9

The Polyakov Proof of Confinement

In a totally surprising result, Polyakov demonstrated that instantons could provide the key to confinement in a particular model in 2+1 dimensions [103]. In this chapter, we will study in detail the Polyakov proof of confinement. We will see that it requires a mild non-Abelian aspect to the theory, but the confinement occurs essentially because of the existence of magnetic monopole solitons in the theory. Purely Abelian gauge theory also contains magnetic monopoles, but they are singular configurations of infinite energy, and hence of no import. The minor non-Abelian excursion simply allows for the existence of finite action (or energy) magnetic monopoles.

9.1 Georgi–Glashow model

We continue our study of quantum electrodynamics in $2+1$ dimensions; however, now we shall consider a theory that is Abelian at low energy but non-Abelian at high energy. This occurs due to spontaneous symmetry breaking. We consider a non-Abelian gauge theory with gauge group $O(3) \sim SU(2)$ spontaneously broken to $U(1)$. The model is the $2+1$-dimensional version of the Georgi–Glashow model [54]. The fields correspond to an iso-triplet of scalar fields interacting via non-Abelian gauge fields and self-interactions, the Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4e^2} F^a_{\mu\nu} F^{a\mu\nu} + |D_\mu \phi|^2 - \frac{1}{4}\lambda \left(|\phi|^2 - a^2\right)^2, \tag{9.1}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu$$

$$\phi = \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \end{pmatrix}, \quad (D_\mu \phi)^a = \partial_\mu \phi^a + \epsilon^{abc} A^b_\mu \phi^c$$

$$|\phi|^2 = \phi^a \phi^a. \tag{9.2}$$

The theory is invariant under local redefinition of the fields by

$$\begin{aligned} \phi^a &\to R^{ab}(x^\nu)\phi^b \\ A^a_\mu &\to R^{ab}(x^\nu)A^b_\mu + \epsilon^{abc}R^{bd}(x^\nu)\partial_\mu R^{cd}(x^\nu), \end{aligned} \tag{9.3}$$

where $R^{ab}(x^\nu)$ is a smooth, $O(3)$-valued gauge transformation.

We may sometimes wish to use the matrix notation, hence we record the corresponding formulae here. The Higgs field is written as ϕ, which is a three-real entry column. The gauge field is a 3×3 real, anti-symmetric matrix A_μ for each spacetime index μ. There are exactly three independent anti-symmetric 3×3 matrices where a basis can be denoted as T^a with components numerically given by $T^a_{bc} = \epsilon^{abc}$ (here the placement of the index as upper or lower is of no import). Then $A_\mu = A^a_\mu T^a$. Then the gauge transformation is written as

$$\begin{aligned} \phi &\to R(x^\nu)\phi \\ A_\mu &\to R(x^\nu)A_\mu + R(x^\nu)\partial_\mu R^T(x^\nu), \end{aligned} \tag{9.4}$$

where $R(x^\nu)$ is a 3×3 orthogonal matrix (hence its inverse is given by its transpose).

We can easily see the perturbative, physical particle spectrum of the theory by making a choice of gauge

$$\phi^1 = \phi^2 = 0. \tag{9.5}$$

To be honest, this is an incomplete gauge-fixing condition: it does not fix the gauge degree of freedom if ϕ is already in the three-direction and it does not fix the gauge transformations which leave ϕ_3 invariant. However, it is sufficient for us to extract the particle spectrum. Then, replacing $\phi^3 = a + \eta$ we have:

$$\begin{aligned} (D_\mu\phi)^1 &= \partial_\mu\phi^1 + \epsilon^{1bc}A^b_\mu\phi^c = \epsilon^{123}A^2_\mu\phi^3 = A^2_\mu(a+\eta) \\ (D_\mu\phi)^2 &= \partial_\mu\phi^2 + \epsilon^{2bc}A^b_\mu\phi^c = \epsilon^{213}A^1_\mu\phi^3 = -A^1_\mu(a+\eta) \\ (D_\mu\phi)^3 &= \partial_\mu\phi^3 + \epsilon^{3bc}A^b_\mu\phi^c = \partial_\mu(a+\eta) = \partial_\mu\eta. \end{aligned} \tag{9.6}$$

Hence

$$|D_\mu\phi|^2 = \partial_\mu\eta\partial^\mu\eta + \left(A^1_\mu A^{1\mu} + A^2_\mu A^{2\mu}\right)\left(a^2 + 2a\eta + \eta^2\right), \tag{9.7}$$

giving the Lagrangian density

$$\begin{aligned} \mathcal{L} = &-\frac{1}{4e^2}F^a_{\mu\nu}F^{a\mu\nu} + \partial_\mu\eta\partial^\mu\eta + \partial_\mu\eta\partial^\mu\eta \\ &+ \left(A^1_\mu A^{1\mu} + A^2_\mu A^{2\mu}\right)\left(a^2 + 2a\eta + \eta^2\right) - \frac{1}{4}\lambda\left(2a\eta + \eta^2\right)^2. \end{aligned} \tag{9.8}$$

This yields the quadratic part

$$\mathcal{L} = \frac{-1}{2e^2}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^{a\nu}\right) + \partial_\nu\eta\partial^\mu\eta + \left(A^1_\mu A^{1\mu} + A^2_\mu A^{2\mu}\right)a^2 - \lambda a\eta^2. \tag{9.9}$$

The physical particle spectrum can now be read off from this equation; it corresponds to a massless vector field A^3_μ, two massive vector fields A^1_μ and A^2_μ

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of mass $M^2 = 4e^2a^2$, and a neutral massive scalar field η (neutral with respect to the gauge field A^3_μ) with mass $m^2 = \lambda a$. η is neutral since it does not couple to A^3_μ, while the massive vector fields A^1_μ and A^2_μ are charged as they do. The two fields ϕ_1 and ϕ_2 are, of course, absent. We might say this is due to our gauge choice; however, the fact that the corresponding physical excitations do not exist is independent of the gauge choice. What we are describing is the classic Higgs mechanism [61], where the putative massless Goldstone bosons associated with spontaneous symmetry-breaking are swallowed by the gauge bosons that correspond to the broken symmetry directions. These gauge bosons consequently become massive. Hence the Goldstone bosons are absent, but their degrees of freedom show up in the additional degrees of freedom of the massive vector bosons (as opposed to massless ones).

We will see in this chapter that, as in the case of the Abelian Higgs model in 1+1 dimensions in Chapter 8, the actual spectrum of the theory does not correspond to this naive spectrum. We will find that the theory in fact confines charged states due to the effects of instantons and that there are no massless states, especially there is no massless photon. The validity of the argument that the Wilson loop is able to subtend an appreciable amount of flux from the instantons, which was used in Chapter 8, becomes critical in $2+1$ dimensions. As the size of the Wilson loop becomes large, it can subtend an arbitrary amount of flux from nearby instantons, and hence the effect of instantons is significant. In 3+1 dimensions we will see that the argument fails.

9.2 Euclidean Theory

Analytically continuing our action to three-dimensional Euclidean space (although much of what we say is trivially generalized to d Euclidean dimensions) gives

$$S_E = \int d^3x \left(\frac{1}{4e^2} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2}(D_\mu\phi)^a (D_\mu\phi)^a + \frac{1}{4}(\phi^a\phi^a - a^2)^2 \right), \tag{9.10}$$

which is again composed of three positive semi-definite terms. We look for finite action configurations: these would correspond to instantons and should be relevant for tunnelling. Finite action requires that the fields behave in such a way that each term in the action goes to zero sufficiently fast at infinity, as each term is positive semi-definite. Sufficiently fast can include $\sim 1/r$ fall off of particular fields or their derivatives, the only condition is that the Euclidean action be finite, and hence each term vanishes sufficiently fast. This then implies that at infinity

$$\phi^a \to R^{ab}_\phi(\Omega)\phi^a_0 \qquad \phi^a_0\phi^a_0 = a^2 \tag{9.11}$$

$$(D_\mu\phi)^a \to 0 \tag{9.12}$$

$$F^a_{\mu\nu} \to 0, \tag{9.13}$$

where Ω are the angular coordinates parametrizing the sphere at infinity. Equation (9.13) requires that the gauge fields approach a configuration that corresponds to a pure gauge transformation of the vacuum, sufficiently fast. We can write the gauge field in a matrix notation

$$A_\mu = A_\mu^a T^a, \tag{9.14}$$

where T^a are 3×3 matrices with components numerically given by $T^a_{bc} = \epsilon^{abc}$. Then Equation (9.13) implies, in this matrix notation, that the gauge field corresponds to a gauge transformation of zero,

$$A_\mu \to R_{A_\mu}(\Omega)\partial_\mu R^\dagger_{A_\mu}(\Omega). \tag{9.15}$$

Then automatically for the covariant derivative of the scalar field we get (suppressing the Ω dependence and its index a)

$$\begin{aligned} D_\mu\phi &\to (\partial_\mu + R_{A_\mu}\partial_\mu R^\dagger_{A_\mu})R_\phi\phi_0 \\ &= R_\phi\left(R^\dagger_\phi\partial_\mu R_\phi + R^\dagger_\phi R_{A_\mu}(\partial_\mu R^\dagger_{A_\mu})R_\phi\right)\phi_0 \\ &= R_\phi\left(R^\dagger_\phi R_{A_\mu}\partial_\mu(R^\dagger_{A_\mu}R_\phi)\right)\phi_0 = 0. \end{aligned} \tag{9.16}$$

This requires that $R^\dagger_\phi R_{A_\mu}\partial_\mu(R^\dagger_{A_\mu}R_\phi)$, which is a Lie algebra element, be in the direction that annihilates ϕ^0 or correspondingly $R^\dagger_{A_\mu}R_\phi$ leaves ϕ^0 invariant, that is $R^\dagger_{A_\mu}R_\phi = H$ where $H\phi^0 = \phi^0$. H may not be globally defined on the sphere at infinity; however, locally it is, and that is all we need. This defines the invariant subgroup or stabilizer of ϕ_0. But now we may redefine $R_\phi \to \tilde{R}_\phi = R_\phi H^{-1}$ as R_ϕ is only defined up to an element of the stabilizer of ϕ_0, as is obvious from Equation (9.11) (we will drop the tilde from now on). Thus we get $R^\dagger_{A_\mu}R_\phi = 1$ at least locally on the sphere at infinity. Although we started with different, independent gauge transformations, R_ϕ and R_{A_μ}, in Equations (9.11) and (9.15), respectively, we see that Equation (9.12) forces the gauge transformations to be the same. We will now call this gauge transformation $R(\Omega)$. We underline that $R(\Omega)$ may not be globally defined, and may actually be singular at some place on the sphere at infinity. In fact, for a non-trivial mapping it must be singular somewhere. However, its action on ϕ_0, which defines the values of the Higgs field at infinity, must be globally defined.

