6
Inclusive weak decay

In this chapter, we will study inclusive weak decays of hadrons containing a b quark. The lowest mass meson or baryon containing a b quark decays weakly, since the strong and electromagnetic interactions preserve quark flavor. One of the main results of this chapter is the demonstration that the parton model picture that inclusive heavy hadron decay is the same as free heavy quark decay is exact in the $m_b \to \infty$ limit. In addition, we will show how to include radiative and nonperturbative corrections to the leading-order formula in a systematic way. The analysis closely parallels that of deep inelastic scattering in Sec. 1.8.

6.1 Inclusive semileptonic decay kinematics

Semileptonic $\bar{B}$-meson decays to final states containing a charm quark arise from matrix elements of the weak Hamiltonian density

$$H_W = \frac{4G_F}{\sqrt{2}}\, V_{cb}\, \bar{c}\gamma^\mu P_L b\, \bar{e}\gamma_\mu P_L \nu_e. \tag{6.1}$$

In exclusive three-body decays such as $\bar{B} \to De\bar{\nu}_e$, one looks at the decay into a definite final state, such as $De\bar{\nu}_e$. The differential decay distribution has two independent kinematic variables, which can be chosen to be E_e and E_{ν_e}, the energy of the electron and antineutrino. The decay distribution depends implicitly on the masses of the initial and final particles, which are constants. In inclusive decays, one ignores all details about the final hadronic state X_c and sums over all final states containing a c quark. Here X_c can be a single-particle state, such as a D meson, or a multiparticle state, such as $D\pi$. In addition to the usual two kinematic variables E_e and E_{ν_e} for exclusive semileptonic decays, there is an additional kinematic variable in $\bar{B} \to X_c e\bar{\nu}_e$ decay since the invariant mass of the final hadronic system can vary. The third variable will be chosen to be q^2, the invariant mass of the virtual W boson. The diagrams for semileptonic b-quark and $\bar{B}$-meson decays are shown in Fig. 6.1.

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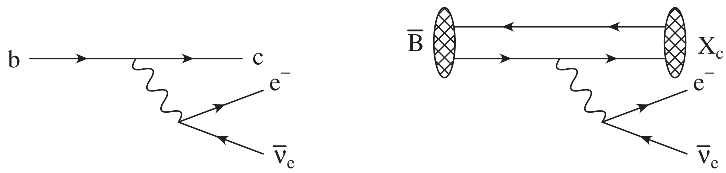

Fig. 6.1. Weak decay diagrams for semileptonic quark and hadron decay.

In the $\bar{B}$ rest frame, the differential decay distribution for inclusive semileptonic decay is

$$\begin{aligned}\frac{\mathrm{d}\Gamma}{\mathrm{d}q^2\,\mathrm{d}E_e\,\mathrm{d}E_{\nu_e}}\\ &= \int \frac{d^4 p_e}{(2\pi)^4} \int \frac{d^4 p_{\nu_e}}{(2\pi)^4}\, 2\pi\,\delta\big(p_e^2\big) 2\pi\,\delta\big(p_{\nu_e}^2\big)\theta\big(p_e^0\big)\theta\big(p_{\nu_e}^0\big)\\ &\quad\times \delta\big(E_e - p_e^0\big)\delta\big(E_{\nu_e} - p_{\nu_e}^0\big)\delta\big[q^2 - \big(p_e + p_{\nu_e}\big)^2\big]\\ &\quad\times \sum_{X_c}\sum_{\substack{\text{lepton}\\ \text{spins}}} \frac{|\langle X_c e \bar{\nu}_e | H_W | \bar{B}\rangle|^2}{2m_B}(2\pi)^4\delta^4\big[p_B - \big(p_e + p_{\nu_e}\big) - p_{X_c}\big], \end{aligned} \tag{6.2}$$

where we have used the familiar formula $d^3p/(2E) = d^4p\,\delta(p^2 - m^2)\theta(p^0)$ and neglected the electron mass. The phase space integrations can be performed in the rest frame of the $\bar{B}$ meson. After summation over final hadronic states X_c, the only relevant angle is that between the electron and the neutrino three momenta. Nothing depends on the direction of the neutrino momentum, and integrating over it gives a factor of 4π. One can then choose the z axis for the electron momentum to be aligned along the neutrino direction. Integrating over the electron azimuthal angle gives a factor of 2π. Consequently, the lepton phase space is

$$d^3p_e d^3p_{\nu_e} = 8\pi^2 |\mathbf{p}_e|^2 d|\mathbf{p}_e| |\mathbf{p}_{\nu_e}|^2 d|\mathbf{p}_{\nu_e}| d\cos\theta, \tag{6.3}$$

where θ is the angle between the electron and neutrino directions. The three remaining integrations are fixed by the three delta functions. Using $\delta(p_e^2) = \delta(E_e^2 - |\mathbf{p}_e|^2)$ to perform the integration over $|\mathbf{p}_e|$, $\delta(p_{\nu_e}^2) = \delta(E_{\nu_e}^2 - |\mathbf{p}_{\nu_e}|^2)$ to perform the integration over $|\mathbf{p}_{\nu_e}|$, and $\delta[q^2 - (p_e + p_{\nu_e})^2] = \delta[q^2 - 2E_e E_{\nu_e}(1 - \cos\theta)]$ to perform the integration over $\cos\theta$ gives

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}q^2\,\mathrm{d}E_e\,\mathrm{d}E_{\nu_e}} = \frac{1}{4}\sum_{X_c}\sum_{\substack{\text{lepton}\\ \text{spins}}} \frac{|\langle X e \bar{\nu}_e | H_W | \bar{B}\rangle|^2}{2m_B}\delta^4\big[p_B - \big(p_e + p_{\nu_e}\big) - p_{X_c}\big]. \tag{6.4}$$

The weak matrix element in Eq. (6.4) can be factored into a leptonic matrix element and a hadronic matrix element, since leptons do not have any strong interactions. Corrections to this result are suppressed by powers of G_F or α, and

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they arise from radiative corrections due to additional electroweak gauge bosons propagating between the quark and lepton lines. The matrix element average is conventionally written as the product of hadronic and leptonic tensors,

$$\frac{1}{4}\sum_{X_c}\sum_{\substack{\text{lepton}\\ \text{spins}}}\frac{|\langle X_c e\bar{\nu}_e|H_W|\bar{B}\rangle|^2}{2m_B}(2\pi)^3\delta^4\left[p_B-\left(p_e+p_{\nu_e}\right)-p_{X_c}\right]$$
$$=2G_F^2|V_{cb}|^2 W_{\alpha\beta}L^{\alpha\beta}, \tag{6.5}$$

where the leptonic tensor is

$$L^{\alpha\beta}=2\left(p_e^\alpha p_{\nu_e}^\beta+p_e^\beta p_{\nu_e}^\alpha-g^{\alpha\beta}p_e\cdot p_{\nu_e}-i\epsilon^{\eta\beta\lambda\alpha}p_{e\eta}p_{\nu_e\lambda}\right) \tag{6.6}$$

and the hadronic tensor is defined by

$$W^{\alpha\beta}=\sum_{X_c}(2\pi)^3\delta^4(p_B-q-p_{X_c})\frac{1}{2m_B}$$
$$\times\langle\bar{B}(p_B)|J_L^{\dagger\alpha}\left|X_c\left(p_{X_c}\right)\right\rangle\left\langle X_c\left(p_{X_c}\right)\right|J_L^\beta|\bar{B}(p_B)\rangle, \tag{6.7}$$

with $J_L^\alpha=\bar{c}\gamma^\alpha P_L b$, the left-handed current. In Eq. (6.7), $q=p_e+p_{\nu_e}$ is the sum of electron and antineutrino four momenta. Here $W_{\alpha\beta}$ is a second-rank tensor that depends on $p_B=m_B v$ and q, the momentum transfer to the hadronic system. The relation $p_B=m_B v$ defines v as the four velocity of the $\bar{B}$ meson. The b quark can have a small three velocity of the order of $1/m_b$ in the $\bar{B}$-meson rest frame, and this effect is included in the $1/m_b$ corrections computed later in this chapter.

The most general tensor $W_{\alpha\beta}$ is

$$W_{\alpha\beta}=-g_{\alpha\beta}W_1+v_\alpha v_\beta W_2-i\epsilon_{\alpha\beta\mu\nu}v^\mu q^\nu W_3+q_\alpha q_\beta W_4+(v_\alpha q_\beta+v_\beta q_\alpha)W_5. \tag{6.8}$$

The scalar structure functions W_j are functions of the Lorentz invariant quantities q^2 and $q\cdot v$. Using Eqs. (6.8), (6.6), and (6.5), we find the differential cross section in Eq. (6.4) becomes

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}q^2\,\mathrm{d}E_e\,\mathrm{d}E_{\nu_e}}=\frac{G_F^2|V_{cb}|^2}{2\pi^3}\left[W_1 q^2+W_2\left(2E_eE_{\nu_e}-q^2/2\right)\right.$$
$$\left.+W_3 q^2\left(E_e-E_{\nu_e}\right)\right]\theta\left(4E_eE_{\nu_e}-q^2\right), \tag{6.9}$$

where we have explicitly included the θ function that sets the lower limit for the E_{ν_e} integration because it will play an important role later in this chapter. The functions W_4 and W_5 do not contribute to the decay rate, since $q_\alpha L^{\alpha\beta}=q_\beta L^{\alpha\beta}=0$ in the limit that the electron mass is neglected. These terms have to be included in decays to the τ.

The neutrino is not observed, and so one integrates the above expression over E_{ν_e} to get the differential spectrum $\mathrm{d}\Gamma/\mathrm{d}q^2\,\mathrm{d}E_e$. For a fixed electron energy the minimum value of q^2 occurs when the electron and neutrino are parallel (i.e.,

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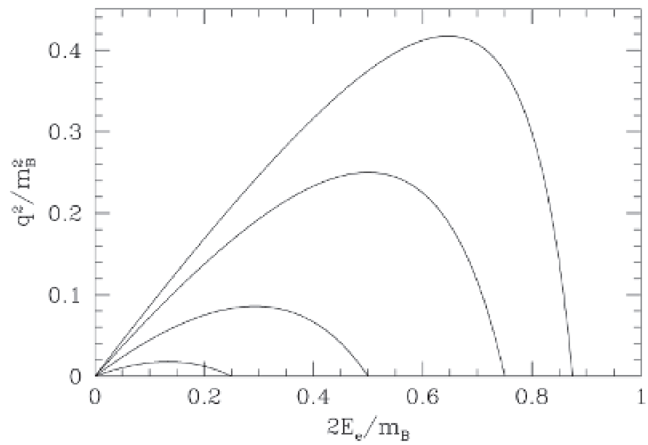

Fig. 6.2. The allowed q^2 values as a function of the electron energy E_e, for different values of the final state hadronic mass m_{X_c}. The entire region inside the curve is allowed. The curves are (from the outermost curve in) for $m_{X_c} = m_D$, $(m_{X_c}/m_B)^2 = 0.25$, 0.5, and 0.75, respectively.

