by a standard argument, we decompose A_s to a product of rank one intertwining operators:

$$A_s = A_s^{\frac{n}{2}}\circ A_s^{\frac{n-1}{2}}\circ\cdots\circ A_s^1,$$

where

$$A_s^j\colon I\big({}^{w_{j-1}w_{j-2}\cdots w_1}(\chi_0^{\mathbf{s}})_\psi\big) \to I\big({}^{w_jw_{j-1}\cdots w_1}(\chi_0^{\mathbf{s}})_\psi\big).$$

is defined by (the meromorphic continuation of)

$$\big(A_s^j(f_{\mathbf{s}})\big)(g) = \int_F f_{\mathbf{s}}\big((w_0^j n^j(x),1)g\big)\,dx.$$

Here

$$w_0^j = \operatorname{diag}\big(I_{n-2j},w',I_{2(j-1)},I_{n-2j},w',I_{2(j-1)}\big),$$

where $w'=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and

$$n^j(x) = \operatorname{diag}\big(I_{n-2j},n'(x),I_{2(j-1)},I_{n-2j},{}^t n(-x),I_{2(j-1)}\big),$$

where $n'(x)=\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)$. Since the $\mathbb{C}^1$ cover of $Sp_{2n}(F)$ splits over the Siegel parabolic subgroup, the proof of (ii) and (iii) is now reduced to well-known $GL_2(F)$ computations. More precisely, by Rao cocycle Formula given in [16, Theorem 5.3],

$$c\Big(\begin{pmatrix}A&B\\0&{}^tA^{-1}\end{pmatrix},\begin{pmatrix}A'&B'\\0&{}^tA'^{-1}\end{pmatrix}\Big) = \big(\det(A),\det(A')\big)_F. \tag{6.3}$$

For $f\in I((\chi^{\mathbf{s}})_\psi)$ and $g\in GL_2(F)$ define

$$\big(C_j(f)\big)(g) = \gamma_\psi(g)|\det(g)|^{\frac{1}{2}-2j} f\big(I_{n-2j},g,I_{2(j-1)},I_{n-2j},{}^tg^{-1},I_{2(j-1)}\big).$$

From (1.1) and (6.3) it follows that

$$C^j(f)\in \operatorname{Ind}_{B(F)}^{GL_2(F)}(\chi^{\mathbf{s}})^j.$$

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Here $B(F)$ is the standard Borel subgroup of $GL_2(F)$ and

$$(\chi^{\mathbf{s}})^j(t_2,t_1) = \chi^{\mathbf{s}}\big(I_{n-2j}, t_2, t_1, I_{2(j-1)}, I_{n-2j}, t_2^{-1}, t_1^{-1}, I_{2(j-1)}\big).$$

(the term $|\det(g)|^{\frac{1}{2}-2j}$ appears here in order to balance the difference between the modular functions of $B_{2n}(F)$ and $B(F)$). The point is that for $f \in I((\chi^{\mathbf{s}})_\psi)$,

$$\big(A^j_{\chi,\mathbf{s}}(f)\big)(I_{2n},1) = \big(A_{(\chi,\mathbf{s})^j}(C_j(f))\big)(I_2),$$

where $A_{(\chi,\mathbf{s})^j}$ is the standard $GL_2(F)$ intertwining integral; see [2, p. 478], for example. This completes the reduction to $GL_2(F)$. (ii) now follows from [2, Proposition 4.5.7], and (iii) follows from [2, Proposition 4.6.7]. ■

Corollary 6.4 *For $f \in I((\chi_{\alpha_0} \boxtimes \xi_\beta)_\psi)$ and $g \in \overline{GSp_{2n}(F)}$, define $(A(f))(g)$ to be the meromorphic continuation of*

$$\big(A(f)\big)(g) = \int_{N_{2n}(F)\cap w_0 N^-_{2n}(F) w_0^{-1}} f\big((w_0 n, 1)g\big)\, dn.$$