The condition of finite action is actually a little more subtle. Indeed, the gauge field must become pure gauge only as fast as $\sim 1/r$ for the $F^a_{\mu\nu}F^a_{\mu\nu}$ to give a finite contribution. Thus we should modify Equation (9.15) to

$$A_\mu \to R(\Omega)\partial_\mu R^\dagger(\Omega) + \tilde{A}_\mu, \tag{9.17}$$

where $\tilde{A}_\mu \sim o(1/r)$ (keeping in mind that the pure gauge terms also behave as $\sim 1/r$). However, such a modification could cause trouble in Equation (9.12), as the covariant derivative of the scalar field must vanish faster than $1/r^2$ for

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finite action. But this can again be solved if these additional possible terms in the gauge field are in the direction of the stabilizer of the Higgs field. Thus we can tolerate additional non-pure gauge terms in the gauge field as long as

$$\tilde{A}_\mu R\phi_0 = 0. \tag{9.18}$$

9.2.1 Topological Homotopy Classes

Thus finite action configurations are characterized by $R(\Omega)$ defined at $|\vec{x}| \to \infty$. This defines a map of the sphere at infinity S^{d-1} (generalizing temporarily to d dimensions) into the space of "vacuum" configurations, $\{\phi^a : \phi^a\phi^a = a^2\} \equiv \mathcal{M} = S^2$. The equivalence classes under homotopy of these maps form the homotopy groups

$$\Pi_{d-1}(\mathcal{M}). \tag{9.19}$$

There is a fascinating and complex set of corresponding homotopy groups [51]:

$$\Pi_{d-1}(\mathcal{M}) = \begin{cases} 0 & \text{d=2} \\ \mathbb{Z} & \text{d=3} \\ \mathbb{Z} & \text{d=4} \\ \mathbb{Z}_2 & \text{d=5} \\ \mathbb{Z}_2 & \text{d=6} \\ \mathbb{Z}_{12} & \text{d=7} \\ \mathbb{Z}_2 & \text{d=8} \\ \mathbb{Z}_2 & \text{d=9} \\ \mathbb{Z}_3 & \text{d=10} \\ \mathbb{Z}_{15} & \text{d=11} \\ \mathbb{Z}_2 & \text{d=12} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{d=13} \\ \mathbb{Z}_{12} \times \mathbb{Z}_2 & \text{d=14} \\ \mathbb{Z}_{84} \times \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{d=15} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{d=16} \\ & . \\ & . \\ & . \end{cases} \tag{9.20}$$

Thus there exist topologically non-trivial configurations in each dimension and the possibility of non-trivial finite Euclidean action configurations. In $d = 3$, the corresponding instantons are actually the 't Hooft–Polyakov magnetic monopole solitons of the $3+1$-dimensional theory.

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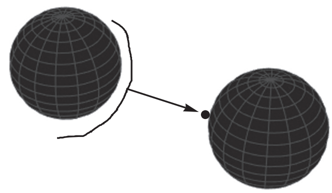

Figure 9.1. Mapping the whole S^2 at ∞ to a point

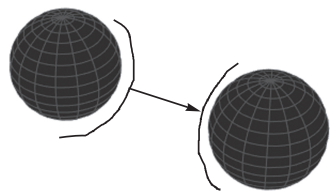

Figure 9.2. Mapping the S^2 at ∞ on to the vacuum manifold S^2

9.2.2 Magnetic Monopole Solutions

For $d = 3$, we have the maps

$$R(\Omega)\phi_0 : S^2 \to S^2, \tag{9.21}$$

where the first S^2 is defined by the set of all Ω's, *i.e.* the sphere at ∞, while the second S^2 is defined by the set of Higgs field values $\phi^2 = \phi^a\phi^a = a^2$. These fall into homotopically inequivalent classes, characterized by the winding number of the map, much like the previous case of maps of $S^1 \to S^1$ in the Abelian Higgs model. Pictured in Figures 9.1 and 9.2 are the trivial map to a point and the onto map, where each point in the first S^2 is mapped to the analogous point on the second S^2. We cannot continuously deform one configuration into another if they have different winding numbers, that is the definition of homotopy classes, and typically this implies that there exists an infinite action barrier between configurations in different classes. We will see that the topological winding number turns out to be associated with the magnetic charge of each sector. The minimum action configuration in each class must solve the equations of motion. The action must be stationary at the minimum action configuration since, if the first-order variation does not vanish, one can find a variation which lowers the action. The equations of motion are therefore satisfied. What is not necessary is that the minimum action configuration is non-trivial; it could, for example, collapse and shrink to a point or, conversely, spread out and dilute infinitely. We will show that it must be non-trivial.

The homotopy class with topological winding number $n = 1$ defines the standard instanton. We can prove that the action is bounded from below in each sector using a method first shown by Bogomolny [17]. We assume that the

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potential $V(\phi)$ is positive semi-definite. Defining the non-Abelian magnetic field as $B_i^a = \frac{1}{2}\epsilon_{ijk}F_{jk}^a$ we have

$$\begin{aligned}
S_E &= \int d^3x \left(\frac{1}{2}\frac{B_i^a}{e}\frac{B_i^a}{e} + \frac{1}{2}(D_i\phi)^a(D_i\phi)^a + V(\phi) \right) \\
&\geq \int d^3x \frac{1}{2}\left(\frac{B_i^a}{e} \mp (D_i\phi)^a \right)^2 \pm \frac{B_i^a}{e}(D_i\phi)^a \\
&\geq \pm \int d^3x \frac{B_i^a}{e}(D_i\phi)^a \\
&= \pm \frac{1}{e}\int d^3x B_i^a \partial_i\phi^a + B_i^a\epsilon^{abc}A_i^b\phi^c \\
&= \pm \frac{1}{e}\int d^3x \partial_i(B_i^a\phi^a) - \left((\partial_i B_i^a)\phi^a - B_i^a\epsilon^{abc}A_i^b\phi^c\right) \\
&= \pm \frac{1}{e}\left(\oint dS^i(B_i^a\phi^a) - \int d^3x \left(\partial_i B_i^a + \epsilon^{abc}A_i^b B_i^c\right)\phi^a \right) \\
&\equiv \pm ga,
\end{aligned} \tag{9.22}$$

where in the second line we have simply completed the square and dropped the potential, in the third line we have dropped the positive semi-definite first term and in the penultimate equation the last term vanishes because of the Jacobi identity. The Jacobi identity is $\epsilon_{ijk}[D_i,[D_j,D_k]] = 0$ which is simply, trivially, algebraically valid (just spell out all of the terms and they cancel pairwise). This gives $D_iB_i^a = \partial_i B_i^a + \epsilon^{abc}A_i^bB_i^c = 0$ as $[D_j^a, D_k^b] = \epsilon_{jkl}\epsilon^{abc}B_l^c$ which is the non-Abelian analogue of Maxwell's equation $\nabla\cdot\vec{B} = 0$. Normally, in the purely Abelian theory, this equation denies the existence of magnetic monopoles. Here the magnetic monopoles do exist, since the non-Abelian divergence of the magnetic field contains inhomogeneous terms. The magnetic monopoles exist as instantons in the Euclideanized 2+1-dimensional theory or as actual static solitons in the 3+1-dimensional theory. g is the magnetic charge

$$g = \frac{1}{ae}\oint dS^i B_i^a\phi^a \tag{9.23}$$

and a is the vacuum expectation value of the scalar field. Clearly, if g is positive we take the plus sign in Equation (9.22), and if g is negative we take the minus sign. This implies that the Euclidean action has a positive definite lower bound in each topological sector. We will show $g \neq 0$ except in the topologically trivial sector. Indeed, for the ansatz

$$\begin{aligned}
\phi^a &= H(aer)\frac{x^a}{er^2} \\
A_i^a &= -\epsilon^{aij}\frac{x^j}{r^2}(1 - K(aer))
\end{aligned} \tag{9.24}$$

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finite action requires

$$
\begin{aligned}
&H(aer) \to aer \quad , \quad r \to \infty \\
&K(aer) \to 0 \quad , \quad r \to \infty \\
&H(aer) < o(aer) \quad , \quad r \to 0 \\
&K(aer) < o(aer) \quad , \quad r \to 0.
\end{aligned}
\tag{9.25}
$$

Thus for large r

$$
\begin{aligned}
\phi^a &\approx a\frac{x^a}{r} = (R\phi_0)^a \\
A_i^a &\approx -\epsilon^{aij}\frac{x^j}{r^2} = R\partial_i R^\dagger + \tilde{A}_i^a
\end{aligned}
\tag{9.26}
$$

giving

$$
F_{ij}^a \approx \epsilon_{ijk}\frac{x^k x^a}{r^4}. \tag{9.27}
$$

Defining the Abelian magnetic field as

$$
B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}^a\frac{\phi^a}{a} \approx \frac{x^i}{r^3} \tag{9.28}
$$

we have

$$
g = \frac{1}{e}\oint dS_i B_i = \frac{4\pi}{e} \neq 0. \tag{9.29}
$$

This is in fact the Dirac quantization condition on magnetic charge, $gq = 2\pi$, for the minimal electric charge $q = e/2$. Not surprisingly, the theory knows that it can, in principle, have fields in the spinor representation of the iso-spin group ($SO(3)$) that do carry charge $e/2$.

For the Higgs field satisfying the conditions of the "Higgs" vacuum

$$
\begin{aligned}
\phi^a\phi^a &= a^2 \\
(D_\mu\phi)^a &= 0
\end{aligned}
\tag{9.30}
$$

we can write the explicit solution, using the iso-vector notation $\vec{\phi}$ for the Higgs field

$$
\begin{aligned}
A_\mu^a &= \frac{1}{a^2}\left(\vec{\phi}\times\partial_\mu\vec{\phi}\right)^a + \frac{1}{a}\phi^a A_\mu \\
F_{\mu\nu}^a &= \frac{1}{a}\phi^a F_{\mu\nu},
\end{aligned}
\tag{9.31}
$$

where

$$
F_{\mu\nu} = \frac{1}{a^3}\phi^a\left(\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi}\right)^a + \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{9.32}
$$

A_μ generates only a source-free magnetic field, but

$$
B_i = \frac{1}{2}\epsilon_{ijk}F_{jk} \tag{9.33}
$$

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can have non-zero magnetic charge due to the first term in Equation (9.32). The magnetic charge in any region is

$$g = \frac{1}{e}\oint_\Sigma \vec{B}\cdot d\vec{S} = \frac{1}{2ea^3}\oint_\Sigma dS_i \epsilon_{abc}\epsilon^{ijk}\phi^a\partial_j\phi^b\partial_k\phi^c. \tag{9.34}$$

We will show that this integral is actually a topological invariant and equal to the result $4\pi/e$ that we found for the configuration in Equation (9.29) above. It counts the winding number of the map from the surface Σ which is topologically S^2 into the S^2 defined by $\phi^a\phi^a = a^2$. Indeed, consider a variation $\delta\phi$ which is of compact support, then $\vec{\phi}\to\vec{\phi}+\delta\vec{\phi}$ but since $\left(\vec{\phi}\cdot\vec{\phi}\right) = 1$ we get

$$\delta\left(\vec{\phi}\cdot\vec{\phi}\right) = 2\vec{\phi}\cdot\delta\vec{\phi} = 0. \tag{9.35}$$