$\cos\theta = 1$), and the maximum value occurs when the electron and neutrino are antiparallel (i.e., $\cos\theta = -1$). Hence

$$0 < q^2 < \frac{2E_e}{(m_B - 2E_e)}\left(m_B^2 - 2E_e m_B - m_{X_c^{\min}}^2\right), \tag{6.10}$$

where $X_c^{\min}$ is the lowest mass state containing a charm quark, i.e., the D meson. The maximum electron energy is

$$E_e^{\max} = \frac{m_B^2 - m_{X_c^{\min}}^2}{2m_B}, \tag{6.11}$$

which occurs at $q^2 = 0$. The allowed q^2 values as a function of E_e are plotted in Fig. 6.2. For a given value of the final hadronic system mass m_{X_c}, electron energy E_e, and q^2, the neutrino energy E_{ν_e} is

$$E_{\nu_e} = \left(\frac{m_B^2 - m_{X_c}^2 + q^2}{2m_B}\right) - E_e. \tag{6.12}$$

Consequently, integrating $\mathrm{d}\Gamma/\mathrm{d}q^2\,\mathrm{d}E_e\,\mathrm{d}E_{\nu_e}$ over E_{ν_e} (at fixed q^2 and E_e) to get $\mathrm{d}\Gamma/\mathrm{d}q^2\,\mathrm{d}E_e$ is equivalent to averaging over a range of final-state hadronic masses. We will see later in this chapter that in some regions of phase space, the validity of the operator product expansion for inclusive decays depends on hadronic mass averaging. For values of q^2 and E_e near the boundary of the allowed kinematic region, $q^2(m_B - 2E_e) - 2E_e(m_B^2 - 2E_e m_B - m_{X_c^{\min}}^2) = 0$, only final hadronic states with masses near $m_{X_c^{\min}}$ get averaged over in the integration over E_{ν_e}.

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The hadronic tensor $W_{\alpha\beta}$ parameterizes all strong interaction physics relevant for inclusive semileptonic $\bar{B}$ decay. It can be related to the discontinuity of a time-ordered product of currents across a cut. Consider the time-ordered product

$$T_{\alpha\beta} = -i \int d^4x e^{-iq\cdot x} \frac{\langle \bar{B}|T\left[J^\dagger_{L\alpha}(x)\, J_{L\beta}(0)\right]|\bar{B}\rangle}{2m_B}. \tag{6.13}$$

Inserting a complete set of states between the currents in each time ordering, using the analogs of Eqs. (1.159), applying the identity

$$\theta(x^0) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega\, \frac{e^{-i\omega x^0}}{\omega + i\varepsilon}, \tag{6.14}$$

and performing the integration over d^4x gives, in the $\bar{B}$ rest frame,

$$\begin{aligned} T_{\alpha\beta} = &\sum_{X_c} \frac{\langle \bar{B}|J^\dagger_{L\alpha}|X_c\rangle\langle X_c|J_{L\beta}|\bar{B}\rangle}{2m_B(m_B - E_X - q^0 + i\varepsilon)}(2\pi)^3\delta^3(\mathbf{q} + \mathbf{p}_X) \\ &- \sum_{X_{\bar{c}bb}} \frac{\langle \bar{B}|J_{L\beta}|X_{\bar{c}bb}\rangle\langle X_{\bar{c}bb}|J^\dagger_{L\alpha}|\bar{B}\rangle}{2m_B(E_X - m_B - q^0 - i\varepsilon)}(2\pi)^3\delta^3(\mathbf{q} - \mathbf{p}_X). \end{aligned} \tag{6.15}$$

Here X_c is a complete set of hadronic states containing a c quark, and $X_{\bar{c}bb}$ is a complete set of hadronic states containing two b quarks and a $\bar{c}$ quark. At fixed $\mathbf{q}$ the time-ordered product of currents $T_{\alpha\beta}$ has cuts in the complex q^0 plane along the real axis. One cut is in the region $-\infty < q^0 < m_B - \sqrt{m^2_{X_c^{\min}} + |\mathbf{q}|^2}$, and the other cut is in the region $\infty > q^0 > \sqrt{m^2_{X_{\bar{c}bb}^{\min}} + |\mathbf{q}|^2} - m_B$. The imaginary part of T (i.e., the discontinuity across the cut) can be evaluated using

$$\frac{1}{\omega + i\varepsilon} = P\frac{1}{\omega} - i\pi\delta(\omega), \tag{6.16}$$

where P denotes the principal value. This gives

$$\begin{aligned} \frac{1}{\pi}\,\mathrm{Im}\, T_{\alpha\beta} = &-\sum_{X_c} \frac{\langle \bar{B}|J^\dagger_{L\alpha}|X_c\rangle\langle X_c|J_{L\beta}|\bar{B}\rangle}{2m_B}(2\pi)^3\delta^4(p_B - q - p_X) \\ &- \sum_{X_{\bar{c}bb}} \frac{\langle \bar{B}|J_{L\beta}|X_{\bar{c}bb}\rangle\langle X_{\bar{c}bb}|J^\dagger_{L\alpha}|\bar{B}\rangle}{2m_B}(2\pi)^3\delta^4(p_B + q - p_X). \end{aligned} \tag{6.17}$$

The first of these two terms is just $-W_{\alpha\beta}$. For values of q and p_B in semileptonic $\bar{B}$ decay, the argument of the δ function in the second term of Eq. (6.17) is never zero, and it does not contribute to the imaginary part of T. It is convenient to express $T_{\alpha\beta}$ in terms of Lorentz scalar structure functions just as we did

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for $W_{\alpha\beta}$:

$$T_{\alpha\beta} = -g_{\alpha\beta}T_1 + v_\alpha v_\beta T_2 - i\epsilon_{\alpha\beta\mu\nu}v^\mu q^\nu T_3 + q_\alpha q_\beta T_4 + (v_\alpha q_\beta + v_\beta q_\alpha)T_5. \tag{6.18}$$

The T_j's are functions of q^2 and $q \cdot v$. One can study T_j in the complex $q \cdot v$ plane for fixed q^2. This is a Lorentz invariant way of studying the analytic structure discussed above. For the cut associated with physical hadronic states containing a c quark, $(p_B - q) - p_X = 0$, which implies that $v \cdot q = (m_B^2 + q^2 - m_{X_c}^2)/2m_B$. This cut is in the region $-\infty < v \cdot q < (m_B^2 + q^2 - m_{X_c^{\rm min}}^2)/2m_B$ (see Fig. 6.3). In contrast, the cut corresponding to physical hadronic states with a $\bar{c}$ quark and two b quarks has $(p_B + q) - p_X = 0$, which implies that $v \cdot q = (m_{X_{\bar{c}bb}}^2 - m_B^2 - q^2)/2m_B$. This cut occurs in the region $(m_{X_{\bar{c}bb}^{\rm min}}^2 - m_B^2 - q^2)/2m_B < v \cdot q < \infty$. These cuts are widely separated for all values of q^2 allowed in $\bar{B} \to X_c e \bar{\nu}_e$ semileptonic decay, $0 < q^2 < (m_B - m_{X_c^{\rm min}})^2$. The minimum separation between the cuts occurs for the maximal value of q^2. Approximating hadron masses by that of the heavy quark they contain (e.g., $m_{X_c^{\rm min}} = m_c$, $m_{X_{\bar{c}bb}^{\rm min}} = m_c + 2m_b$, etc.), we find the minimum separation between the two cuts is, $4m_c$, which is much greater than the scale $\Lambda_{\rm QCD}$ of nonperturbative strong interactions. The discontinuity across the left-hand cut gives the structure functions for inclusive semileptonic decay:

$$-\frac{1}{\pi}\,{\rm Im}\,T_j = W_j \quad \text{(left-hand cut only)}. \tag{6.19}$$

The double differential decay rate $d\Gamma/dq^2\,dE_e$ can be obtained from the triple differential rate $d\Gamma/dq^2\,dE_e\,dE_{\nu_e}$, or equivalently, $d\Gamma/dq^2\,dE_e\,dv \cdot q$, by integrating over $q \cdot v = E_e + E_{\nu_e}$. Integrals of the structure functions $W_j(q^2, v \cdot q)$ over $v \cdot q$ are then related to integrals of T_j over the contour $\mathcal{C}$ shown in Fig. 6.3.

The situation is similar for $b \to u$ decays. The results for this case can be obtained from our previous discussion just by changing the subscript c to u. However, since the u quark mass is negligible, the separation between the two cuts is not large compared with the scale of the strong interactions, $\Lambda_{\rm QCD}$, when q^2 is near its maximal value for $b \to u$ decays. The significance of this will be commented on later in this chapter.

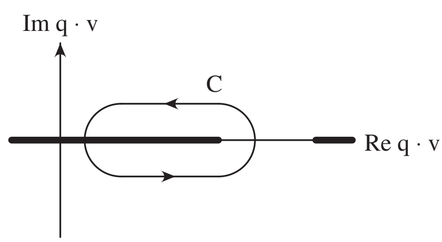

Fig. 6.3. Contour for the T_j integral.

6.2 The operator product expansion

The structure functions T_j can be expressed in terms of matrix elements of local operators using the operator product expansion to simplify the time-ordered product of currents,

$$-i \int d^4x e^{-iq\cdot x} T\left[J^\dagger_{L\alpha}(x)\, J_{L\beta}(0)\right], \tag{6.20}$$

whose $\bar{B}$-meson matrix element is $T_{\alpha\beta}$. The coefficients of the operators that occur in this expansion can be reliably computed by using QCD perturbation theory, in any region of $v \cdot q$ that is far away (compared with $\Lambda_{\rm QCD}$) from the cuts. We compute the coefficients of the operators that occur in the operator product expansion by using quark and gluon matrix elements of Eq. (6.20). These operators will involve the b-quark field, covariant derivatives D, and the gluon field strength $G^A_{\mu\nu}$. At dimension six and above, the light quark fields also occur.

At lowest order in perturbation theory the matrix element of Eq. (6.20) between b-quark states with momentum $m_b v + k$ is (see Fig. 6.4)

$$\frac{1}{(m_b v - q + k)^2 - m_c^2 + i\varepsilon} \bar{u}\gamma_\alpha P_L (m_b \not{v} - \not{q} + \not{k})\gamma_\beta P_L u. \tag{6.21}$$

In the matrix elements of interest, q is usually of the order of m_b, but k is of the order of $\Lambda_{\rm QCD}$. Expanding in powers of k gives an expansion in powers of $\Lambda_{\rm QCD}/m_b$, and thus an expansion in $1/m_b$ of the form factors T_j.