The map

$$A\colon I\big((\chi_{\alpha_0} \boxtimes \xi_\beta)_\psi\big) \to I\big((\chi_{-\alpha_0} \boxtimes \xi_\beta)_\psi\big)$$

is a well defined $\overline{GSp_{2n}(F)}$ intertwining map and

$$A\big(f^\eta_{(\chi_{\alpha_0}\boxtimes\xi_\beta)_\psi}\big) = \Big(\frac{L(\eta_{u_0},0)}{L(\eta_{u_0},1)}\Big)^{\frac{n}{2}} f^\eta_{(\chi_{-\alpha_0}\boxtimes\xi_\beta)_\psi}.$$

Proof First, by using the fact that $\overline{GSp^+_{2n}(F)} = Z(\overline{GSp^+_{2n}(F)})\overline{Sp_{2n}(F)}$, we extend $A_{\mathbf{s}}$ to be a $\overline{GSp^+_{2n}(F)}$ intertwining map from $I'((\chi_{\alpha_0} \boxtimes \xi_\beta)_\psi)$ to $I'((\chi_{-\alpha_0} \boxtimes \xi_\beta)_\psi)$ (the notation I' was defined in (3.8)). Then, using the same induction by stages argument used in the beginning of proof of Theorem 3.4, we push $A_{\mathbf{s}}$ to be the $\overline{GSp_{2n}(F)}$ intertwining map defined in this corollary. Lemma 6.3(ii) guarantees that A is holomorphic at $(\chi_{\alpha_0} \boxtimes \xi_\beta)_\psi$.

Since the $K^\eta_{2n}(F)$ invariant space inside $I((\chi_\alpha \boxtimes \xi_\beta)_\psi)$ is one dimensional it follows that

$$A\big(f^\eta_{(\chi_{\alpha_0}\boxtimes\xi_\beta)_\psi}\big) = c_{\alpha_0} f^\eta_{(\chi_{-\alpha_0}\boxtimes\xi_\beta)_\psi},$$

where c_{α_0} is the analytic continuation of

$$\int_{N_{2n}(F)\cap w_0 N^-_{2n}(F) w_0^{-1}} f^\eta_{(\chi_{\alpha_0}\boxtimes\xi_\beta)_\psi}(w_0 n)\, dn.$$

Note that since for both quadratic characters of $\mathbb{O}^*_F$, η, the restriction of $f^\eta_{(\chi_\alpha\boxtimes\xi_\beta)_\psi}$ to $\overline{Sp_{2n}(F)}$ is $f^0_{(\chi_\alpha)_\psi}$, it follows that c_{α_0} does not depend on η. In fact, by Lemma 6.3(iii)

$$c_{\alpha_0} = \Big(\frac{L(\eta_{u_0},0)}{L(\eta_{u_0},1)}\Big)^{\frac{n}{2}} \neq 0.$$ ■

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Recall that η_1 and η_π are respectively the trivial and nontrivial quadratic characters of $\mathbb{O}_F^*$. We shall denote

$$K_{2n}^+(F) = K_{2n}^{\eta_1}(F), \quad K_{2n}^-(F) = K_{2n}^{\eta_{u_0}}(F).$$

We shall also denote by $I^+(\omega)$ and $I^-(\omega)$ the (isomorphism classes) of the subrepresentations of $I(\omega)$ generated by the $K_{2n}^+(F)$ and $K_{2n}^-(F)$ invariant subspaces.

Theorem 6.5 *Assume that $I((\chi_\alpha)_\psi)$ is an irreducible $\overline{Sp_{2n}(F)}$ module. Then $I(\omega)$ is reducible if and only if $R(\omega) > 1$. In this case $I^+(\omega)$ and $I^-(\omega)$ are irreducible, and*

$$I(\omega) \simeq I^+(\omega) \oplus I^-(\omega)$$

is a decomposition of $I(\omega)$ into a direct sum of two irreducible nonisomorphic generic subspaces, each having a space of Whittaker functionals of dimension 2.