Then

$$\begin{aligned}
\delta\left(\vec{\phi}\cdot(\partial_j\vec{\phi}\times\partial_k\vec{\phi})\right) &= \delta\vec{\phi}\cdot\left(\partial_j\vec{\phi}\times\partial_k\vec{\phi}\right) + \vec{\phi}\cdot\left(\partial_j\delta\vec{\phi}\times\partial_k\vec{\phi}\right) + \vec{\phi}\cdot\left(\partial_j\vec{\phi}\times\partial_k\delta\vec{\phi}\right) \\
&= \delta\vec{\phi}\cdot\left(\partial_j\vec{\phi}\times\partial_k\vec{\phi}\right) + \partial_j\left(\vec{\phi}\cdot(\delta\vec{\phi}\times\partial_k\vec{\phi})\right) - \partial_j\vec{\phi}\cdot\left(\delta\vec{\phi}\times\partial_k\vec{\phi}\right) \\
&\quad \vec{\phi}\cdot\left(\delta\vec{\phi}\times\partial_j\partial_k\vec{\phi}\right) + \partial_k\left(\vec{\phi}\cdot(\partial_j\vec{\phi}\times\delta\vec{\phi})\right) \\
&\quad \partial_j\vec{\phi}\cdot\left(\delta\vec{\phi}\times\partial_k\vec{\phi}\right) - \vec{\phi}\cdot\left(\partial_j\partial_k\vec{\phi}\times\delta\vec{\phi}\right) \\
&= 3\,\delta\vec{\phi}\cdot\left(\partial_j\vec{\phi}\times\partial_k\vec{\phi}\right) + 2\partial_j\left(\vec{\phi}\cdot(\delta\vec{\phi}\times\partial_k\vec{\phi})\right),
\end{aligned} \tag{9.36}$$

where, in the last step, we use that the expression is contracted with ϵ^{ijk}. The total derivative terms give no contribution to any integral since $\delta\phi$ is of compact support. Now $\partial_j\phi$ and $\partial_k\phi$ are both orthogonal to ϕ, thus $\partial_j\vec{\phi}\times\partial_k\vec{\phi}$ is parallel to ϕ, giving

$$\delta\vec{\phi}\cdot\left(\partial_j\vec{\phi}\times\partial_k\vec{\phi}\right) = 0 \tag{9.37}$$

hence

$$\delta\left(\vec{\phi}\cdot(\partial_j\vec{\phi}\times\partial_k\vec{\phi})\right) = 2\partial_j\left(\vec{\phi}\cdot(\delta\vec{\phi}\times\partial_k\vec{\phi})\right). \tag{9.38}$$

Therefore the integral, Equation (9.34), is invariant under arbitrary continuous deformation of ϕ, since these are built up from a sequence of infinitesimal deformations of compact support. A continuous deformation of the surface over which the field is defined can also be interpreted as a continuous deformation of the ϕ field, thus g is also invariant under continuous deformations of the integration surface (remember that we are only in the Higgs vacuum).

Finally we can calculate g for

$$\phi^a = a\hat{x}^a = a\frac{x^a}{r}, \tag{9.39}$$

asymptotically, which corresponds to the winding number equal to one map. Then

$$\partial^i\phi^a = a\left(\frac{\delta^{ai}}{r} - \frac{x^a x^i}{r^3}\right) = \frac{a}{r}\left(\delta^{ai} - \hat{x}^a\hat{x}^i\right), \tag{9.40}$$

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which gives

$$\epsilon^{ijk}\epsilon_{abc}\phi^a\partial_j\phi^b\partial_k\phi^c = a^3\epsilon_{ijk}\epsilon_{abc}\frac{x^a}{r^3}\left(\delta^{jb}-\hat{x}^j\hat{x}^b\right)\left(\delta^{kc}-\hat{x}^k\hat{x}^c\right) = \frac{2a^3}{r^2}\hat{x}^i. \quad (9.41)$$

Hence

$$g = \frac{1}{2ea^3}\oint_\Sigma dS^i\frac{2a^3}{r^2}\hat{x}^i = \frac{1}{2ea^3}8\pi a^3 = \frac{4\pi}{e}. \quad (9.42)$$

This answer is robust, in that it does not change for any infinitesimal changes and hence for any continuous change in the Higgs field. If we use the winding number 2 map, the answer for the integral will be $2\times 4\pi/e$, and so on. If we write $\phi = R\phi_0$, then the winding number N map is obtained by taking $\phi = R^N\phi_0$.

If we transform $\phi^a \to \tilde{\phi}^a = \delta^{a3}a$, we cannot define the gauge transformation globally over any surface containing the core. We get the usual Dirac string singularity,

$$A_i^a = \delta^{a3}\frac{1}{er}\frac{(1-\cos\theta)}{\sin\theta}\hat{\varphi}_i, \quad (9.43)$$

where $\hat{\varphi}$ is the unit vector in the azimuthal direction.

9.3 Monopole Ansatz with Maximal Symmetry

The solution follows from the most general ansatz

$$\phi^a = H(aer)\frac{x^a}{er^2}$$
$$A_i^a = -\epsilon^{aij}\frac{x^j}{e^2r^2}(1-K(aer)) + \frac{r^2\delta^{ai}-x^ix^a}{e^2r^3}B(aer) + \frac{x^ix^a}{e^2r^3}C(aer), \quad (9.44)$$

which is symmetric under the diagonal subgroup of the group $SO(3)_{\text{rot.}} \times SO(3)_{\text{iso-rot.}}$ of rotations and iso-rotations. If we had imposed invariance only under the $SO(3)_{\text{rot.}}$, the rotation subgroup alone, we would have to impose that ϕ^a is a constant on each spatial sphere, giving trivial asymptotic topology. On the other hand, the configuration that is invariant only under $SO(3)_{\text{iso-rot.}}$, the iso-rotational group, has the only possibility $\phi^a = 0$, which also has trivial topology. However, we can impose invariance under the next subgroup available, $SO(3)_{\text{diagonal}}$, the diagonal subgroup of rotations and iso-rotations, which the fields in Equation (9.44) satisfy.

Parity corresponds to the transformation

$$P:\ \phi^a(x^j,t)\to\phi^a(-x^j,t), \quad A_i^a(x^j,t)\to -A_i^a(-x^j,t) \quad (9.45)$$

and there is also the discrete transformation

$$Z:\ \phi^a(x^j,t)\to -\phi^a(x^j,t), \quad A_i^a(x^j,t)\to A_i^a(x^j,t). \quad (9.46)$$

P and Z individually reverse the magnetic charge, thus we cannot impose invariance under each separately. However, their product leaves the magnetic

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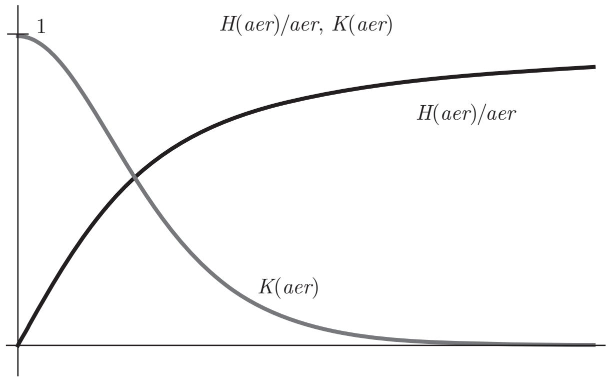

Figure 9.3. The curves of $H(aer)/aer$ and $K(aer)$

charge invariant. Hence, in the spirit of imposing the maximum symmetry on the solution without making it trivial, we impose that the ansatz be invariant under PZ. This implies $B(aer) = C(aer) = 0$.

9.3.1 Monopole Equations

We find, then, that $H(aer)$ and $K(aer)$ satisfy the system of equations

$$\begin{aligned} r^2 \frac{d^2}{dr^2} K(r) &= K(r)H^2(r) + K(r)(K^2(r) - 1) \\ r^2 \frac{d^2}{dr^2} H(r) &= 2K(r)^2 H(r) + \frac{\lambda}{e^2} H(r)(H^2(r) - a^2 r^2). \end{aligned} \tag{9.47}$$

They have numerical solutions as depicted in Figure 9.3. In the Prasad–Sommerfield limit [104], $\lambda \to 0$, we know the exact solution

$$\begin{aligned} H(aer) &= aer \coth(aer) - 1 \\ K(aer) &= \frac{aer}{\sinh(aer)}. \end{aligned} \tag{9.48}$$

This solution corresponds to the famous 't Hooft–Polyakov magnetic monopole. In $3+1$ dimensions it is a static, stable, finite-energy solution to the equations of motion. In $2+1$ dimensions, but Euclideanized, it serves equally well as a finite-action, Euclidean space instanton, where it mediates tunnelling between different classical vacua, as we will see below.

9.4 Non-Abelian Gauge Field Theories

We must examine in some more detail what it means to have a quantum non-Abelian gauge theory.

9.4.1 Classical Non-Abelian Gauge Invariance

First we will consider non-Abelian gauge invariance more generally, and then apply it to our specific case. A non-Abelian gauge theory admits fields which transform according to given representations of a non-Abelian group,

$$\phi \to \mathcal{U}(g)\phi \quad \mathcal{U}(g) \in G, \tag{9.49}$$

where $\mathcal{U}(g)\mathcal{U}^\dagger(g) = \mathcal{U}^\dagger(g)\mathcal{U}(g) = 1$. If g does not depend on the spacetime point, we call the gauge transformation global, otherwise it is a local gauge transformation. However, the allowed variation of the gauge transformation is restricted to a region of compact support. It is easy to write a Lagrangian that is invariant under global gauge transformations, we simply construct it out of invariant polynomials of the fields. Spacetime derivatives commute with the gauge-transforming field $\mathcal{U}(g)$ and hence cause no problems. Now if we want to generalize the invariance to include local gauge transformations, we must introduce new fields. For our case

$$\begin{aligned} \phi^a &\to (\mathcal{U}(g)\phi)^a \\ (\partial^\mu\phi)^a &\to \partial^\mu (\mathcal{U}(g)\phi)^a \\ &= (\mathcal{U}(g)\partial^\mu\phi)^a + ((\partial^\mu\mathcal{U}(g))\,\phi)^a . \end{aligned} \tag{9.50}$$

That is, if $\mathcal{U}(g)$ depends on the spacetime point, the derivative does not commute with it. We must introduce a new field, the gauge field A^a_μ, with an inhomogeneous transformation property which will exactly cancel the extra term generated by the derivative. We replace all derivatives by

$$\partial_\mu \to \partial_\mu + A_\mu, \tag{9.51}$$

where A_μ is a vector field with values in the Lie algebra of the representation under which ϕ transforms. In our case

$$A_\mu = A^b_\mu \left(-\epsilon^{bac}\right), \tag{9.52}$$

thus

$$\begin{aligned} (D_\mu\phi)^a &= \partial_\mu\phi^a - A^b_\mu\epsilon^{bac}\phi^c \\ &= \partial_\mu\phi^a + \epsilon^{abc}A^b_\mu\phi^c. \end{aligned} \tag{9.53}$$

A_μ is given the transformation property such that the covariant derivative transform covariantly:

$$D_\mu\phi \to \mathcal{U}(g)D_\mu\phi. \tag{9.54}$$

This is satisfied if

$$A_\mu \to \mathcal{U}(g)\,(A_\mu + \partial_\mu)\mathcal{U}^\dagger(g). \tag{9.55}$$