6.2.1 Lowest order

The order k^0 terms in the expansion of Eq. (6.21) are

$$\begin{aligned}\frac{1}{\Delta_0}\bar{u}[&(m_b v - q)_\alpha \gamma_\beta + (m_b v - q)_\beta \gamma_\alpha - (m_b \not{v} - \not{q}) g_{\alpha\beta} \\ &- i\epsilon_{\alpha\beta\lambda\eta}(m_b v - q)^\lambda \gamma^\eta] P_L u,\end{aligned} \tag{6.22}$$

where

$$\Delta_0 = (m_b v - q)^2 - m_c^2 + i\varepsilon, \tag{6.23}$$

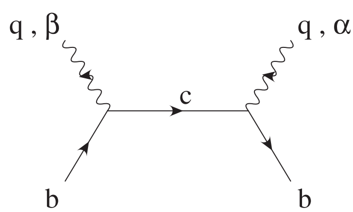

Fig. 6.4. Leading-order diagrams in the OPE.

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and we have used the identity in Eq. (1.119). The matrix elements of the dimension-three operators $\bar{b}\gamma^\lambda b$ and $\bar{b}\gamma^\lambda\gamma_5 b$ between b-quark states are $\bar{u}\gamma^\lambda u$ and $\bar{u}\gamma^\lambda\gamma_5 u$, respectively, so the operator product expansion is obtained by replacing u and $\bar{u}$ in Eq. (6.22) by the fields b and $\bar{b}$, respectively. Finally, to get the T_j we take the hadronic matrix elements of the operators,

$$\langle \bar{B}(p_B)|\bar{b}\gamma_\lambda b|\bar{B}(p_B)\rangle = 2p_{B\lambda} = 2m_B v_\lambda \tag{6.24}$$

and

$$\langle \bar{B}(p_B)|\bar{b}\gamma_\lambda\gamma_5 b|\bar{B}(p_B)\rangle = 0. \tag{6.25}$$

The latter matrix element vanishes because of the parity invariance of the strong interactions. Equation (6.24) follows because $\bar{b}\gamma_\lambda b$ is the conserved b-quark number current. The b-quark number charge $Q_b = \int d^3x \bar{b}\gamma_0 b$ acts on $\bar{B}$-meson states as $Q_b|\bar{B}\rangle = |\bar{B}\rangle$, since they have unit b-quark number. Note that Eqs. (6.24) and (6.25) are exact. There are no corrections of order Λ_{QCD}/m_b to these relations and hence at this level in the OPE there is no need to make a transition to the heavy quark effective theory.

The T_j's that follow from Eqs. (6.24), (6.25), and (6.22) are

$$\begin{aligned} T_1^{(0)} &= \frac{1}{2\Delta_0}(m_b - q\cdot v), \\ T_2^{(0)} &= \frac{1}{\Delta_0} m_b, \\ T_3^{(0)} &= \frac{1}{2\Delta_0}. \end{aligned} \tag{6.26}$$

At this level in the operator product expansion, the entire cut reduces to a simple pole. The W_j's that follow from Eq. (6.26) are

$$\begin{aligned} W_1^{(0)} &= \frac{1}{4}\left(1 - \frac{q\cdot v}{m_b}\right)\delta\left[v\cdot q - \left(\frac{q^2 + m_b^2 - m_c^2}{2m_b}\right)\right], \\ W_2^{(0)} &= \frac{1}{2}\delta\left[v\cdot q - \left(\frac{q^2 + m_b^2 - m_c^2}{2m_b}\right)\right], \\ W_3^{0} &= \frac{1}{4m_b}\delta\left[v\cdot q - \left(\frac{q^2 + m_b^2 - m_c^2}{2m_b}\right)\right]. \end{aligned} \tag{6.27}$$

Putting these expressions into Eq. (6.9) and performing the integration over neutrino energies using the δ function in Eq. (6.27) gives

$$\frac{d\Gamma}{d\hat{q}^2\, dy} = \frac{G_F^2 |V_{cb}|^2 m_b^5}{192\pi^3} 12(y - \hat{q}^2)(1 + \hat{q}^2 - \rho - y)\theta(z), \tag{6.28}$$

where

$$y = 2E_e/m_b, \qquad \hat{q}^2 = q^2/m_b^2, \qquad \rho = m_c^2/m_b^2, \tag{6.29}$$

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and

$$z = 1 + \hat{q}^2 - \rho - \hat{q}^2/y - y \tag{6.30}$$

are convenient dimensionless variables. This is the same result one obtains from calculating the decay of a free b quark. Integrating over $\hat{q}^2$ gives the lepton energy spectrum

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}y} = \frac{G_F^2 |V_{cb}|^2 m_b^5}{192\pi^3}\left[2(3-2y)y^2 - 6y^2\rho - \frac{6y^2\rho^2}{(1-y)^2} + \frac{2(3-y)y^2\rho^3}{(1-y)^3}\right], \tag{6.31}$$

which also is the same as obtained from free quark decay. Including perturbative QCD corrections to the coefficient of the operator $\bar{b}\gamma_\lambda b$ in the operator product expansion would reproduce the perturbative QCD corrections to the b-quark decay rate.

At linear order in k, Eq. (6.21) contains the terms

$$\begin{aligned}&\frac{1}{\Delta_0}\bar{u}(k_\alpha\gamma_\beta + k_\beta\gamma_\alpha - g_{\alpha\beta}\not{k} - i\epsilon_{\alpha\beta\lambda\eta}k^\lambda\gamma^\eta)P_L u\\ &\quad - \frac{2k\cdot(m_b v - q)}{\Delta_0^2}\bar{u}[(m_b v - q)_\alpha\gamma_\beta + (m_b v - q)_\beta\gamma_\alpha\\ &\quad - (m_b \not{v} - \not{q})g_{\alpha\beta} - i\epsilon_{\alpha\beta\lambda\eta}(m_b v - q)^\lambda\gamma^\eta]P_L u.\end{aligned} \tag{6.32}$$

These produce terms in the operator product expansion of the form $\bar{b}\gamma_\lambda(iD_\tau - m_b v_\tau)b$ and $\bar{b}\gamma_\lambda\gamma_5(iD_\tau - m_b v_\tau)b$. Converting the b-quark fields in QCD to those in the heavy quark effective theory gives, at leading order in $1/m_b$, the operators $\bar{b}_v\gamma_\lambda iD_\tau b_v = v_\lambda\bar{b}_v iD_\tau b_v$ and $\bar{b}_v\gamma_\lambda\gamma_5 iD_\tau b_v$. The second of these has a vanishing $\bar{B}$-meson matrix element by parity invariance of the strong interactions. The first has a matrix element that can be written in the form

$$\langle\bar{B}(v)|\bar{b}_v iD_\tau b_v|\bar{B}(v)\rangle = X v_\tau. \tag{6.33}$$

Contracting both sides with v^τ, we find that the equation of motion in HQET, $(iv\cdot D)b_v = 0$, implies that $X = 0$. There are no matrix elements of dimension-four operators that occur in the OPE for the T_j's. This means that, when the differential semileptonic $\bar{B}$-meson decay rate is expressed in terms of the bottom and charm quark masses, there are no corrections suppressed by a single power of Λ_{QCD}/m_b.

6.2.2 Dimension-five operators

There are several sources of contributions from dimension-five operators to the operator product expansion. At order k^1 we found in the previous subsection that the operators $\bar{b}\gamma_\lambda(iD_\tau - m_b v_\tau)b$ and $\bar{b}\gamma_\lambda\gamma_5(iD_\tau - m_b v_\tau)b$ occur. Including

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$1/m_b$ corrections to the relationship between QCD and HQET operators gives rise to dimension-five operators in HQET. Recall from Chapter 4 that at order $1/m_b$, the relationship between the b-quark field in QCD and in HQET is (to zeroth order in α_s)

$$b(x) = e^{-im_b v\cdot x}\left(1+\frac{i\not{D}}{2m_b}\right)b_v(x), \tag{6.34}$$

and the order $1/m_b$ HQET Lagrange density is

$$\mathcal{L}_1 = -\bar{b}_v\frac{D^2}{2m_b}b_v - \bar{b}_v g\frac{G_{\alpha\beta}\sigma^{\alpha\beta}}{4m_b}b_v. \tag{6.35}$$

As was noted in Chapter 4, one can drop the $\perp$ subscript on D at this order. Equations (6.34) and (6.35) imply that at order $1/m_b$ (and zeroth order in α_s),

$$\begin{aligned}\bar{b}\gamma_\lambda(iD_\tau - m_b v_\tau)b &= \bar{b}_v\gamma_\lambda iD_\tau b_v + i\int d^4x\, T[\bar{b}_v\gamma_\lambda iD_\tau b_v(0)\,\mathcal{L}_1(x)]\\ &+\bar{b}_v\left(\frac{-i\overleftarrow{\not{D}}}{2m_b}\right)\gamma_\lambda iD_\tau b_v + \bar{b}_v\gamma_\lambda iD_\tau\frac{i\not{D}}{2m_b}b_v.\end{aligned} \tag{6.36}$$

Equation (6.36) is an operator matching condition. The matrix element of the left-hand side is to be taken in QCD, and of the right-hand side in HQET between hadrons states constructed using the lowest order Lagrangian. The effects of the $1/m_b$ corrections to the Lagrangian have been explicitly included as a time-ordered product term in the operator. Equation (6.36) is valid at a subtraction point $\mu = m_b$, with corrections of order $\alpha_s(m_b)$.

Let us consider the $\bar{B}$-matrix element of the various terms that occur on the right-hand side of Eq. (6.36). We have already shown that the equations of motion of HQET imply that $\bar{b}_v\gamma_\lambda iD_\tau b_v$ has zero $\bar{B}$-meson matrix elements. For the time-ordered product, we note that γ_λ can be replaced by v_λ and write

$$\langle\bar{B}(v)|i\int d^4x\, T[\bar{b}_v iD_\tau b_v(0)\,\mathcal{L}_1(x)]|\bar{B}(v)\rangle = Av_\tau. \tag{6.37}$$

Contracting with v_τ yields

$$\langle\bar{B}(v)|i\int d^4x T[\bar{b}_v(iv\cdot D)b_v(0)\,\mathcal{L}_1(x)]|\bar{B}(v)\rangle = A. \tag{6.38}$$

At tree level, the time-ordered product is evaluated by using $(v\cdot D)S_h(x-y) = \delta^4(x-y)$, where S_h is the HQET propagator. Consequently,

$$A = -\langle\bar{B}(v)|\mathcal{L}_1(0)|\bar{B}(v)\rangle = -\frac{\lambda_1}{m_b} - \frac{3\lambda_2}{m_b}, \tag{6.39}$$

where λ_1 and λ_2 were defined in Eqs. (4.23). There is another way to evaluate the $\bar{B}$-matrix element of the first two terms on the right-hand side of Eq. (6.36). Instead of including the time-ordered product, one evaluates the matrix element

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of the first term by using the equations of motion that include $\mathcal{O}(1/m_Q)$ terms in the Lagrangian, i.e., $\bar{b}_v\,(iv\cdot D)\,b_v = -\mathcal{L}_1$.