Proof Given Theorem 3.4 and Lemma 6.2, we only need to show that $I^+(\omega) \neq I^-(\omega)$. Since $I((\chi_\alpha)_\psi)$ is irreducible, we can conjugate the inducing character by $w \in W_{2n}$ without changing the isomorphism class of $I((\chi_\alpha)_\psi)$. Hence, since $R(\omega) > 1$, it follows from Lemma 6.1 that we can assume $\alpha = \alpha_0$. Let

$$f^+_{(\chi_\alpha \boxtimes \xi_\beta)_\psi} = f^{\eta_1}_{(\chi_\alpha \boxtimes \xi_\beta)_\psi} \quad \text{and} \quad f^-_{(\chi_\alpha \boxtimes \xi)_\psi} = f^{\eta_\pi}_{(\chi_\alpha \boxtimes \xi)_\psi}.$$

We shall construct here a self intertwining map T on $I((\chi_\alpha \boxtimes \xi_\beta)_\psi)$ and show that $f^{\pm}_{(\chi_\alpha \boxtimes \xi_\beta)_\psi}$ are eigenvectors of T corresponding to two different eigenvalues. We define

$$B\colon I\big((\chi_{-\alpha} \boxtimes \xi_\beta)_\psi\big) \to I\big((\chi_\alpha \boxtimes \xi_\beta)_\psi\big)$$

by

$$B\big(f\big)(g) = f\big((i(u_0, 1)g\big).$$

By the same argument as in Corollary 6.4,

$$B\big(f^{\eta}_{(\chi_{-\alpha} \boxtimes \xi_\beta)_\psi}\big) = d_\eta f^{\eta}_{(\chi_\alpha \boxtimes \xi_\beta)_\psi},$$

where this time d_η does depend on η and is defined by

$$d_\eta = f^{\eta}_{(\chi_{-\alpha} \boxtimes \xi_\beta)_\psi}\big((i(u_0, 1)\big) = \eta(u_0).$$

Let A be the intertwining map defined in Corollary 6.4. Then

$$B \circ A\colon I\big((\chi_\alpha \boxtimes \xi_\beta)_\psi\big) \to I\big((\chi_\alpha \boxtimes \xi_\beta)_\psi\big)$$

is a self intertwining operator, and by Corollary 6.4

$$B \circ A\big(f^+_{(\chi_\alpha \boxtimes \xi_\beta)_\psi}\big) = \Big(\frac{L(\eta_{u_0}, 0)}{L(\eta_{u_0}, 1)}\Big)^{\frac{n}{2}} f^+_{(\chi_\alpha \boxtimes \xi_\beta)_\psi},$$

$$B \circ A\big(f^-_{(\chi_\alpha \boxtimes \xi_\beta)_\psi}\big) = -\Big(\frac{L(\eta_{u_0}, 0)}{L(\eta_{u_0}, 1)}\Big)^{\frac{n}{2}} f^-_{(\chi_\alpha \boxtimes \xi_\beta)_\psi}. \qquad \blacksquare$$

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Recall that if the restriction of ω to $Z(\overline{T'_{2n}(F)}) \cap \overline{Sp_{2n}(F)}$ is unitary, then $I((\chi_\alpha)_\psi)$ is irreducible. Hence, we have proved the following corollary.

Corollary 6.6 *All of the reducible unramified principal series representations of $\overline{GSp_{2n}(F)}$ induced from a unitary data are a direct sum of two nonisomorphic irreducible generic subspaces. Each has a two-dimensional space of Whittaker functionals. For each of the two quadratic characters of $\mathbb{O}_F^*$, η, exactly one of the summands has a one-dimensional $K_{2n}^{\eta}(F)$ invariant vector.*

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References

[1] W. D. Banks, *Exceptional representations on the metaplectic group*. Ph.D. Thesis, Stanford University. ProQuest LLC, Ann Arbor, MI, 1994.

[2] D. Bump, *Automorphic forms and representations*. Cambridge Studies in Advanced Mathematics, 55, Cambridge University Press, Cambridge, 1997.

[3] D. Bump, S. Friedberg, and J. Hoffstein, *p-adic Whittaker functions on the metaplectic group*. Duke Math. J. **63**(1991), no. 2, 379–397. http://dx.doi.org/10.1215/S0012-7094-91-06316-7

[4] I. N. Bernstein and A. V. Zelevinsky, *Representations of the group $GL(n, F)$ where F is a non-archimedean local field.* Russian Math. Surveys **31**(1976), 1–68.