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Evidently

$$\begin{aligned} D_\mu\phi = (\partial_\mu + A_\mu)\,\phi \to & \left(\partial_\mu + \mathcal{U}(g)\,(A_\mu + \partial_\mu)\mathcal{U}^\dagger(g)\right)\mathcal{U}(g)\phi \\ &= (\partial_\mu\mathcal{U}(g))\,\phi + \mathcal{U}(g)\partial_\mu\phi + \mathcal{U}(g)A_\mu\phi + \mathcal{U}(g)\left(\partial_\mu\mathcal{U}^\dagger(g)\right)\mathcal{U}(g)\phi \\ &= \mathcal{U}(g)\,(\partial_\mu + A_\mu)\,\phi + \left(\partial_\mu\mathcal{U}(g) + \mathcal{U}(g)\left(\partial_\mu\mathcal{U}^\dagger(g)\right)\mathcal{U}(g)\right)\phi \\ &= \mathcal{U}(g)\,(\partial_\mu + A_\mu)\,\phi \\ &= \mathcal{U}(g)D_\mu\phi. \end{aligned} \tag{9.56}$$

The covariant derivative has the geometrical interpretation as the parallel transport in a fibre bundle with connection A_μ. For each infinitesimal path, $x^\mu \to x^\mu + dx^\mu$, we introduce the gauge field $A_\mu(x^\nu)$ and an element of the group,

$$g(x + dx, A_\mu) = 1 + dx^\mu A_\mu. \tag{9.57}$$

Then for a finite path $\mathcal{C}$ we integrate this as

$$g(\mathcal{C}, A) = P\left(\exp\left\{\int_{\mathcal{C}} dx^\mu A_\mu\right\}\right), \tag{9.58}$$

where the P symbol means the path-ordered integral. Intuitively this corresponds to the limit taken by multiplying the group elements of the form (9.57) for a finitely discretized approximation to the finite curve $\mathcal{C}$, in the order corresponding to the direction of the curve, and taking the limit that the discretization becomes infinitely fine. The other definition, which yields the same result, is to expand the exponential and then perform the multiple integral at each order, after applying the path-ordering to the integrand. A field is considered to have been transported in parallel in the connection A_μ if

$$\begin{aligned} \phi\,(x + dx) = \phi^{g(x+dx,A_\mu)}(x) &= \mathcal{U}(g(x + dx, A_\mu))\phi \\ &= \phi(x) + dx^\mu A_\mu\phi(x). \end{aligned} \tag{9.59}$$

Then, in general,

$$\begin{aligned} \phi(x + dx) - \phi^{g(x+dx,A_\mu)}(x) &= dx^\mu\,(\partial_\mu + A_\mu(x))\,\phi(x) \\ &= dx^\mu D_\mu\phi(x) \end{aligned} \tag{9.60}$$

defines the covariant derivative in the connection A_μ. Here $A_\mu = A_\mu^a t^a$, where t^a are the generators of the group in the representation that $\phi(x)$ transforms under.

9.4.2 The Field Strength

To construct the non-Abelian field strength we must consider a generalization of the Abelian version,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{9.61}$$

This is invariant under Abelian gauge transformations

$$A_\mu \to A_\mu + i\partial_\mu\Lambda$$

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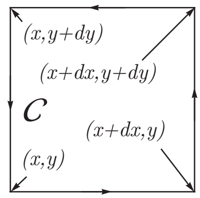

Figure 9.4. An infinitesimal closed loop $\mathcal{C}$

$$\delta F_{\mu\nu} = (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\,\Lambda = 0. \tag{9.62}$$

We can write this as

$$F_{\mu\nu} \to e^{-i\Lambda} F_{\mu\nu} e^{i\Lambda} = F_{\mu\nu}. \tag{9.63}$$

For Abelian phases, $F_{\mu\nu}$ is invariant, but if we generalize this formula to non-Abelian groups, $F_{\mu\nu}$ does transform, but homogeneously. We construct $F_{\mu\nu}$ via parallel transport. The same construction works as in the abelian case. Consider a closed loop $\mathcal{C}$ drawn in Figure 9.4, and

$$\begin{aligned} g(\mathcal{C},x,A) &= P\left(\exp\left\{-\oint dx^\mu A_\mu\right\}\right) \\ &= 1 - \oint dx^\mu A_\mu + \oint dx_1 \oint_{x_2 > x_1} dx_2^\mu A_\mu(x_2)^\nu A_\nu(x_1) + \cdots. \end{aligned} \tag{9.64}$$

This group element transforms covariantly. Infinitesimally for each segment of the curve $\mathcal{C}$, we find

$$\begin{aligned} g(x+dx,A^g) &= 1 - dx^\mu A^g_\mu \\ &= 1 - dx^\mu \mathcal{U}(g)(A_\mu + \partial_\mu)\mathcal{U}^\dagger(g) \\ &= \mathcal{U}(g)\left(1 - dx^\mu\left(A_\mu + \left(\partial_\mu \mathcal{U}^\dagger(g)\right)\mathcal{U}(g)\right)\right)\mathcal{U}^\dagger(g). \end{aligned} \tag{9.65}$$

Now,

$$\begin{aligned} &\mathcal{U}(g(x))\left(1 - dx^\mu\left(\partial_\mu \mathcal{U}^\dagger(g(x))\right)\mathcal{U}(g(x))\right) \\ &\quad = \mathcal{U}(g(x)) - dx^\mu \mathcal{U}(g(x))\partial_\mu \mathcal{U}^\dagger(g(x))\mathcal{U}(g(x)) \\ &\quad = \mathcal{U}(g(x)) + dx^\mu \partial_\mu \mathcal{U}(g(x)) = \mathcal{U}(g(x+dx)) \end{aligned} \tag{9.66}$$

hence

$$\begin{aligned} g(x+dx,A^g) &= \mathcal{U}(g(x+dx))\,(1 - dx^\mu A_\mu)\,\mathcal{U}^\dagger(g(x)) \\ &= \mathcal{U}(g(x+dx))g(x+dx,A)\mathcal{U}^\dagger(g(x)). \end{aligned} \tag{9.67}$$

Thus for the infinitesimal closed loop, as in Figure 9.4, starting and ending at x

$$g(\mathcal{C},x,A^g) = \mathcal{U}(g(x))g(\mathcal{C},x,A)\mathcal{U}^\dagger(g(x)), \tag{9.68}$$

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which is exactly the covariant transformation property. Considering the second-order term in the expansion in Equation (9.64), we have for each straight line path part of the contour of direction l_μ

$$\int dx^\mu A_\mu = \int_0^1 dt l^\mu A_\mu(x^\nu + l^\nu t) = \int_0^1 dt l^\mu \left(A_\mu(x^\nu) + l^\sigma t \partial_\sigma A_\mu(x^\nu)\right) + o(l^3)$$
$$= l^\mu A_\mu(x^\nu) + \frac{1}{2} l^\mu l^\sigma \partial_\sigma A_\mu(x^\nu) + \cdots . \tag{9.69}$$

Thus for the closed path we get to second-order contribution

$$\oint dx^\mu A_\mu(x^\nu) = \Big\{ \Big(dx^\mu A_\mu(x^\nu) + \frac{1}{2} dx^\mu dx^\sigma \partial_\sigma A_\mu(x^\nu) \Big)$$
$$+ \Big(dy^\mu A_\mu(x^\nu + dx^\nu) + \frac{1}{2} dy^\mu dy^\sigma \partial_\sigma A_\mu(x^\nu) \Big)$$
$$+ \Big(-dx^\mu A_\mu(x^\nu + dx^\nu + dy^\nu) + \frac{1}{2} dx^\mu dx^\sigma \partial_\sigma A_\mu(x^\nu) \Big)$$
$$+ \Big(-dy^\mu A_\mu(x^\nu + dy^\nu) + \frac{1}{2} dy^\mu dy^\sigma \partial_\sigma A_\mu(x^\nu) \Big) \Big\}$$
$$= \Big\{ \Big(dx^\mu A_\mu(x^\nu) + \frac{1}{2} dx^\mu dx^\sigma \partial_\sigma A_\mu(x^\nu) \Big)$$
$$+ \Big(dy^\mu \left[A_\mu(x^\nu) + dx^\sigma \partial_\sigma A_\mu(x^\nu)\right] + \frac{1}{2} dy^\mu dy^\sigma \partial_\sigma A_\mu(x^\nu) \Big)$$
$$+ (-dx^\mu \left[A_\mu(x^\nu) \ + \ dx^\sigma \partial_\sigma A_\mu(x^\nu) + dy^\sigma \partial_\sigma A_\mu(x^\nu)\right] + \frac{1}{2} dx^\mu dx^\sigma \partial_\sigma A_\mu(x^\nu) \Big)$$
$$+ \Big(-dy^\mu \left[A_\mu(x^\nu) + dy^\sigma \partial_\sigma A_\mu(x^\nu)\right] + \frac{1}{2} dy^\mu dy^\sigma \partial_\sigma A_\mu(x^\nu) \Big) \Big\}$$
$$= dx^\sigma dy^\mu \left(\partial_\sigma A_\mu(x^\nu) - \partial_\mu A_\sigma(x^\nu)\right). \tag{9.70}$$

Notice that this term contributes with a minus sign in Equation (9.64). When integrating along one side in Equation (9.64), the second-order term gives directly

$$\int_x^{x+dx} dx_2^\mu \int_x^{x_2} dx_1^\mu A_\mu(x_2^\nu) A_\mu(x_1^\nu) = \int_0^1 dt \Big(l^\mu A_\mu(x^\nu + l^\nu t) \int_0^t ds\, l^\sigma A_\sigma(x^\nu + l^\nu s) \Big)$$
$$= \int_0^1 dt \Big(l^\mu A_\mu(x^\nu) \int_0^t ds\, l^\sigma A_\sigma(x^\nu) \Big)$$
$$= \int_0^1 dt \left(l^\mu A_\mu(x^\nu) t\, l^\sigma A_\sigma(x^\nu) \right)$$
$$= \frac{1}{2} l^\mu l^\sigma A_\mu(x^\nu) A_\sigma(x^\nu). \tag{9.71}$$

The two integrals simply factorize when the integrations are on two different segments and no factor of one half is generated. Hence adding up the

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contributions around the loop, substituting for l^μ with dx^μ or dy^μ gives

$$
\begin{aligned}
&\oint dx_1^\nu \oint_{x_2>x_1} dx_2^\mu A_\mu(x_2)A_\nu(x_1) = \Big\{ -dy^\mu A_\mu(x^\nu+dy^\nu) \\
&\times \left[\frac{1}{2}(-dy^\sigma)A_\sigma(x^\nu+dy^\nu) - dx^\sigma A_\sigma(x^\nu+dx^\nu+dy^\nu) \right.\\
&\left. +dy^\sigma A_\sigma(x^\nu+dx^\nu) + dx^\sigma A_\sigma(x^\nu)\right] \\
&-dx^\mu A_\mu(x^\nu+dx^\nu+dy^\nu)\left[\frac{1}{2}(-dx^\sigma)A_\sigma(x^\nu+dx^\nu+dy^\nu)\right. \\
&\left. +dy^\sigma A_\sigma(x^\nu+dx^\nu) + dx^\sigma A_\sigma(x^\nu)\right] \\
&+dy^\mu A_\mu(x^\nu+dx^\nu)\left[\frac{1}{2}dy^\sigma A_\sigma(x^\nu+dx^\nu) + dx^\sigma A_\sigma(x^\nu)\right] \\
&+dx^\mu A_\mu(x^\nu)\left[\frac{1}{2}dx^\sigma A_\sigma(x^\nu)\right]\Big\} \\
&+\Big\{-\frac{1}{2}dy^\mu A_\mu(x^\nu)dy^\sigma A_\sigma(x^\nu) - \frac{1}{2}dx^\mu A_\mu(x^\nu)dx^\sigma A_\sigma(x^\nu) \\
&-dx^\mu A_\mu(x^\nu)dy^\sigma A_\sigma(x^\nu) \\
&+\frac{1}{2}dy^\mu A_\mu(x^\nu)dy^\sigma A_\sigma(x^\nu) + dy^\mu A_\mu(x^\nu)dx^\sigma A_\sigma(x^\nu) + \frac{1}{2}dx^\mu A_\mu(x^\nu)dx^\sigma A_\sigma(x^\nu)\Big\} \\
&= -dx^\sigma dy^\mu \left[A_\sigma(x^\nu)A_\mu(x^\nu) - A_\mu(x^\nu)A_\sigma(x^\nu)\right]. \qquad (9.72)
\end{aligned}
$$