Using the operator identity $[D_\alpha, D_\beta] = igG_{\alpha\beta}$, we find the last two terms on the right side of Eq. (6.36) become

$$\bar{b}_v \frac{i\not{D}}{2m_b}\gamma_\lambda iD_\tau b_v + \bar{b}_v\gamma_\lambda iD_\tau \frac{i\not{D}}{2m_b} b_v = \bar{b}_v \frac{iD_{(\lambda}iD_{\tau)}}{m_b} b_v - \bar{b}_v g\frac{G_{\alpha\tau}\sigma^\alpha{}_\lambda}{2m_b} b_v, \tag{6.40}$$

where parentheses around indices denote that they are symmetrized, i.e.,

$$a^{(\alpha}b^{\beta)} = \frac{1}{2}(a^\alpha b^\beta + a^\beta b^\alpha).$$

For the operator with symmetrized covariant derivatives we write

$$\langle \bar{B}(v)|\bar{b}_v iD_{(\lambda}iD_{\tau)}b_v|\bar{B}(v)\rangle = Y(g_{\lambda\tau} - v_\lambda v_\tau). \tag{6.41}$$

The tensor structure on the right-hand side of this equation follows from the HQET equation of motion $(iv\cdot D)b_v = 0$, which implies that it must vanish when either index is contracted with the b quark's four velocity. To fix Y we contract both sides with $g^{\lambda\tau}$, giving

$$Y = \frac{1}{3}\langle \bar{B}(v)|\bar{b}_v (iD)^2 b_v|\bar{B}(v)\rangle = \frac{2}{3}\lambda_1. \tag{6.42}$$

Finally we need

$$\langle \bar{B}(v)|\bar{b}_v g G_{\alpha\tau}\sigma^\alpha{}_\lambda b_v|\bar{B}(v)\rangle = Z(g_{\lambda\tau} - v_\lambda v_\tau), \tag{6.43}$$

where again the tensor structure on the right-hand side follows from the fact that contracting v^λ into it must vanish, since $\bar{b}_v\sigma^\alpha{}_\lambda v^\lambda b_v = 0$. Contracting both sides of Eq. (6.43) with the metric tensor yields

$$Z = \frac{1}{3}\langle \bar{B}(v)|\bar{b}_v g G_{\alpha\beta}\sigma^{\alpha\beta} b_v|\bar{B}(v)\rangle = -4\lambda_2. \tag{6.44}$$

Combining these results we have that the order k^1 terms in Eq. (6.32) give the following contribution to the T_j's:

$$\begin{aligned}
T_1^{(1)} &= -\frac{1}{2m_b}(\lambda_1 + 3\lambda_2)\left\{\frac{1}{6\Delta_0} - \frac{(m_b - q\cdot v)^2}{\Delta_0^2} + \frac{2}{3}\frac{[q^2 - (q\cdot v)^2]}{\Delta_0^2}\right\},\\
T_2^{(1)} &= -\frac{1}{2m_b}(\lambda_1 + 3\lambda_2)\left[\frac{5}{3\Delta_0} - \frac{2m_b(m_b - v\cdot q)}{\Delta_0^2} + \frac{4}{3}\frac{m_b v\cdot q}{\Delta_0^2}\right],\\
T_3^{(1)} &= \frac{1}{2m_b}(\lambda_1 + 3\lambda_2)\frac{5}{3}\left(\frac{m_b - v\cdot q}{\Delta_0^2}\right).
\end{aligned} \tag{6.45}$$

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6.2.3 Second order

The order k^2 terms in Eq. (6.21) are

$$-2\frac{k\cdot(m_b v-q)}{\Delta_0^2}\bar{u}(k_\alpha\gamma_\beta+k_\beta\gamma_\alpha-g_{\alpha\beta}\not{k}-i\epsilon_{\alpha\beta\lambda\eta}k^\lambda\gamma^\eta)P_L u$$
$$+\left\{\frac{4[k\cdot(m_b v-q)]^2}{\Delta_0^3}-\frac{k^2}{\Delta_0^2}\right\}\bar{u}[(m_b v-q)_\alpha\gamma_\beta+(m_b v-q)_\beta\gamma_\alpha$$
$$-(m_b\not{v}-\not{q})g_{\alpha\beta}-i\epsilon_{\alpha\beta\lambda\eta}(m_b v-q)^\lambda\gamma^\eta]P_L u. \tag{6.46}$$

These can be expressed in terms of matrix elements of the operators $\bar{b}\gamma^\lambda(iD-m_b v)^{(\alpha}(iD-m_b v)^{\beta)}b$ and $\bar{b}\gamma^\lambda\gamma_5(iD-m_b v)^{(\alpha}(iD-m_b v)^{\beta)}b$. The operator involving γ_5 will not contribute to $\bar{B}$-meson matrix elements by parity. Rewriting the result using HQET operators, we find the only operator that occurs is $v^\lambda\bar{b}_v iD^{(\alpha}iD^{\beta)}b_v$. Its matrix element is given by Eqs. (6.41) and (6.42). So we find that the terms with two k's give the following contribution to the structure functions:

$$T_1^{(2)}=\frac{1}{6}\lambda_1(m_b-v\cdot q)\left\{\frac{4}{\Delta_0^3}[q^2-(v\cdot q)^2]-\frac{3}{\Delta_0^2}\right\},$$
$$T_2^{(2)}=\frac{1}{3}\lambda_1 m_b\left\{\frac{4}{\Delta_0^3}[q^2-(v\cdot q)^2]-\frac{3}{\Delta_0^2}-\frac{2v\cdot q}{m_b\Delta_0^2}\right\}, \tag{6.47}$$
$$T_3^{(2)}=\frac{1}{6}\lambda_1\left\{\frac{4}{\Delta_0^3}[q^2-(v\cdot q)^2]-\frac{5}{\Delta_0^2}\right\}.$$

At zeroth order in α_s, the b-quark matrix element of the operator $\bar{b}\sigma_{\alpha\beta}G^{\alpha\beta}b$ vanishes. To find the part of the operator product expansion proportional to this operator, we need to consider the $b\to b+$ gluon matrix element of the time-ordered product. At tree level it is given by the Feynman diagram in Fig. 6.5. The matrix element has the initial b quark with residual momentum $p/2$, a final b quark with residual momentum $-p/2$, and the gluon with outgoing momentum p. This choice is convenient since the denominators of the c-quark propagators do not contribute to the p dependence at linear order in p. The part of this Feynman diagram with no factors of the gluon four momentum, p, is from the $b\to b+$gluon matrix element of operators we have already found, with the gluon

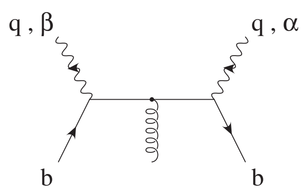

Fig. 6.5. The one-gluon matrix element in the OPE.

field coming from the covariant derivative $D = \partial + igA$. The part linear in p is

$$gT^A\varepsilon^{A\lambda *}(p)\frac{1}{2\Delta_0^2}\bar{u}\gamma_\alpha[-\not{p}\gamma_\lambda(m_b\not{v} - \not{q}) + (m_b\not{v} - \not{q})\gamma_\lambda\not{p}]\gamma_\beta P_L u, \tag{6.48}$$

where $\varepsilon^{A\lambda}$ is the gluon polarization vector. Only the part of this antisymmetric under interchange $p \leftrightarrow \varepsilon^*$ contributes to the operator we are considering. Equation (1.119) is used to reexpress the product of three-gamma matrices in the square brackets of Eq. (6.48) in terms of a single-gamma matrix. Only the part proportional to the Levi-Civita tensor survives. Applying the identity of Eq. (1.119) one more time shows that the term linear in p is reproduced by the matrix element of the operator

$$\frac{g}{2\Delta_0^2}\bar{b}G_{\mu\nu}\epsilon^{\mu\nu\lambda\sigma}(m_b v - q)_\lambda(g_{\alpha\sigma}\gamma_\beta + g_{\beta\sigma}\gamma_\alpha - g_{\alpha\beta}\gamma_\sigma + i\epsilon_{\alpha\sigma\beta\tau}\gamma^\tau\gamma_5)P_L b. \tag{6.49}$$

Here we have used the replacement

$$p^\beta T^A\varepsilon^{A\lambda *} \to -\frac{i}{2}G^{\beta\lambda}$$

for the part antisymmetric in β and λ.

The transition to HQET is made by replacing b-quark fields in the above by b_v. The operators that occur are $\bar{b}_v G^{\mu\nu}\gamma^\lambda\gamma_5 b_v$ and $\bar{b}_v G^{\mu\nu}\gamma^\lambda b_v$. Because of the antisymmetry on the indices μ and ν, parity invariance of the strong interaction forces the latter operator to have a zero matrix element between $\bar{B}$-meson states. The matrix element of the other operator can be written as

$$\langle\bar{B}(v)|\bar{b}_v g G^{\mu\nu}\gamma^\lambda\gamma_5 b_v|\bar{B}(v)\rangle = N\epsilon^{\mu\nu\lambda\tau}v_\tau. \tag{6.50}$$

Contracting both sides of this equation with $\epsilon_{\mu\nu\lambda\rho}v^\rho$ and using the identity

$$\epsilon_{\mu\nu\lambda\rho}v^\rho\bar{b}_v\gamma^\lambda\gamma_5 b_v = -\bar{b}_v\sigma_{\mu\nu}b_v \tag{6.51}$$

yields

$$N = -2\lambda_2. \tag{6.52}$$

Consequently, the $b \to b +$ gluon matrix element gives these additional contributions to the structure functions:

$$\begin{aligned} T_1^{(g)} &= \lambda_2\frac{(m_b - v\cdot q)}{2\Delta_0^2}, \\ T_2^{(g)} &= -\lambda_2\frac{m_b}{\Delta_0^2}, \\ T_3^{(g)} &= \lambda_2\frac{1}{2\Delta_0^2}. \end{aligned} \tag{6.53}$$

Summing the three contributions we have discussed,

$$T_j = T_j^{(1)} + T_j^{(2)} + T_j^{(g)}, \tag{6.54}$$

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gives the complete contribution of dimension-five operators in HQET to the structure functions. At this order in the operator product expansion only two matrix elements occur, λ_1 and λ_2. Furthermore, one of them, $\lambda_2 \simeq 0.12\ \mathrm{GeV}^2$, is known from $B^* - B$ mass splitting. The results for T_j determine the nonperturbative $\Lambda^2_{\rm QCD}/m_b^2$ corrections to the inclusive semileptonic decay rate.