[5] W. Casselman and J. Shalika, *The unramified principal series of p-adic groups. II. The Whittaker function.* Compositio Math. **41**(1980), no. 2, 207–231.

[6] G. Chinta and O. Offen, *A metaplectic Casselman-Shalika formula for GL(r).* Amer. J. Math. **135**(2013), no. 2, 403–441. http://dx.doi.org/10.1353/ajm.2013.0013

[7] W. T. Gan and G. Savin, *Representations of metaplectic groups I: epsilon dichotomy and local Langlands correspondence.* Compositio Math. **148**(2012), no. 6, 1655–1694. http://dx.doi.org/10.1112/S0010437X12000486

[8] S. Gelbart, R. Howe, I. Piatetski-Shapiro, *Uniqueness and existence of Whittaker models for the metaplectic group.* Israel J. Math. **34**(1979), no. 1–2, 21–37. http://dx.doi.org/10.1007/BF02761822

[9] S. Gelbart and I. Piatetski-Shapiro, *Some remarks on metaplectic cusp forms and the correspondences of Shimura and Waldspurger.* Israel J. Math. **44**(1983), no. 2, 97–126. http://dx.doi.org/10.1007/BF02760615

[10] M. Hanzer and I. Matic, *Irreducibility of the unitary principal series of p-adic $\widetilde{Sp(n)}$*. Manuscripta Math. **132**(2010), no. 3-4, 539–547. http://dx.doi.org/10.1007/s00229-010-0362-4

[11] D. A. Kazhdan and S. J. Patterson, *Metaplectic forms*. Inst. Hautes Études Sci. Publ. Math. **59**(1984) 35–142.

[12] C. D. Keys and F. Shahidi, *Artin L-functions and normalization of intertwining operators.* Ann. Sci. École Norm. Sup. (4) **21**(1988), no. 1, 67–89.

[13] P. J. McNamara, *Principal series representations of metaplectic groups over local fields.* In: Multiple Dirichlet series, *L*-functions and automorphic forms, Progr. Math., 300, Birkhäuser/Springer, New York, 2012.

[14] C. Mœglin, M.-F. Vignéras, and J. L. Waldspurger, *Correspondances de Howe sur un corps p-adique.* Lecture Notes in Mathematics, 1291, Springer-Verlag, Berlin, 1987.

[15] C. Mœglin and J. L. Waldspurger, *Spectral decomposition and Eisenstein series.* Cambridge Tracts in Mathematics, 113, Cambridge University Press, Cambridge, 1995.

[16] R. R. Rao, *On some explicit formulas in the theory of Weil representation.* Pacific J. Math. **157**(1993), no. 2, 335–371. http://dx.doi.org/10.2140/pjm.1993.157.335

[17] B. Roberts, *The theta correspondence for similitudes.* Israel J. Math. **94**(1996), 285–317. http://dx.doi.org/10.1007/BF02762709

[18] F. Shahidi, *On certain L-functions.* Amer. J. Math. **103**(1981), no. 2, 297–355. http://dx.doi.org/10.2307/2374219

[19] D. Szpruch, *Uniqueness of Whittaker model for the metaplectic group.* Pacific J. Math. **232**(2007), no. 2, 453–469. http://dx.doi.org/10.2140/pjm.2007.232.453
[20] ———, *Computation of the local coefficients for principal series representations of the metaplectic double cover of $SL_2(F)$.* J. Number Theory, **129**(2009), no. 9, 2180–2213. http://dx.doi.org/10.1016/j.jnt.2009.01.024
[21] ———, *Some irreducibility theorems of parabolic induction on the metaplectic group via the Langlands-Shahidi method.* Israel J. Math. **195**(2013), no. 2, 897–971. http://dx.doi.org/10.1007/s11856-012-0140-y
[22] ———, *Some results in the theory of genuine representations of the metaplectic double cover of GSp(2,F) over p-adic fields.* J. Algebra **388**(2013), 160–193. http://dx.doi.org/10.1016/j.jalgebra.2013.05.001
[23] A. Weil, *Sur certains groupes d'opérateurs unitaires.* Acta Math **111**(1964), 143–211. http://dx.doi.org/10.1007/BF02391012

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