Adding up the two contributions, Equations (9.72) and (9.70), simply gives

$$
\begin{aligned}
P\exp\{dx^\mu A_\mu(x^\nu)\} &= -dx^\sigma dy^\mu \left(\partial_\sigma A_\mu(x^\nu) - \partial_\mu A_\sigma(x^\nu)\right. \\
&\quad \left. + \left[A_\sigma(x^\nu), A_\mu(x^\nu)\right]\right) + o(dx)^3 \\
&\equiv -dx^\sigma dy^\mu F_{\sigma\mu} + o(dx)^3, \qquad (9.73)
\end{aligned}
$$

which must transform covariantly. Actually we can write $F_{\mu\nu}$ as the commutator of two covariant derivatives,

$$
\begin{aligned}
F_{\mu\nu} &= [D_\mu, D_\nu] = [\partial_\mu + A_\mu, \partial_\nu + A_\nu] \\
&= [\partial_\mu, A_\nu] + [A_\mu, \partial_\nu] + [A_\mu, A_\nu] \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \qquad (9.74)
\end{aligned}
$$

Then, due to the algebraic structure of $F_{\mu\nu}$, we immediately know that the Jacobi identity will be satisfied,

$$
\begin{aligned}
&[D_\mu,[D_\nu,D_\sigma]] + [D_\sigma,[D_\mu,D_\nu]] + [D_\nu,[D_\sigma,D_\mu]] = 0 \\
\Rightarrow\ &[D_\mu, F_{\nu\sigma}] + [D_\sigma, F_{\mu\nu}] + [D_\nu, F_{\sigma\mu}] = 0, \qquad (9.75)
\end{aligned}
$$

which in four dimensions is exactly the Bianchi identity,

$$
\partial_\mu \epsilon^{\mu\nu\sigma\tau} F_{\sigma\tau} + [A_\mu, \epsilon^{\mu\nu\sigma\tau} F_{\sigma\tau}] = 0. \qquad (9.76)
$$

Thus $F_{\mu\nu}$ is the appropriate covariant generalization of the usual Abelian definition of the field strength.

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9.5 Quantizing Gauge Field Configurations

The physical (non-gauge) zero modes of the action come from translations of the positions of the monopoles and rotations of the monopoles in iso-space. This gives simply the volume of spacetime and the volume of the gauge group as a Jacobian factor. However, things are not so simple, since in a gauge theory there are lots of unphysical zero modes associated with gauge-equivalent configurations. The naive functional integral for a gauge theory is not well-defined, even in Euclidean space.

The Lagrangian of a gauge theory is called a singular Lagrangian, the equations of motion do not give rise to a well-defined initial value problem for the gauge fields. Obviously, if we fix the initial data, and find a solution of the equations of motion, there actually exist an infinite number of solutions of the equations of motion that satisfy the initial conditions, which are simply gauge transforms of the original solutions. The gauge transformations, of course, must be time-dependent, so that they do nothing to the gauge fields on the initial hyper-surface, but they do modify the gauge fields afterwards. The freedom to do time-dependent gauge transformations allow for this, and the solution of the initial value problem is not unique. Thus fixing the gauge becomes essential to define even the classical dynamics. Correspondingly, the quantum dynamics also requires gauge fixing in order to be well-defined. The important point is that, because of the gauge invariance, the actual physical content of the theory does not depend on the choice of gauge fixing.

The action is invariant under the infinite dimensional group of gauge transformations, $\mathcal{G}$. Thus

$$\mathcal{N}\int \mathcal{D}(A,\phi)\,e^{\frac{-S_E}{\hbar}} = (\text{volume}\,(\mathcal{G}))\left(\mathcal{N}\int_{\substack{\text{gauge}\\ \text{inequivalent}}} \mathcal{D}(A,\phi)\,e^{\frac{-S_E}{\hbar}}\right), \qquad (9.77)$$

as geometrically drawn in Figure 9.5. The volume $\mathcal{G}$ is, of course, infinite, it is not just a few zero modes which arise as in the propagator, but an infinity of zero modes due to arbitrary local gauge transformations. This infinite volume should cancel between numerator and denominator; however, we must realize how to define

$$\mathcal{N}\int_{\substack{\text{gauge}\\ \text{inequivalent}}} \mathcal{D}(A,\phi)\,e^{\frac{-S_E}{\hbar}} \qquad (9.78)$$

properly, *i.e.* in a gauge-invariant manner. The method for defining this integral is to begin in a canonical gauge, where the quantization is understood and well-defined, and then transform to any other gauge in an invariant way. This procedure was first spelled out by Faddeev and Popov [44].

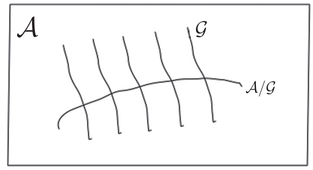

Figure 9.5. The space of all gauge fields, corresponding to the space $\mathcal{A}/G$ with leaves, foliated by the group of gauge transformation $\mathcal{G}$

9.5.1 The Faddeev–Popov Determinant

We will start with the gauge choice

$$A^3 = 0. \tag{9.79}$$

This gauge condition is complete, which means that we may not make any further gauge transformations whose derivatives are of compact support. These are the so-called local gauge transformations, those that go sufficiently fast (often taken to be of compact support), to a constant at infinity. This constant is usually taken to be the identity. We insist on the gauge choice, that is, $A^3 = 0$, then any gauge transformation must satisfy

$$A^3 \to g^{-1}\partial_3 g = 0 \quad \Rightarrow \partial_3 g = 0. \tag{9.80}$$

But then g must be a global constant, everywhere equal to its value at infinity, chosen to be the identity. It is easy to convince ourselves that no local gauge transformation can be non-trivial and still be independent of x^3. Hence Equation (9.79) is a complete gauge-fixing condition as far as the group of local gauge transformations is concerned. We define

$$\mathcal{I} = \mathcal{N}\int \mathcal{D}(A,\phi)\,\delta(A_3)\,e^{\frac{-S_E(A,\phi)}{\hbar}}. \tag{9.81}$$

For any other gauge choice such that $F(A_i) = 0$ there must exist a gauge transformation $g_0(A)$ such that

$$(A_3)^{g_0(A)} = 0, \tag{9.82}$$

since it is understood that the set of gauge orbits of a given gauge slice must span the entire space of gauge fields at least locally.[1]

We define $\Delta(A)$ by

$$1 = \Delta(A)\int \mathcal{D}g\,\delta\left(F(A_i^g)\right), \tag{9.83}$$

[1] The Gribov ambiguity maintains that this is not exactly true. There do exist multiple gauge field configurations that respect the same gauge condition. However, these configurations are typically a finite distance away from each other. Thus the configurations that satisfy the gauge-fixing condition and their gauge orbits certainly give a complete foliation of the local neighbourhood of the space of gauge fields.

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where $\mathcal{D}g$ corresponds to the integration measure for integration over the full group of local gauge transformations. This measure is defined in an invariant way, formally, the metric on the space of gauge transformations is defined as (in d dimensions)

$$(\delta g)^2 = -\int d^d x \mathrm{tr}\left((g^{-1}\delta g)(g^{-1}\delta g)\right). \tag{9.84}$$

Here δg corresponds to an element of the tangent space of the group of gauge transformations, this is called its Lie algebra. If h is an arbitrary fixed element of the group of gauge transformations, then the left multiplication by h in the group gives left multiplication of the algebra, $\delta(hg) = h\delta g$ and the 1-form $g^{-1}\delta g$ is left-invariant, as is the metric Equation (9.84). The metric is actually also invariant under right multiplication, since $\delta(gh) = (\delta g)h$, but then $\mathrm{tr}((gh)^{-1}\delta(gh)(gh)^{-1}\delta(gh)) = \mathrm{tr}((h^{-1}g^{-1}(\delta g)hh^{-1}g^{-1}(\delta g)h) = \mathrm{tr}\left((g^{-1}\delta g)(g^{-1}\delta g)\right)$. $\mathcal{D}g$ is then formally the corresponding volume form. We will mostly need to integrate over an infinitesimal neighbourhood of the identity. Here, with $g = 1+\alpha$, where α is an infinitesimal element of the Lie algebra, we have, since $g^{-1}\delta g = \alpha$ to first order, and the analogue of the Euclidean geometry in the space of all α's

$$|\alpha|^2 = \int d^d x \mathrm{tr}\left(\alpha^2\right). \tag{9.85}$$

This then allows for the replacement $\mathcal{D}g \to \mathcal{D}\alpha$ with free, linear integration over α.