6.3 Differential decay rates

The inclusive $\bar{B}$ semileptonic differential decay rate is calculated by using Eqs. (6.9) and (6.54), with the W_j's obtained from the imaginary part of the T_j's. The identity

$$-\frac{1}{\pi}\,\mathrm{Im}\left(\frac{1}{\Delta_0}\right)^{n+1} = \frac{(-1)^n}{n!}\delta^{(n)}\left[(m_b v - q)^2 - m_c^2\right], \tag{6.55}$$

where the superscript denotes the nth derivative of the δ function with respect to its argument, is useful in computing the W_j's. Terms with derivatives of the δ function are evaluated by first integrating by parts to take the derivatives off the δ function. In using this procedure, one must be careful to include the factor $\theta(4E_e E_{\nu_e} - q^2)$, which sets the lower limit of the E_{ν_e} integration, in the differential decay rate, since the derivative can act on this term. Differentiating the θ function with respect to E_{ν_e} gives

$$\delta\left[\left(\frac{m_b^2 - m_c^2 + q^2}{2m_b} - E_e\right) - \frac{q^2}{4E_e}\right], \tag{6.56}$$

which, in terms of the variables y, $\hat{q}^2$, and z defined in Eqs. (6.29) and (6.30), is the δ function $2\delta(z)/m_b$. This procedure gives for the differential decay rate

$$\begin{aligned}
\frac{\mathrm{d}\Gamma}{\mathrm{d}\hat{q}^2\,\mathrm{d}y} = \frac{G_F^2 m_b^5}{192\pi^3}|V_{cb}|^2 \Bigg\{ & \theta(z)\Bigg[12(y-\hat{q}^2)(1+\hat{q}^2-\rho-y) \\
& - \frac{2\lambda_1}{m_b^2}(4\hat{q}^2 - 4\hat{q}^2\rho + 4\hat{q}^4 - 3y + 3\rho y - 6\hat{q}^2 y) \\
& - \frac{6\lambda_2}{m_b^2}(-2\hat{q}^2 - 10\hat{q}^2\rho + 10\hat{q}^4 - y + 5\rho y - 10\hat{q}^2 y)\Bigg] \\
& + \frac{\delta(z)}{y^2}\Bigg[-\frac{2\lambda_1}{m_b^2}(2\hat{q}^6 + \hat{q}^4 y^2 - 3\hat{q}^2 y^3 - \hat{q}^2 y^4 + y^5) \\
& - \frac{6\lambda_2}{m_b^2}\hat{q}^2(\hat{q}^2 - y)(5\hat{q}^2 - 8y + y^2)\Bigg] \\
& + \frac{\delta'(z)}{y^3}\left[-\frac{2\lambda_1}{m_b^2}\hat{q}^2(y^2-\hat{q}^2)^2(y-\hat{q}^2)\right]\Bigg\},
\end{aligned} \tag{6.57}$$

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where the dimensionless variable $\hat{q}^2$, y, ρ, and z are defined in Eqs. (6.29) and (6.30). Experimentally, the electron energy spectrum $d\Gamma/dy$ is easier to study than the doubly differential decay rate. Integration of Eq. (6.57) over the allowed region $0 < \hat{q}^2 < y(1-y-\rho)/(1-y)$ gives

$$\begin{aligned}\frac{d\Gamma}{dy} = \frac{G_F^2 m_b^5}{192\pi^3}|V_{cb}|^2\Bigg\{&\left[2(3-2y)y^2 - 6y^2\rho - \frac{6y^2\rho^2}{(1-y)^2} + \frac{2(3-y)y^2\rho^3}{(1-y)^3}\right]\\ &- \frac{2\lambda_1}{m_b^2}\left[-\frac{5}{3}y^3 - \frac{y^3(5-2y)\rho^2}{(1-y)^4} + \frac{2y^3(10-5y+y^2)\rho^3}{3(1-y)^5}\right]\\ &- \frac{6\lambda_2}{m_b^2}\Bigg[-y^2\frac{(6+5y)}{3} + \frac{2y^2(3-2y)\rho}{(1-y)^2}\\ &+ \frac{3y^2(2-y)\rho^2}{(1-y)^3} - \frac{5y^2(6-4y+y^2)\rho^3}{3(1-y)^4}\Bigg]\Bigg\}. \end{aligned} \tag{6.58}$$

Integrating over the allowed electron energy $0 < y < 1-\rho$ yields the total $\bar{B} \to X_c e \bar{\nu}_e$ decay rate,

$$\begin{aligned}\Gamma = \frac{G_F^2 m_b^5}{192\pi^3}|V_{cb}|^2\Bigg[&(1 - 8\rho + 8\rho^3 - \rho^4 - 12\rho^2 \ln\rho)\\ &+ \frac{\lambda_1}{2m_b^2}(1 - 8\rho + 8\rho^3 - \rho^4 - 12\rho^2 \ln\rho)\\ &- \frac{3\lambda_2}{2m_b^2}(3 - 8\rho + 24\rho^2 - 24\rho^3 + 5\rho^4 + 12\rho^2 \ln\rho)\Bigg], \end{aligned} \tag{6.59}$$

which can be written in the compact form

$$\Gamma = \frac{G_F^2 m_b^5}{192\pi^3}|V_{cb}|^2\left[1 + \frac{\lambda_1}{2m_b^2} + \frac{3\lambda_2}{2m_b^2}\left(2\rho\frac{d}{d\rho} - 3\right)\right] f(\rho), \tag{6.60}$$

where

$$f(\rho) = 1 - 8\rho + 8\rho^3 - \rho^4 - 12\rho^2 \ln\rho. \tag{6.61}$$

The first term is the leading term in the $m_b \to \infty$ limit and is equal to the free quark decay rate. The next two terms are $1/m_b^2$ corrections. The $1/m_b$ correction vanishes. Note that the ρ dependence of the coefficient of λ_1 is the same as that in the free quark decay rate. We will give a simple physical reason for this result in the next section.

Results for semileptonic $\bar{B}$-meson decays from the $b \to u$ transition are obtained from Eqs. (6.57), (6.58), and (6.59) by taking the limit $\rho \to 0$. Taking this limit is straightforward, except in the case of the electron spectrum in Eq. (6.58).

Suppose the electron energy spectrum in the $b \to c$ case contains a term of the form

$$g_\rho(y) = \frac{\rho^{n-1}}{(1-y)^n}. \tag{6.62}$$

The limit as $\rho \to 0$ of $g_\rho(y)$ is not zero. The problem is that the maximum value of y is $1-\rho$ and hence at maximum electron energy the denominator in Eq. (6.62) goes to zero as $\rho \to 0$. Imagine integrating $g_\rho(y)$ against a smooth test function $t(y)$. Integrating by parts

$$\begin{aligned}\lim_{\rho\to 0}\int_0^{1-\rho} dy\, t(y) g_\rho(y) &= \frac{1}{(n-1)}\left[t(1) - \lim_{\rho\to 0}\int_0^{1-\rho} dy\, \frac{\mathrm{d}t}{\mathrm{d}y}(y)\frac{\rho^{n-1}}{(1-y)^{n-1}}\right]\\ &= \frac{1}{(n-1)}t(1).\end{aligned} \tag{6.63}$$

Hence we conclude that

$$\lim_{\rho\to 0} g_\rho(y) = \frac{1}{(n-1)}\delta(1-y). \tag{6.64}$$

Differentiating the above gives

$$\lim_{\rho\to 0}\frac{\rho^{n-1}}{(1-y)^{n+1}} = -\frac{1}{n(n-1)}\delta'(1-y). \tag{6.65}$$

The $\rho \to 0$ limit of the electron spectrum in Eq. (6.58) is the $\bar{B} \to X_u e \bar{\nu}_e$ electron energy spectrum,

$$\begin{aligned}\frac{\mathrm{d}\Gamma}{\mathrm{d}y} = \frac{G_F^2 m_b^5 |V_{ub}|^2}{192\pi^3}\bigg\{&2(3-2y)y^2\theta(1-y)\\ &-\frac{2\lambda_1}{m_b^2}\left[-\frac{5}{3}y^3\theta(1-y) + \frac{1}{6}\delta(1-y) + \frac{1}{6}\delta'(1-y)\right]\\ &-\frac{2\lambda_2}{m_b^2}\left[-y^2(6+5y)\theta(1-y) + \frac{11}{2}\delta(1-y)\right]\bigg\},\end{aligned} \tag{6.66}$$

and the total decay width is

$$\Gamma = \frac{G_F^2 m_b^5 |V_{ub}|^2}{192\pi^3}\left(1 + \frac{\lambda_1}{2m_b^2} - \frac{9\lambda_2}{2m_b^2}\right). \tag{6.67}$$

6.4 Physical interpretation of $1/m_b^2$ corrections

The corrections to the decay rate proportional to λ_1 have a simple physical interpretation. They arise from the motion of the b quark inside the $\bar{B}$ meson. At

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leading order in the $1/m_b$ expansion, the b quark is at rest in the $\bar{B}$-meson rest frame, and the $\bar{B}$-meson differential decay rate is equal to the b-quark, decay rate, $\mathrm{d}\Gamma^{(0)}(v_r, m_b)$. However, in a $\bar{B}$ meson the b quark really has (in the $\bar{B}$-meson rest frame) a four momentum $p_b = m_b v_r + k$. We can consider this as a b quark with an effective mass m'_b and an effective four velocity v' satisfying

$$m'_b v' = m_b v_r + k. \tag{6.68}$$

Including effects of the b-quark motion in the $\bar{B}$ meson, we find the fully differential semileptonic decay rate $\mathrm{d}\Gamma$ is

$$\mathrm{d}\Gamma = \left\langle \mathrm{d}\Gamma^{(0)}(v', m'_b)/v'^0 \right\rangle, \tag{6.69}$$

where v'^0 is the time-dilation factor, the fences denote averaging over k, and $\mathrm{d}\Gamma^{(0)}$ is the free b-quark differential decay rate. This averaging is done by expanding Eq. (6.69) to quadratic order in k and using

$$\langle k^\alpha \rangle = -\frac{\lambda_1}{2m_b} v_r^\alpha, \qquad \langle k^\alpha k^\beta \rangle = \frac{\lambda_1}{3}\left(g^{\alpha\beta} - v_r^\alpha v_r^\beta\right). \tag{6.70}$$

More powers of k would correspond to higher dimension operators in the OPE than those we have considered so far. In expanding Eq. (6.69) one can use

$$m_b'^2 = (m'_b v')^2 = (m_b v_r + k)^2 = m_b^2 + 2m_b v_r \cdot k + k^2. \tag{6.71}$$

Note that Eqs. (6.70) and (6.71) imply that $\langle m_b'^2 \rangle = \langle m_b^2 \rangle$. Since $v_{r\alpha} \langle k^\alpha k^\beta \rangle = 0$, we can replace m'_b by m_b in Eq. (6.69) without worrying about cross terms in the average where one factor of k arises from expanding m'_b and the other from expanding v'. The effective four velocity v' is related to v_r and k by

$$v'_\alpha = v_{r\alpha} + \frac{1}{m'_b} k_\alpha = v_{r\alpha} + \frac{k_\alpha}{m_b}, \tag{6.72}$$

so the time-dilation factor is

$$v'_0 = v_r \cdot v' = 1 + v_r \cdot k/m_b. \tag{6.73}$$

Averaging this yields $\langle v'_0 \rangle = 1 - \lambda_1/2m_b^2$, and since $v_{r\alpha}\langle k^\alpha k^\beta \rangle = 0$ we can replace the factor $1/v'^0$ in Eq. (6.69) by $(1 + \lambda_1/2m_b^2)$. The fully differential decay rate can be taken to be $\mathrm{d}\Gamma/\mathrm{d}\hat{q}^2\, \mathrm{d}y\, \mathrm{d}x$, where we have introduced the dimensionless neutrino energy variable

$$x = \frac{2E_{\nu_e}}{m_b}. \tag{6.74}$$

The variables x and y depend on the four velocity v_r of the b quark through $y = 2v_r \cdot p_e/m_b$, $x = 2v_r \cdot p_{\nu_e}/m_b$, and consequently, under the replacement

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$v_r \to v'$,

$$y \to y' = y + \frac{2k \cdot p_e}{m_b^2}, \qquad x \to x' = x + \frac{2k \cdot p_{\nu_e}}{m_b^2}. \tag{6.75}$$

Hence Eq. (6.69) implies that

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\hat{q}^2\,\mathrm{d}y\,\mathrm{d}x} = \left[1 - \frac{\lambda_1}{2m_b^2}\left(-1 + y\frac{\partial}{\partial y} + x\frac{\partial}{\partial x} + \frac{1}{3}y^2\frac{\partial^2}{\partial y^2} + \frac{1}{3}x^2\frac{\partial^2}{\partial x^2} + \frac{2}{3}(xy - 2\hat{q}^2)\frac{\partial^2}{\partial x\,\partial y}\right)\right]\frac{\mathrm{d}\Gamma^{(0)}}{\mathrm{d}\hat{q}^2\,\mathrm{d}y\,\mathrm{d}x}. \tag{6.76}$$

Integrating over x yields

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\hat{q}^2\,\mathrm{d}y} = \left[1 - \frac{\lambda_1}{2m_b^2}\left(-\frac{4}{3} + \frac{1}{3}y\frac{\partial}{\partial y} + \frac{1}{3}y^2\frac{\partial^2}{\partial y^2}\right)\right]\frac{\mathrm{d}\Gamma^{(0)}}{\mathrm{d}\hat{q}^2\,\mathrm{d}y}, \tag{6.77}$$

where the free b-quark differential decay rate $\mathrm{d}\Gamma^{(0)}/\mathrm{d}\hat{q}^2\,\mathrm{d}y$ is given in Eq. (6.28). Integrating over $\hat{q}^2$ and y, we find for the total decay rate

$$\Gamma = \left(1 + \frac{\lambda_1}{2m_b^2}\right)\Gamma^{(0)}. \tag{6.78}$$

Equations (6.77) and (6.78) give the correct λ_1 dependence of the $\bar{B}$-meson differential decay rates. Unfortunately, the dependence of the $\bar{B}$-meson differential decay rates in Eqs. (6.57)–(6.59) on λ_2 does not seem to have as simple a physical interpretation.

6.5 The electron endpoint region

The predictions that follow from the operator product expansion for the differential $\bar{B} \to Xe\bar{\nu}_e$ semileptonic decay rate cannot be compared directly with experiment in all regions of the phase space. For example, the expression for the differential cross section $\mathrm{d}\Gamma/\mathrm{d}\hat{q}^2\,\mathrm{d}y$ in Eq. (6.57) contains singular terms on the boundary of the Dalitz plot, $z = 0$. Rigorously, predictions based on the operator product expansion and perturbative QCD can be compared with experiment only when averaged over final hadronic state masses m_X with a smooth weighting function. Very near the boundary of the Dalitz plot, only the lower-mass final hadronic states can contribute, and the integration over neutrino energies does not provide the smearing over final hadronic masses needed to compare the operator product expansion results with experiment. In fact, since m_X is necessarily less than m_B, the weighting function is never truly smooth. As a result the contour integral over $v \cdot q$ needed to recover the structure functions W_j from those associated with the time-ordered product T_j necessarily pinches the cut at one point. Near the cut the use of the OPE cannot be rigorously justified because there will be propagators that have denominators close to zero. This is not considered a

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problem in a region where the final hadronic states are above the ground state by a large amount compared with the nonperturbative scale of the strong interactions, because threshold effects that are present in nature but not in the OPE analysis are very small. In inclusive $\bar{B}$ decay we assume that threshold effects associated with the limit on the maximum available hadronic mass $m_{X^{\max}}$ are negligible as long as $m_{X^{\max}} - m_{X^{\min}} \gg \Lambda_{\rm QCD}$. Near the boundary of the Dalitz plot this inequality is not satisfied. Note that at the order in α_s to which we have worked, the singularities in T_j are actually poles located at the ends of what we have called cuts. When α_s corrections are included, the singularities become the cuts we have described. Hence when radiative corrections are neglected, the contour in Fig. 6.3 need not be near a singularity in the $b \to c$ decay case. For $b \to u$ at q^2 near $q^2_{\max}$, the contour necessarily comes near singularities because the ends of the cuts are close together.

The endpoint region of the electron spectrum in inclusive semileptonic $\bar{B}$ decay has played an important role in determining the value of the element of the CKM matrix $|V_{ub}|$. For a given hadronic final state mass m_X, the maximum electron energy is $E_e^{\max} = (m_B^2 - m_X^2)/2m_B$. Consequently, electrons with energies greater than $E_e = (m_B^2 - m_D^2)/2m_B$ must necessarily come from the $b \to u$ transition. However, this endpoint region is precisely where the singular contributions proportional to $\delta(1-y)$ and $\delta'(1-y)$ occur in the $b \to u$ electron energy spectrum. Note that these singular terms occur at the endpoint set by quark–gluon kinematics, $E_e = m_b/2$, which is smaller than the true maximum $E_e^{\max} = m_B/2$. Clearly, in this region we must average over electron energies before comparing the predictions of the OPE and perturbative QCD with experiment.

To quantify the size of the averaging region in electron energies needed, we examine the general structure of the OPE. The most singular terms in the endpoint region result from expanding the k dependence in the denominator of the charm quark propagator. A term with power k^p produces an operator with p covariant derivatives and gives a factor of $1/\Delta_0^{p+1}$ in the T_j. This results in a factor of $\delta^{(p-1)}(1-y)$ in the electron energy spectrum. Matrix elements of operators with p covariant derivatives are of the order of $\Lambda_{\rm QCD}^p$ and so the general structure of the OPE prediction for the electron energy spectrum is

$$\begin{aligned}\frac{{\rm d}\Gamma}{{\rm d}y} \propto\, & \theta(1-y)(\varepsilon^0 + 0\,\varepsilon + \varepsilon^2 + \cdots)\\ & + \delta(1-y)(0\,\varepsilon + \varepsilon^2 + \cdots)\\ & + \delta'(1-y)(\varepsilon^2 + \varepsilon^3 + \cdots)\\ & \vdots\\ & + \delta^{(n)}(1-y)(\varepsilon^{n+1} + \varepsilon^{n+2} + \cdots)\\ & \vdots \end{aligned} \tag{6.79}$$

where ε^n denotes a quantity of the order of $(\Lambda_{\rm QCD}/m_b)^n$. It may contain smooth

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y dependence. The zeroes are the coefficients of the dimension-four operators, which vanish by the equations of motion. Although the theoretical expression for $\mathrm{d}\Gamma/\mathrm{d}y$ is singular near the b-quark decay endpoint $y = 1$, the total semileptonic decay rate is not. The contribution to the total rate of a term of order $\varepsilon^m \delta^{(n)}(1-y)$ is order ε^m, and so the semileptonic width has a well-behaved expansion in powers of $1/m_b$:

$$\Gamma \propto (\varepsilon^0 + 0\,\varepsilon + \varepsilon^2 + \varepsilon^3 + \cdots). \tag{6.80}$$

In the endpoint region consider integrating $\mathrm{d}\Gamma/\mathrm{d}y$ against a normalized function of y that has a width σ. This provides a smearing of the electron energy spectrum near $y = 1$ and corresponds to examining the energy spectrum with resolution in y of σ (i.e., a resolution in electron energy of $m_b\sigma$). A meaningful prediction for the endpoint spectrum can be made when the smearing width σ is large enough that terms that have been neglected in Eq. (6.79) are small in comparison to the terms that have been retained. The singular term $\varepsilon^m \delta^{(n)}(1-y)$ (where $m > n$) smeared over a region of width σ gives a contribution of order $\varepsilon^m/\sigma^{n+1}$. If the smearing width σ is of the order of ε^p, the generic term $\varepsilon^m \delta^{(n)}(1-y)$ yields a contribution to the smeared spectrum of the order of $\varepsilon^{m-p(n+1)}$. Even though $m > n$, higher-order terms in the $1/m_b$ expansion get more important than lower-order ones unless $p \leq 1$.

If the smearing in y is chosen of order ε (i.e., a region of electron energies of the order of Λ_{QCD}), then all terms of the form $\theta(1-y)$ and $\varepsilon^{n+1}\delta^{(n)}(1-y)$ contribute equally to the smeared electron energy spectrum, with less singular terms being suppressed. For example, all terms of order $\varepsilon^{n+2}\delta^{(n)}(1-y)$ are suppressed by ε, and so on. Thus one can predict the endpoint region of the electron energy spectrum, with a resolution in electron energies of the order of Λ_{QCD}, if these leading singular terms are summed. The sum of these leading singularities produces a contribution to $\mathrm{d}\Gamma/\mathrm{d}y$ of width ε but with a height of the same order as the free quark decay spectrum.