Notice that $\Delta(A)$ is gauge-invariant, for an arbitrary gauge transformation h,

$$\Delta(A^h) = \Delta(A). \tag{9.86}$$

This is because the integration measure over the group of gauge transformations is expected to be and can be defined to be gauge-invariant, that is,

$$\begin{aligned}\frac{1}{\Delta(A^h)} &= \int \mathcal{D}g\,\delta\left(F((A_i^h)^g)\right) = \int \mathcal{D}(g)\,\delta\left(F(A_i^{gh})\right)\\ &= \int \mathcal{D}(gh)\,\delta\left(F(A_i^{gh})\right) = \int \mathcal{D}g\,\delta\left(F(A_i^g)\right)\\ &= \frac{1}{\Delta(A)}.\end{aligned} \tag{9.87}$$

$\Delta(A)$ is called the Faddeev–Popov factor. (We call to your attention that $\left(A_i^h\right)^g = A_i^{gh}$ as the group action works by left multiplication.) Then

$$\begin{aligned}\mathcal{I} &= \mathcal{N}\int \mathcal{D}(A,\phi)\,\delta(A_3)e^{\frac{-S_E}{\hbar}}\left(\Delta(A)\int \mathcal{D}g\delta\left(F(A_i^g)\right)\right)\\ &= \mathcal{N}\int \mathcal{D}g\int \mathcal{D}(A,\phi)\,\delta(A_3)e^{\frac{-S_E}{\hbar}}\Delta(A)\delta\left(F(A_i^g)\right)\\ &= \mathcal{N}\int \mathcal{D}g\int \mathcal{D}(A,\phi)\,\delta(A_3^{g^{-1}})e^{\frac{-S_E}{\hbar}}\Delta(A^{g^{-1}})\delta\left(F(A_i)\right)\\ &= \mathcal{N}\int \mathcal{D}(A,\phi)\,\delta\left(F(A_i)\right)e^{\frac{-S_E}{\hbar}}\Delta(A)\left(\int \mathcal{D}g\delta(A_3^{g^{-1}})\right).\end{aligned} \tag{9.88}$$

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Now let

$$g^{-1} = g'^{-1} g_0(A) \tag{9.89}$$

such that

$$(A_3)^{g_0(A)} = 0. \tag{9.90}$$

For a given g, g'^{-1} will depend on A; however, the integration over all g' will not, as the integration measure is invariant under left or right multiplication, as explained in the discussion after Equation (9.84). That is

$$\int \mathcal{D}g\delta\left(A_3^{g^{-1}}\right) = \int \mathcal{D}g'\delta\left(\left(A_3^{g_0}\right)^{g'^{-1}}\right) = \int \mathcal{D}g'\delta\left((0)^{g'^{-1}}\right) \tag{9.91}$$

is a constant, independent of A, and so we can absorb it into the normalization. Thus we get

$$\mathcal{I} = \mathcal{N}' \int \mathcal{D}(A,\phi)\delta\left(F(A_i)\right)\Delta(A)e^{\frac{-S_E}{\hbar}}. \tag{9.92}$$

We see how to change the gauge from the choice $A_3 = 0$ to an arbitrary gauge choice $F(A_i) = 0$, the integration measure must be appended with the Faddeev–Popov factor. The Faddeev–Popov factor,

$$\Delta^{-1}(A) = \int \mathcal{D}g\, \delta\left(F\left(A^g\right)\right) \tag{9.93}$$

will only get contributions from the infinitesimal neighbourhood of A around the point where $F(A) = 0$. Thus for A satisfying the gauge condition, we have, with $g = 1+\alpha$, where α is an infinitesimal element of the the Lie algebra,

$$F\left(A^{1+\alpha}\right) = F(A) + \int d^3y \frac{\delta F}{\delta A_i(y)} D_i\alpha(y) + o(\alpha^2), \tag{9.94}$$

since the change in the gauge field is exactly $\delta A_i(y) = D_i\alpha(y)$ and the integration is over α with measure $\mathcal{D}g \to \mathcal{D}\alpha$. Then generalizing the standard property of the integration over a delta function $\int d^n x\delta(M\cdot x) = (\det M)^{-1}$, we get

$$\begin{aligned}\Delta^{-1}(A) &= \int \mathcal{D}\alpha\delta\left(\int d^3y\left(-D_i\frac{\delta F}{\delta A_i(y)}\right)\alpha(y)\right)\\ &= \det{}^{-1}\left(-D_i\frac{\delta F}{\delta A_i(y)}\right)\left(\int \mathcal{D}\alpha\delta(\alpha(y))\right).\end{aligned} \tag{9.95}$$

The last factor is 1, thus

$$\Delta(A) = \det\left(-D_i\frac{\delta F}{\delta A_i(y)}\right). \tag{9.96}$$

This expression is usually re-expressed as a fermionic functional integral over the so-called Faddeev–Popov ghost fields, which formally gives the determinant; however, for our analysis, we will not require or implement this step.

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9.6 Monopoles in the Functional Integral

We want to calculate the functional integral

$$\langle 0| e^{-\frac{T\hat{H}}{\hbar}} |0\rangle = \mathcal{N} \int \mathcal{D}(A,\phi)\, e^{-\frac{S_E(A,\phi)}{\hbar}}. \tag{9.97}$$

We will calculate it in Gaussian approximation about the critical points of $S_E(A,\phi)$. This corresponds to integrating over the space of fields in the infinitesimal neighbourhood of the classical critical points, the monopole solutions. The usual understanding is that the contribution from the fields that are not in the infinitesimal neighbourhood of the monopole solutions will be suppressed by the exponential of the action. Knowing monopole solutions exist and are the critical points, we will get a result of the form

$$\langle 0| e^{-\frac{T\hat{H}}{\hbar}} |0\rangle = \mathcal{N} \sum_{n=-\infty}^{\infty} e^{-\frac{S_E(n \text{ monopoles})}{\hbar}} \det{}^{-\frac{1}{2}} \left[\left(\frac{\delta^2 S_E}{\delta \phi_i^2} \right)\bigg|_{\text{crit.}} \right]. \tag{9.98}$$

To make this expression quantitative, we must do three further calculations:

1. Find the action for N instantons (n_1 monopoles and n_2 anti-monopoles with $n_1 + n_2 = N$).
2. Identify and separate the zero modes in the spectrum of Gaussian fluctuations.
3. Define the measure of functional integration to make the determinant in Equation (9.98) well-defined.

9.6.1 The Classical Action

As usual

$$\begin{aligned} \mathcal{N} \left(\frac{\delta^2 S_E}{\delta \phi_i^2} \right)\bigg|_{\text{crit.}} &= \left(\frac{\left(\frac{\delta^2 S_E}{\delta \phi_i^2}\right)\Big|_{\text{crit.}}}{\left(\frac{\delta^2 S_E}{\delta \phi_i^2}\right)\Big|_{\text{vac.}}} \right) \mathcal{N} \left(\frac{\delta^2 S_E}{\delta \phi_i^2} \right)\bigg|_{\text{vac.}} \\ &= K^n \cdot 1, \end{aligned} \tag{9.99}$$

where "crit." stands for the critical point of n instantons, and "vac." stands for the vacuum configuration. The last factor is equal to 1 which serves to define $\mathcal{N}$

$$\mathcal{N} \left(\frac{\delta^2 S_E}{\delta \phi_i^2} \right)\bigg|_{\text{vac.}} \equiv 1. \tag{9.100}$$

The action for n widely separated instantons is n times that of one instanton. The number of such configurations behaves like

$$\sim \frac{(V\beta)^n}{n!}. \tag{9.101}$$

This "entropy" factor is, as usual, much larger than the corresponding factor when any subset of these n instantons are constrained to be close together, *i.e.* multi-monopole configurations. Even though the contribution of n widely separated

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instantons is suppressed by the exponential of its action $e^{-nS_E^0}$, the "entropy" factor can be big for a large finite spacetime volume $(V\beta)^n$, until eventually the $1/n!$ takes over as it will always eventually dominate.

The action for a single monopole is defined by a function $\epsilon\left(\frac{\lambda}{e^2}\right)$:

$$S_E^0 = \frac{m_W}{e^2}\,\epsilon\left(\frac{\lambda}{e^2}\right). \tag{9.102}$$

$m_W \sim a$ and the function ϵ can, in general, only be calculated numerically; however, in the Prasad–Sommerfield limit, $\epsilon(0) = 4\pi$, S_E^0 comes almost entirely from the integration over the core region

$$\int_{|\vec{x}|<R} d^3x \mathcal{L}^E = \frac{m_W}{e^2}\epsilon\left(\frac{\lambda}{e^2}\right)\left(1 + o\left(\frac{1}{m_W R}\right)\right). \tag{9.103}$$

The correction to the action from fields outside the core behaves like $\frac{1}{R}$, exactly the classical Coulomb self-energy of a magnetic charge.

For n well-separated monopoles of charge $\frac{4\pi q_a}{e}$, in addition to the Coulomb self-energy of each monopole, there is also a Coulomb interaction energy, a correction that is additive

$$S_E|_{\text{Coulomb}} = \frac{\pi}{2e^2}\sum_{a\neq b}\frac{q_a q_b}{|\vec{x}_a - \vec{x}_b|}, \tag{9.104}$$

with $q_a = \pm 1$. Then

$$S_E(n \text{ monopoles}) = \frac{m_W}{e^2}\epsilon\left(\frac{\lambda}{e^2}\right)\sum_a q_a^2 + \frac{\pi}{2e^2}\sum_{a\neq b}\frac{q_a q_b}{|\vec{x}_a - \vec{x}_b|} + o\left(\frac{1}{m_W R}\right), \tag{9.105}$$

where the small corrections exist because the monopoles are not point charges but spread out over regions of size $\frac{1}{m_W R}$. The additional Coulomb interaction energy term is non-negligible and has profound consequences.

9.6.2 Monopole Contribution: Zero Modes

Now we are in a position to analyse the zero-mode spectrum. If we write

$$A_i = A_i^{\text{cl}} + a_i \quad \phi = \phi^{\text{cl}} + \varphi, \tag{9.106}$$

where a_i and φ are quantum fluctuations about the classical values, we have the expansion of the action to second order in the fluctuations,

$$S_E = (S_E)_{\text{cl}} + (S_E)_2 + \cdots. \tag{9.107}$$

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The first-order term vanishes because the equations of motion are satisfied for the classical fields, and $(S_E)_2$ is given by

$$(S_E)_2 = \int d^3x \mathrm{tr}\left[\frac{1}{4e^2}\left(D_i^{\mathrm{cl}}a_j - D_j^{\mathrm{cl}}a_i\right)^2 + \frac{1}{2e^2}\left([a_i,a_j]F_{ij}^{\mathrm{cl}}\right) + \frac{1}{2}\left[a_i,\phi^{\mathrm{cl}}\right]^2 + \frac{1}{2}\left(D_i^{\mathrm{cl}}\varphi\right)^2 + \frac{1}{2}\varphi\mu^2\left(\phi^{\mathrm{cl}}\right)\varphi + \phi^{\mathrm{cl}}\left[D_i^{\mathrm{cl}}\varphi, a_i\right] + D_i^{\mathrm{cl}}\phi^{\mathrm{cl}}\left[a_i,\varphi\right]\right] \quad (9.108)$$

with

$$D_i^{\mathrm{cl}} = \partial_i + \left[A_i^{\mathrm{cl}}, \,.\right. \quad (9.109)$$

This is a bilinear expression in a_i and φ, thus integration over these fields will give $\det^{-\frac{1}{2}}(\mathcal{O})$, where the operator $\mathcal{O}$ is the hermitean, linear, second-order differential operator appearing between these fields in Equation (9.108). We expect $\mathcal{O}$ to have eigenfunctions as (although they generally will be a continuous set)

$$\mathcal{O}\left(A^{\mathrm{cl}},\phi^{\mathrm{cl}}\right)\begin{pmatrix} a_i^n \\ \phi^n \end{pmatrix} = \Omega_n^2 \begin{pmatrix} a_i^n \\ \phi^n \end{pmatrix}. \quad (9.110)$$

We expect the eigenvalues to be positive or zero, since the classical solution about which we expand the action is a minimum of the action. It is important to see that for any n such that $\Omega_n^2 > 0$ the corresponding eigenfunctions satisfy

$$D_i^{\mathrm{cl}}a_i^n + \left[\phi^{\mathrm{cl}},\phi^n\right] = 0. \quad (9.111)$$

We will prove this from the hermiticity of the operator $\mathcal{O}$, and the evident fact that

$$a_i^0 = D_i^{\mathrm{cl}}\alpha(x), \quad \phi^0 = \left[\phi^{\mathrm{cl}},\alpha(x)\right] \quad (9.112)$$

is a zero mode of $\mathcal{O}$ for every choice of $\alpha(x)$. a_i^0 and ϕ^0 are simply the changes induced by a gauge transformation, hence $S_E\left(A_i^{\mathrm{cl}}+a_i^0,\phi^{\mathrm{cl}}+\phi^0\right) = S_E\left(A_i^{\mathrm{cl}},\phi^{\mathrm{cl}}\right)$, which is valid order by order. This implies

$$S_2^E = \int dx (a_i^0,\phi^0)\mathcal{O}\begin{pmatrix} a_i^0 \\ \phi^0 \end{pmatrix} = 0. \quad (9.113)$$