We can easily get the general form of the most singular contributions to the operator product expansion for the electron spectrum by using the physical picture of smearing over b-quark momenta discussed in the previous section. We want to continue the process to arbitrary orders in k, but only the most singular y dependence is needed. It arises only from the dependence of y on m_b and v. Shifting to m_b' and v',

$$y \to y' = \frac{2v' \cdot p_e}{m_b'} = y + k^\mu \frac{2}{m_b}(\hat{p}_{e\mu} - y v_\mu) + \cdots, \tag{6.81}$$

where the ellipsis denotes terms higher order in k, and $\hat{p}_e = p_e/m_b$. The term proportional to $y v_\mu$ in Eq. (6.81) arose from the dependence of m_b' on k. The most singular terms come from the y dependence in the factor $\theta(1-y)$, and so

$$\frac{d\Gamma}{dy} = \frac{d\Gamma^{(0)}}{dy}\Bigg[1 + \langle k^{\mu_1}\rangle \left(\frac{2}{m_b}\right)(\hat{p}_e - v)_{\mu_1}\frac{\partial}{\partial y} + \cdots + \frac{1}{n!}\langle k^{\mu_1}\cdots k^{\mu_n}\rangle \left(\frac{2}{m_b}\right)^n (\hat{p}_e - v)_{\mu_1}\cdots(\hat{p}_e - v)_{\mu_n}\frac{\partial^n}{\partial y^n} + \cdots\Bigg]\theta(1-y). \tag{6.82}$$

Equation (6.82) sums the most singular nonperturbative corrections in the endpoint region, provided one interprets the averaging over residual momenta as

$$\langle k_{\mu_1}\cdots k_{\mu_n}\rangle = \frac{1}{2}\langle \bar{B}(v)|\bar{b}_v iD_{(\mu_1}\cdots iD_{\mu_n)} b_v|\bar{B}(v)\rangle. \tag{6.83}$$

There is no operator ordering ambiguity because $\langle k^{\mu_1}\cdots k^{\mu_n}\rangle$ is contracted with a tensor completely symmetric in $\mu_1\cdots\mu_n$. Finally, only the part of the matrix element $\langle \bar{B}(v)|\bar{b}_v iD_{(\mu_1}\cdots iD_{\mu_n)} b_v|\bar{B}(v)\rangle$ proportional to $v_{\mu_1}\cdots v_{\mu_n}$ contributes to the most singular terms. A dependence on the metric tensor $g_{\mu_i\mu_j}$ would result in a factor of $(\hat{p}_e - v)^2$ that vanishes at $y=1$. So writing

$$\frac{1}{2}\langle \bar{B}(v)|\bar{b}_v iD_{(\mu_1}\cdots iD_{\mu_n)} b_v|\bar{B}(v)\rangle = A_n v_{\mu_1}\cdots v_{\mu_n} + \cdots, \tag{6.84}$$

we find the differential decay spectrum near $y=1$ is

$$\frac{d\Gamma}{dy} = \frac{d\Gamma^{(0)}}{dy}[\theta(1-y) + S(y)], \tag{6.85}$$

where the shape function $S(y)$ is

$$S(y) = \sum_{n=1}^{\infty}\frac{A_n}{m_b^n n!}\delta^{(n-1)}(1-y). \tag{6.86}$$

In Sec. 6.2.2, we showed that $A_1 = 0$ and $A_2 = -\frac{1}{3}\lambda_1$. At present, one must use phenomenological models for the shape function to extract $|V_{ub}|$ from semileptonic decay data in the endpoint region. This yields $|V_{ub}| \approx 0.1|V_{cb}|$. The perturbative QCD corrections to $d\Gamma/dy$ also become singular as $y \to 1$. These singular terms must also be summed to make a prediction for the shape of the electron spectrum in the endpoint region.

For inclusive $b\to c$ semileptonic decay, the $1/m_b^2$ corrections are not singular at the endpoint of the electron spectrum, but they are large because $m_c^2/m_b^2 \simeq 1/10$ is small. (At order $1/m_b^3$ singular terms occur even for $b\to c$ semileptonic decay.) It is instructive to plot the $b\to c$ electron spectrum including the $1/m_b^2$ corrections. This is done in Fig. 6.6. One can clearly see that the $1/m_b^2$ corrections become large near the endpoint. The OPE analysis gives the electron spectrum in Eq. (6.58), which depends on the heavy quark masses m_b and m_c. In particular,

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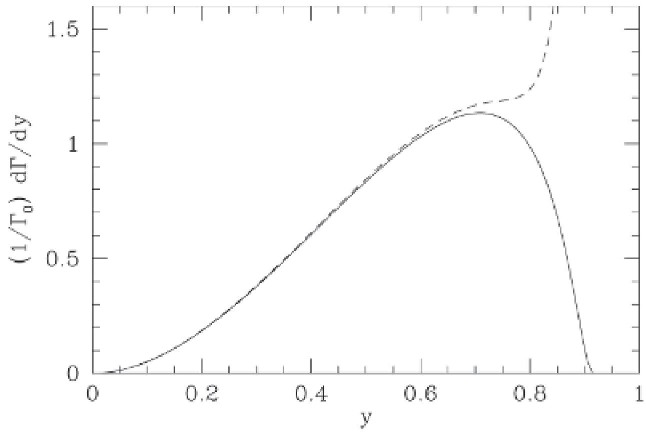

Fig. 6.6. The electron energy spectrum in inclusive semileptonic $\bar{B} \to X_c$ decay at lowest order (solid curve) and including the $1/m_b^2$ corrections (dashed curve), with $\lambda_1 = -0.2\ \text{GeV}^2$, $m_b = 4.8$ GeV, and $m_c = 1.4$ GeV. Here $\Gamma_0 = G_F^2 |V_{cb}|^2 m_b^5 / 192\pi^3$.

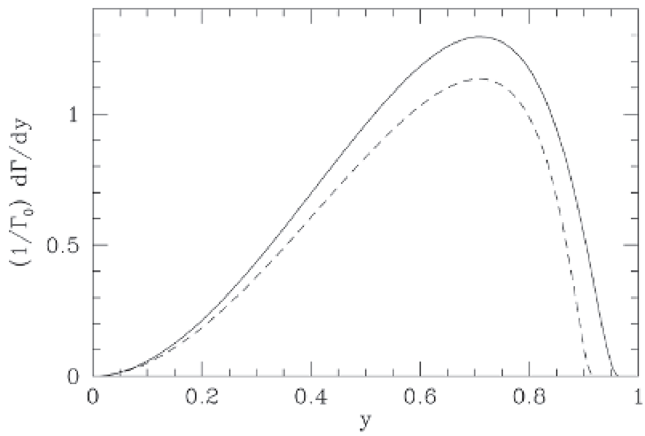

Fig. 6.7. The electron energy spectrum in inclusive semileptonic $\bar{B} \to X_c$ decay, using the lowest-order formula with quark masses (dashed curve) and with hadron masses (solid curve); y is defined as $2E_e/m_b$ in both plots, and $\Gamma_0 = G_F^2 |V_{cb}|^2 m_b^5 / 192\pi^3$.

the electron endpoint energy is $(m_b^2 - m_c^2)/2m_b$. The true kinematic endpoint for the electron spectrum is $(m_B^2 - m_D^2)/2m_B$, and it depends on the hadron masses. In Fig. 6.7, the lowest-order electron spectrum using quark masses has been compared with the same spectrum in which quark masses have been replaced by hadron masses. Over most of the phase space, this is close to the true spectrum, but very near the maximum value of E_e there is no theoretical basis to believe that the lowest-order spectrum with hadron masses has any connection with the actual electron spectrum. Nevertheless, the spectrum with hadron masses ends

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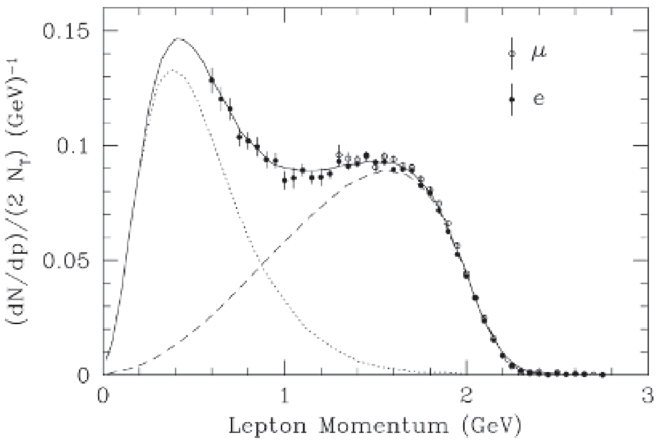

Fig. 6.8. The inclusive lepton energy spectrum for semileptonic $\bar{B} \to X_c$ decay, as measured by the CLEO Collaboration. The data are from the Ph.D thesis of R. Wang. The filled dots are the electron spectrum, and the open dots are the muon spectrum. The dashed curve is a model fit to the primary leptons from $b \to c$ semileptonic decay, which should be compared with theoretical predictions in Figs. 6.6 and 6.7. The dotted curve is a model fit to the secondary leptons from semileptonic decay of the c quark produced in b decay, and the solid curve is the sum of the two.

at the true kinematic endpoint of the allowed electron spectrum. The measured inclusive lepton spectrum in semileptonic B decay is shown in Fig. 6.8.