Since $\mathcal{O}$ is hermitean, the modes for $\Omega_n^2 > 0$ are orthogonal to the zero modes hence

$$\begin{aligned} 0 &= \int d^3x \mathrm{tr}\left(a_i^n a_i^0 + \phi^n\phi^0\right) \\ &= \int d^3x \mathrm{tr}\left(a_i^n D_i^{\mathrm{cl}}\alpha(x) + \phi^n\left[\phi^{\mathrm{cl}},\alpha(x)\right]\right) \\ &= \int d^3x \mathrm{tr}\left(\partial_i(a_i^n\alpha(x)) - (D_i^{\mathrm{cl}}a_i^n)\alpha(x) + \left[\phi^n,\phi^{\mathrm{cl}}\right]\alpha(x)\right) \\ &= -\int d^3x \mathrm{tr}\left(\left(D_i^{\mathrm{cl}}a_i^n + \left[\phi^{\mathrm{cl}},\phi^n\right]\right)\alpha(x)\right) \\ &\Rightarrow D_i^{\mathrm{cl}}a_i^n + \left[\phi^{\mathrm{cl}},\phi^n\right] = 0. \end{aligned} \quad (9.114)$$

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The conclusion in the last equation is reached since the integral must vanish for any choice of $\alpha(x)$. The $\alpha(x)$ zero modes in Equation (9.112) are not physical zero modes, they arise from the gauge invariance. If we impose the gauge choice

$$D_i^{\mathrm{cl}} A_i + \left[\phi^{\mathrm{cl}}, \phi\right] = 0 \tag{9.115}$$

with the understanding that the classical fields are assumed to satisfy this gauge condition, we can show that the unphysical gauge zero modes simply do not exist. Indeed, the gauge condition implies

$$\begin{aligned} 0 = D_i^{\mathrm{cl}} A_i + \left[\phi^{\mathrm{cl}}, \phi\right] &= D_i^{\mathrm{cl}}(A_i^{\mathrm{cl}} + a_i) + \left[\phi^{\mathrm{cl}}, \phi^{\mathrm{cl}} + \varphi\right] \\ &= D_i^{\mathrm{cl}} A_i^{\mathrm{cl}} + \left[\phi^{\mathrm{cl}}, \phi^{\mathrm{cl}}\right] + D_i^{\mathrm{cl}} a_i + \left[\phi^{\mathrm{cl}}, \varphi\right] \\ &= D_i^{\mathrm{cl}} a_i + \left[\phi^{\mathrm{cl}}, \varphi\right]. \end{aligned} \tag{9.116}$$

Then we see that the norm of the putative zero mode that satisfies the gauge condition Equation (9.115), that is $D_i^{\mathrm{cl}} a_i^0 + \left[\phi^{\mathrm{cl}}, \phi^0\right] = 0$, simply vanishes:

$$\begin{aligned} \int d^3x \mathrm{tr}\left(a_i^0 a_i^0 + \phi^0 \phi^0\right) &= \int d^3x \mathrm{tr}\left(D_i^{\mathrm{cl}}\alpha(x) D_i^{\mathrm{cl}}\alpha(x) + (\left[\phi^{\mathrm{cl}}, \alpha(x)\right])^2\right) \\ &= -\int d^3x \mathrm{tr}\left(\left(D_i^{\mathrm{cl}} D_i^{\mathrm{cl}}\alpha(x) + \left[\phi^{\mathrm{cl}}, \left[\phi^{\mathrm{cl}}, \alpha(x)\right]\right]\right)\alpha(x)\right) \\ &= -\int d^3x \mathrm{tr}\left(\left(D_i^{\mathrm{cl}} a_i^0 + \left[\phi^{\mathrm{cl}}, \phi^0\right]\right)\alpha(x)\right) = 0. \end{aligned} \tag{9.117}$$

This requires $a_i^0 = \phi^0 = 0$, that is, the pure gauge zero mode that satisfies the gauge condition simply does not exist.

The Faddeev–Popov factor comes from the gauge-fixing condition

$$F(A,\phi) = D_i^{\mathrm{cl}}(A_i) + \left[\phi^{\mathrm{cl}}, \phi\right] = 0. \tag{9.118}$$

Then following Equation (9.95) we have

$$\begin{aligned} F(A^{1+\alpha}, \phi^{1+\alpha}) &= D_i^{\mathrm{cl}}(A_i + D_i^A \alpha(x)) + \left[\phi^{\mathrm{cl}}, \phi + [\phi, \alpha(x)]\right] \\ &= D_i^{\mathrm{cl}} A_i + D_i^{\mathrm{cl}} D_i^A \alpha(x) + \left[\phi^{\mathrm{cl}}, \phi\right] + \left[\phi^{\mathrm{cl}}, [\phi, \alpha(x)]\right] \\ &= D_i^{\mathrm{cl}} A_i + \left[\phi^{\mathrm{cl}}, \phi\right] + D_i^{\mathrm{cl}} D_i^A \alpha(x) + \left[\phi^{\mathrm{cl}}, [\phi, \alpha(x)]\right]. \end{aligned} \tag{9.119}$$

Thus from Equation (9.96)

$$\begin{aligned} \Delta(A,\phi) &= \det\left(D_i^{\mathrm{cl}} D_i^A + \left[\phi^{\mathrm{cl}}, [\phi, \ \right)\right. \\ &= \det\left(D_i^{\mathrm{cl}} D_i^{\mathrm{cl}} + \left[\phi^{\mathrm{cl}}, \left[\phi^{\mathrm{cl}}, \ \right)\right.\right.(1 + o(a_i, \varphi)). \end{aligned} \tag{9.120}$$

9.6.3 Defining the Integration Measure

We can go further by defining the metric and integration measure on function space. We will integrate over an infinitesimal neighbourhood of the classical fields.

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With $\delta A_i \equiv a_i = A_i - A_i^{\rm cl}$ and $\delta\phi \equiv \varphi = \phi - \phi^{\rm cl}$ to emphasize that we are in an infinitesimal neighbourhood of the classical fields, we can write the metric as

$$(\delta l)^2 = -\int d^3x \mathrm{tr}\left((\delta A_i)^2 + (\delta\phi)^2\right). \tag{9.121}$$

The minus sign is to take into account the anti-hermitean generators of the Lie algebra of the gauge group taken in the definition of the gauge fields and scalar fields. This metric is gauge-invariant since the infinitesimal change in the fields transform homogeneously under gauge transformations, and hence the gauge transformation cancels out due to the cyclicity of the trace. We parametrize the space of all gauge fields as a sub-manifold which corresponds to those gauge fields that satisfy the gauge condition, which is called the gauge slice, and orthogonal directions which correspond to gauge transformations. These lead to those gauge fields that do not satisfy the gauge condition but lie along the gauge orbit of the gauge fields on the gauge slice. We can expand the variations δA_i and $\delta\phi$ in terms of an arbitrary, linear combination of the eigenmodes of the operator $\mathcal{O}$, which respect the gauge condition, plus an arbitrary linearized gauge transformation. The eigenmodes translate us along the gauge slice while an arbitrary deformation off the gauge slice corresponds to a gauge transformation. Hence expanding to first order in ξ^n and $\alpha(x)$ gives

$$\begin{aligned} A_i &= A_i^{\rm cl} + \sum_n \xi^n a_i^n + D_i^{\rm cl}\alpha(x) \\ \phi &= \phi^{\rm cl} + \sum_n \xi^n \phi^n + \left[\phi^{\rm cl}, \alpha(x)\right] \end{aligned} \tag{9.122}$$

hence

$$\begin{aligned} (\delta l)^2 &= \sum_n (\xi_n)^2 - \int d^3x \mathrm{tr}\left(\left(D_i^{\rm cl}\alpha(x)\right)^2 + \left[\phi^{\rm cl}, \alpha(x)\right]^2\right) \\ &= \sum_n (\xi_n)^2 - \int d^3x \mathrm{tr}\left(\alpha(x)\left(-D_i^{\rm cl}D_i^{\rm cl} - , \phi^{\rm cl}\right]\left[\phi^{\rm cl}, \right)\alpha(x)\right) \\ &= \sum_n (\xi_n)^2 - \int d^3x \mathrm{tr}\left(\alpha(x)\left(-D_i^{\rm cl}D_i^{\rm cl} - \left[\phi^{\rm cl}, \left[\phi^{\rm cl}, \right)\alpha(x)\right)\right.\right. . \end{aligned} \tag{9.123}$$

Thus the measure is given by

$$\begin{aligned} \mathcal{D}(A_i,\phi) = \prod_x \mathcal{D}A_i(x)\mathcal{D}\phi(x) &= \prod_n d\xi_n \prod_x d\alpha(x) \det{}^{\frac{1}{2}}\left(\left(D_i^{\rm cl}D_i^{\rm cl} + \left[\phi^{\rm cl}, \left[\phi^{\rm cl}, \right)\right)\right.\right. \\ &\equiv \prod_n d\xi_n \mathcal{D}\alpha(x) \det{}^{\frac{1}{2}}\left(\left(D_i^{\rm cl}D_i^{\rm cl} + \left[\phi^{\rm cl}, \left[\phi^{\rm cl}, \right)\right)\right.\right. \end{aligned} \tag{9.124}$$

using a direct generalization of the corresponding volume measure for a finite dimensional system, if $ds^2 = \sum_{ij} g_{ij} dx^i dx^j$ then the volume measure is $dV = d^n x\sqrt{g}$, where $g = \det[g_{ij}]$. Then the integration giving rise to the Euclidean generating functional Equation (9.92) is given by

$$\begin{aligned}\mathcal{I} &= \mathcal{N}' \int \mathcal{D}(A,\phi)\delta\left(F(A_i,\phi)\right)\Delta(A,\phi)e^{\frac{-S_E}{\hbar}} \\ &= \mathcal{N}' \int \prod_n d\xi_n \mathcal{D}\alpha(x) \det{}^{\frac{1}{2}}\left(-\left(D_i^{\rm cl} D_i^{\rm cl} + \left[\phi^{\rm cl},\left[\phi^{\rm cl}, \ \right)\right)\delta\left(F(A_i,\phi)\right)\Delta(A,\phi)e^{\frac{-S_E}{\hbar}}.\right.\right.\end{aligned} \tag{9.125}$$

But $\delta(F(A_i,\phi)) = \delta\left(D_i^{\rm cl}(A_i) + \left[\phi^{\rm cl},\phi\right]\right)$ and then using the expansion Equation (9.122) gives

$$\begin{aligned}\int \mathcal{D}\alpha(x)\delta\left(F(A_i,\phi)\right) &= \int \mathcal{D}\alpha(x)\delta\left(-\left(D_i^{\rm cl} D_i^{\rm cl} + \left[\phi^{\rm cl},\left[\phi^{\rm cl}, \right) \ \alpha(x)\right)\right.\right. \\ &= \int \mathcal{D}\alpha(x) \det{}^{-1}\left(-\left(D_i^{\rm cl} D_i^{\rm cl} + \left[\phi^{\rm cl},\left[\phi^{\rm cl}, \ \right)\right)\left(\delta(\alpha(x))\right)\right.\right. \\ &= \det{}^{-1}\left(-\left(D_i^{\rm cl} D_i^{\rm cl} + \left[\phi^{\rm cl},\left[\phi^{\rm cl}, \ \right)\right).\right.\right.\end{aligned} \tag{9.126}$$

We notice that this factor will actually neatly cancel out the Faddeev–Popov determinant. Indeed, we get

$$\begin{aligned}\mathcal{D}(A_i,\phi)\Delta(A_i,\phi) &= \prod_n d\xi_n \det\left(D_i^{\rm cl} D_i^{A} + \left[\phi^{\rm cl},\left[\phi, \ \right)\frac{\det^{\frac{1}{2}}\left(D_i^{\rm cl} D_i^{\rm cl} + \left[\phi^{\rm cl},\left[\phi^{\rm cl}, \ \right)\right.\right.}{\det\left(D_i^{\rm cl} D_i^{\rm cl} + \left[\phi^{\rm cl},\left[\phi^{\rm cl}, \ \right)\right.\right.}\right.\right. \\ &\approx \prod_n d\xi_n \det{}^{\frac{1}{2}}\left(D_i^{\rm cl} D_i^{\rm cl} + \left[\phi^{\rm cl},\left[\phi^{\rm cl}, \ \right),\right.\right.\end{aligned} \tag{9.127}$$

where in the first line we have retained the full Faddeev–Popov factor multiplied by the factor coming from the measure and the integration over the gauge-fixing delta function.

There are still the physical zero modes corresponding to translation and internal rotational symmetries. The rotations give a finite constant volume factor which eventually cancels. Naively these are for translations

$$\begin{aligned}\tilde{a}_i^{(k,0)} &= N^{-\frac{1}{2}}\partial_k A_i^{\rm cl} \\ \tilde{\phi}^{(k,0)} &= N^{-\frac{1}{2}}\partial_k \phi^{\rm cl}\end{aligned} \tag{9.128}$$

however, these expressions do not satisfy the gauge condition. Augmenting by a gauge transformation gives (with $\alpha^k = -A_k^{\rm cl}$)

$$\begin{aligned}a_i^{(k,0)} &= N^{-\frac{1}{2}}\left(\partial_k A_i^{\rm cl} - D_i^{\rm cl} A_k^{\rm cl}\right) = N^{-\frac{1}{2}} F_{ki}^{\rm cl} \\ \phi^{(k,0)} &= N^{-\frac{1}{2}}\left(\partial_k \phi^{\rm cl} + \left[A_k^{\rm cl},\phi^{\rm cl}\right]\right) = N^{-\frac{1}{2}} D_k^{\rm cl}\phi^{\rm cl}\end{aligned} \tag{9.129}$$

with $N = -\int d^3x \operatorname{tr}\left(\left(F_{ki}^{\rm cl}\right)^2 + \left(D_k^{\rm cl}\phi^{\rm cl}\right)^2\right)$. The gauge condition

$$D_i^{\rm cl} F_{ki} + \left[\phi^{\rm cl}, D_k \phi^{\rm cl}\right] = 0 \tag{9.130}$$

is just the equation of motion. Under a translation

$$\delta A_i = A_i^{\rm cl}(x+\delta R) - D_i^{\rm cl}(\delta R_j A_j) = \delta R_k F_{ki} = N^{\frac{1}{2}}\delta R_k a_i^{(k,0)} \tag{9.131}$$

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thus

$$d\xi_0^k = N^{\frac{1}{2}} dR_k \tag{9.132}$$

and

$$d^3\xi_0^k = N^{\frac{3}{2}} d^3\vec{R}. \tag{9.133}$$

So finally the integration measure is

$$\mathcal{D}(A_i,\phi)\Delta = N^{\frac{3}{2}} d^3\vec{R} \prod_{n\neq 0} d\xi_n \det^{\frac{1}{2}}\left(D_i^{\text{cl}}D_i^{\text{cl}} + [\phi^{\text{cl}}, [\phi^{\text{cl}}, \,)\right). \tag{9.134}$$

For one monopole we have

$$Z_1 = \int N^{\frac{3}{2}} d^3\vec{R} \det^{\frac{1}{2}}\left(D_i^{\text{cl}}D_i^{\text{cl}} + [\phi^{\text{cl}}, [\phi^{\text{cl}}, \,)\right) \prod_{n\neq 0}\left(\frac{\Omega_n^0}{\Omega_n}\right) e^{-\frac{(S_E)_0}{\hbar}}$$

$$= \int \frac{m_W^{\frac{7}{2}}}{e}\alpha\left(\frac{\lambda}{e^2}\right) e^{-\epsilon\left(\frac{\lambda}{e^2}\right)\frac{m_W}{e^2}} d^3\vec{R} \tag{9.135}$$

from dimensional analysis and α is a function that can, in principle, be calculated. For N (not to be confused with the normalization above) instantons, n_1 monopoles and n_2 anti-monopoles,

$$Z_N = \frac{\zeta^N}{N!}\int \prod_{j=1}^N d^3\vec{R}_j \sum_{q_a=\pm 1} e^{-\frac{\pi}{2e^2}\sum_{a\neq b}\frac{q_a q_b}{|\vec{R}_a-\vec{R}_b|}} \tag{9.136}$$

and

$$Z = \sum_{N,q_a=\pm 1}\frac{\zeta^N}{N!}\int \prod_{j=1}^N d^3\vec{R}_j e^{-\frac{\pi}{2e^2}\sum_{a\neq b}\frac{q_a q_b}{|\vec{R}_a-\vec{R}_b|}}, \tag{9.137}$$

where

$$\zeta = \frac{m_W^{\frac{7}{2}}}{e}\alpha\left(\frac{\lambda}{e^2}\right) e^{-\epsilon\left(\frac{\lambda}{e^2}\right)\frac{m_W}{e^2}}. \tag{9.138}$$

9.7 Coulomb Gas and Debye Screening

This is exactly the partition function of a Coulomb gas. We know that such a gas has the property of screening. This is the same as confinement. Any electric fields will be cancelled exactly by a complete rearrangement of the particles in the gas.

If we re-express Z as a functional integral

$$Z = \int \mathcal{D}\chi e^{-\frac{\pi e^2}{2}\int d^3x(\nabla\chi)^2} \sum_{N,q_a=\pm 1}\frac{\zeta^N}{N!}\int \prod_{j=1}^N d^3\vec{R}_j e^{i\sum_a q_a\chi(\vec{R}_a)}. \tag{9.139}$$

Indeed,

$$\int \mathcal{D}\chi e^{-\frac{\pi e^2}{2}\int d^3x(\nabla\chi)^2 + i\sum_a q_a\chi(\vec{R}_a)} =$$

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$$= \int \mathcal{D}\chi e^{-\frac{\pi e^2}{2}\int d^3x\left(-\chi\nabla^2\chi+i\frac{2}{\pi e^2}\sum_a q_a\delta(\vec{x}-\vec{R}_a)\chi(\vec{x})\right)}$$
$$= \int \mathcal{D}\chi e^{-\frac{\pi e^2}{2}\int d^3x\left(\chi+\frac{i}{\pi e^2}\sum_a q_a\delta(\vec{x}-\vec{R}_a)\left(\frac{1}{-\nabla^2}\right)\right)\left(-\nabla^2\right)\left(\chi+\frac{i}{\pi e^2}\left(\frac{1}{-\nabla^2}\right)\sum_b q_b\delta(\vec{x}-\vec{R}_b)\right)}$$
$$\times e^{-\frac{\pi e^2}{2}\int d^3x\frac{1}{\left(\pi e^2\right)^2}\sum_a q_a\delta(\vec{x}-\vec{R}_a)\left(\frac{1}{-\nabla^2}\right)\left(-\nabla^2\right)\left(\frac{1}{-\nabla^2}\right)\sum_b q_b\delta(\vec{x}-\vec{R}_b)}$$
$$= \mathcal{C}e^{-\frac{1}{2\pi e^2}\int d^3x\sum_a q_a\delta(\vec{x}-\vec{R}_a)\left(\frac{1}{-\nabla^2}\right)\sum_b q_b\delta(\vec{x}-\vec{R}_b)}$$
$$= \mathcal{C}e^{-\frac{1}{2\pi e^2}\int d^3x\sum_a q_a\delta(\vec{x}-\vec{R}_a)\frac{1}{4\pi}\int d^3y\frac{1}{|\vec{x}-\vec{y}|}\sum_b q_b\delta(\vec{y}-\vec{R}_b)}$$
$$= \mathcal{C}e^{-\frac{1}{2\pi e^2}\int d^3x\sum_a q_a\delta(\vec{x}-\vec{R}_a)\sum_b q_b\frac{1}{4\pi|\vec{x}-\vec{R}_b|}}$$
$$= \mathcal{C}e^{-\frac{1}{8\pi e^2}\sum_{a,b,a\neq b} q_a q_b\frac{1}{|\vec{R}_a-\vec{R}_b|}}, \tag{9.140}$$

where we absorb a harmless divergence at $a=b$ into the constant.[2] Thus (using $e\to e/2\pi$ in Equation (9.139)) we have

$$Z = \int \mathcal{D}\chi e^{-\frac{e^2}{8\pi}\int d^3x(\nabla\chi)^2}\sum_N\frac{\zeta^N}{N!}\int\prod_{j=1}^N d^3\vec{R}_j\left(e^{i\chi(\vec{R}_j)}+e^{-i\chi(\vec{R}_j)}\right)$$
$$= \int \mathcal{D}\chi e^{-\frac{e^2}{8\pi}\int d^3x(\nabla\chi)^2}\sum_N\frac{\zeta^N}{N!}\left(\int d^3\vec{R}\,2\cos(\chi(\vec{R}))\right)^N$$
$$= \int \mathcal{D}\chi e^{-\frac{e^2}{8\pi}\frac{\pi e^2}{2}\int d^3x(\nabla\chi)^2}e^{2\zeta\int d^3x\cos(\chi(x))}$$
$$= \int \mathcal{D}\chi e^{-\frac{e^2}{8\pi}\int d^3x\left((\nabla\chi)^2-M^2\cos(\chi(x))\right)} \tag{9.141}$$

with $M^2=\frac{16\pi\zeta}{e^2}$.

There are no massless modes. The coupling constant, nominally taken as ζ, satisfies $\zeta\propto e^{-\left(\frac{m_W}{e^2}\right)\epsilon\left(\frac{\lambda}{e^2}\right)}<<1$ as $e\to 0$. This means that there are no massless gauge bosons, the low-energy Abelian theory is confined due to the effects of instantons. This is an incredible result; the theory is confining. Unfortunately, the result will not go over to four dimensions. However, in three dimensions, where the general arguments concerning the flux subtended by a large Wilson loop are critical, we find that the theory nevertheless favours confinement.

[2] We have a slight discrepancy with respect to Polyakov's paper [103]. We find that in Equation (9.139) we should replace $e\to e/2\pi$. This does not change the behaviour of the theory. We implement the change from now on.

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