6.6 $|V_{cb}|$ from inclusive decays

The expression for the inclusive differential semileptonic decay rate in Eq. (6.57) can be used to deduce the HQET parameters $\bar{\Lambda}$ and λ_1. In addition, it provides a determination of the CKM matrix element V_{cb}. For comparison with experiment, it is useful to eliminate the c- and b-quark masses in favor of hadron masses. The average D- and B-meson masses are

$$\bar{m}_D = \frac{m_D + 3m_{D^*}}{4} = 1.975\ \text{GeV}, \qquad \bar{m}_B = \frac{m_B + 3m_{B^*}}{4} = 5.313\ \text{GeV}. \tag{6.87}$$

Using the results of Chapter 4, we find

$$\begin{aligned} m_c &= \bar{m}_D - \bar{\Lambda} + \frac{\lambda_1}{2\bar{m}_D} + \cdots, \\ m_b &= \bar{m}_B - \bar{\Lambda} + \frac{\lambda_1}{2\bar{m}_B} + \cdots, \end{aligned} \tag{6.88}$$

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where the ellipses denote terms higher order in the $1/m_Q$ expansion. This gives, for example,

$$\frac{m_c}{m_b} = \frac{\bar{m}_D}{\bar{m}_B} - \frac{\bar{\Lambda}}{\bar{m}_B}\left(1 - \frac{\bar{m}_D}{\bar{m}_B}\right) - \frac{\bar{\Lambda}^2}{\bar{m}_B^2}\left(1 - \frac{\bar{m}_D}{\bar{m}_B}\right) + \frac{\lambda_1}{2\bar{m}_B\bar{m}_D}\left(1 - \frac{\bar{m}_D^2}{\bar{m}_B^2}\right)$$
$$\simeq 0.372 - 0.63\frac{\bar{\Lambda}}{\bar{m}_B} - 0.63\frac{\bar{\Lambda}^2}{\bar{m}_B^2} + 1.2\frac{\lambda_1}{\bar{m}_B^2}. \qquad (6.89)$$

Applying this procedure to the inclusive semileptonic decay rate in Eq. (6.59) and including the perturbative QCD corrections to the terms not suppressed by powers of Λ_{QCD}/m_Q gives

$$\Gamma_{\text{SL}}(B) = \frac{G_F^2|V_{cb}|^2 m_B^5}{192\pi^3}\, 0.369\left[\eta_\Gamma - 1.65\frac{\bar{\Lambda}}{\bar{m}_B} - 1.0\frac{\bar{\Lambda}^2}{\bar{m}_B^2} - 3.2\frac{\lambda_1}{\bar{m}_B^2}\right]. \qquad (6.90)$$

Note that m_B^5 has been factored out instead of $\bar{m}_B^5$. This choice makes the coefficient of $\lambda_2/\bar{m}_B^2$ very small, and it has been neglected in the square brackets in Eq. (6.90).

The perturbative corrections to the leading term in the $1/m_Q$ expansion are known to order α_s^2:

$$\eta_\Gamma = 1 - 1.54\frac{\alpha_s(m_b)}{\pi} - 12.9\left[\frac{\alpha_s(m_b)}{\pi}\right]^2 = 0.83. \qquad (6.91)$$

Using Eq. (6.90), the measured semileptonic branching ratio $\text{BR}(B \to Xe\bar{\nu}_e) = (10.41 \pm 0.29)\,\%$, and the B lifetime $\tau(B) = (1.60 \pm 0.04) \times 10^{-12}$ s, one finds

$$|V_{cb}| = \frac{[\,39 \pm 1\,(\text{exp})\,] \times 10^{-3}}{\sqrt{1 - 2.0\frac{\bar{\Lambda}}{\bar{m}_B} - 1.2\left(\frac{\bar{\Lambda}}{\bar{m}_B}\right)^2 - 3.9\frac{\lambda_1}{\bar{m}_B^2}}}. \qquad (6.92)$$

The differential decay rate constrains the values of $\bar{\Lambda}$ and λ_1. An analysis of the electron energy spectrum gives (at order α_s^2) $\bar{\Lambda} \simeq 0.4$ GeV and $\lambda_1 \simeq -0.2\ \text{GeV}^2$, with a large uncertainty. These values imply that $|V_{cb}| = 0.042$. Note that this is close to the value extracted from semileptonic $\bar{B} \to D^* e\bar{\nu}_e$ decay in Chapter 4 (see Eq. (4.65)). Theoretical uncertainty in this determination of V_{cb} arises from the values of $\bar{\Lambda}$ and λ_1 and possible violations of quark hadron duality.

In Eq. (6.91) the order α_s^2 term is ~60% of the order α_s term. There are two reasons for this. First, recall from Chapter 4 that $\bar{\Lambda}$ is not a physical quantity and has a renormalon ambiguity of the order of Λ_{QCD}. Using HQET, we can relate $\bar{\Lambda}$ to a measurable quantity, for example $\langle \delta s_H \rangle$, the average value of $\delta s_H = s_H - \bar{m}_D^2$, where s_H is the hadronic invariant mass squared in semileptonic $\bar{B}$ decay. This relation involves a perturbative series in α_s. If one eliminates $\bar{\Lambda}$ in Eq. (6.90) in favor of $\langle \delta s_H \rangle$, then the combination of the perturbative series in the relation between $\bar{\Lambda}$ and $\langle \delta s_H \rangle$ and the series η_Γ will replace η_Γ in Eq. (6.90). This modified series has no Borel singularity at $u = 1/2$ and is somewhat better

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behaved. Second, the typical energy of the decay products in $b \to ce\bar{\nu}_e$ quark decay is not m_b, but rather $E_{\text{typ}} \sim (m_b - m_c)/3 \sim 1.2$ GeV. Using this scale instead of m_b to evaluate the strong coupling at in Eq. (6.91) leads to a series in which the order α_s^2 term is 25% of the order α_s term. Note that for this one uses $\alpha_s(m_b) = \alpha_s(E_{\text{typ}}) - \alpha_s^2(E_{\text{typ}})\beta_0 \ln m_b^2/E_{\text{typ}}^2 + \cdots$ in Eq. (6.91) and expands η_Γ to quadratic order in $\alpha_s(E_{\text{typ}})$.

6.7 Sum rules

One can derive a set of sum rules that restrict exclusive $\bar{B} \to D^{(*)}e\bar{\nu}_e$ form factors by comparing the inclusive and exclusive semileptonic $\bar{B}$ decay rates. The basic ingredient is the simple result that the inclusive $\bar{B}$ decay rate must always be greater than or equal to the exclusive $\bar{B} \to D^{(*)}$ decay rate.

The analysis uses $T_{\alpha\beta}$ considered as a function of q_0 with $\mathbf{q}$ held fixed. It is convenient not to focus on just the left-handed current, which is relevant for semileptonic decay, but rather to allow J to be the axial vector or vector currents or a linear combination of these. Also we change variables from q_0 to

$$\varepsilon = m_B - q_0 - E_{X_c^{\min}}, \tag{6.93}$$

where $E_{X_c^{\min}} = \sqrt{m_{X_c^{\min}}^2 + |\mathbf{q}|^2}$ is the minimal possible energy of the hadronic state. With this definition, $T_{\alpha\beta}(\varepsilon)$ has a cut in the complex ε plane along $0 < \varepsilon < \infty$, corresponding to physical states with a c quark. $T_{\mu\nu}$ has another cut for $2m_B - E_{X_{\bar{c}bb}^{\min}} - E_{X_c^{\min}} > \varepsilon > -\infty$ corresponding to physical states with two b quarks and a $\bar{c}$ quark.* This cut will not be important for the results in this section. Contracting $T_{\mu\nu}$ with a fixed four vector a^ν yields

$$a^{*\mu} T_{\mu\nu}(\varepsilon) a^\nu = -\sum_{X_c} (2\pi)^3 \delta^3(\mathbf{q} + \mathbf{p}_X) \frac{\langle \bar{B}| J^\dagger \cdot a^* |X_c\rangle\langle X_c| J \cdot a |\bar{B}\rangle}{2m_B\left(E_{X_c} - E_{X_c^{\min}} - \varepsilon\right)} + \cdots, \tag{6.94}$$

where the ellipsis denotes the contribution from the cut corresponding to two b quarks and a $\bar{c}$ quark. Consider integrating the product of a weight function $W_\Delta(\varepsilon)$ and $T_{\mu\nu}(\varepsilon)$ along the contour C shown in Fig. 6.9. Assuming W_Δ is analytic in the region enclosed by this contour, we get

$$\frac{1}{2\pi i}\int_C d\varepsilon\, W_\Delta(\varepsilon)\, a^{*\mu} T_{\mu\nu}(\varepsilon) a^\nu$$
$$= \sum_{X_c} W_\Delta\left(E_{X_c} - E_{X_c^{\min}}\right)(2\pi)^3 \delta^3\left(\mathbf{q} + \mathbf{p}_X\right) \frac{|\langle X_c| J \cdot a |\bar{B}\rangle|^2}{2m_B}. \tag{6.95}$$

* Note that the left-hand and right-hand cuts are exchanged when switching from q^0 to ε because of the minus sign in Eq. (6.93).

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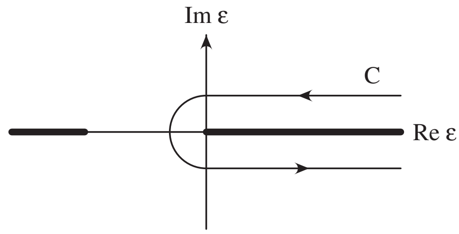

Fig. 6.9. The sum-rule cut.

We want the weight function $W_\Delta(\varepsilon)$ to be positive semidefinite along the cut so the contribution of each term in the sum over X_c above is nonnegative. For convenience we impose the normalization condition $W_\Delta(0) = 1$. We also assume W_Δ is flat near $\varepsilon = 0$ and falls off rapidly to zero for $\varepsilon \gg \Delta$. If the operator product expansion and perturbative QCD are used to evaluate the left-hand side of Eq. (6.95), it is crucial that W_Δ is flat in a region of ϵ much bigger than $\Lambda_{\rm QCD}$. Otherwise, higher-order terms in the operator product expansion and perturbative corrections will be large.

The positivity of each term in the sum over states X in Eq. (6.95) implies the bound

$$\frac{1}{2\pi i}\int_C d\varepsilon\, W_\Delta(\varepsilon)\, a^{*\mu} T_{\mu\nu}(\varepsilon) a^\nu > \frac{\left|\langle X_c^{\rm min}\right| J\cdot a \left|\bar{B}\rangle\right|^2}{4m_B E_{X_c}^{\rm min}}. \tag{6.96}$$

To derive this we note that the sum over X_c includes an integral over $d^3p/(2\pi)^3 2E$ for each particle in the final state. For the one-particle state, $X_c^{\rm min}$, performing the integral over its three momentum by using the delta function leaves the factor $(2E_{X_c^{\rm min}})$ in the denominator of Eq. (6.96). All the other states make a nonnegative contribution, leading to the inequality Eq. (6.96).

A set of possible weight functions is

$$W_\Delta^{(n)} = \frac{\Delta^{2n}}{\varepsilon^{2n} + \Delta^{2n}}. \tag{6.97}$$

For $n > 2$ the integral Eq. (6.96) is dominated by states with a mass less than Δ. These weight functions have poles at $\varepsilon = (-1)^{1/2n}\Delta$. Therefore, if n is not too large and Δ is much greater than the QCD scale, the contour in Fig. 6.9 is far from the cut. As $n \to \infty$, $W_\Delta^{(n)} \to \theta(\Delta - \varepsilon)$ for positive ε, which corresponds to summing over all final hadronic resonances with equal weight up to excitation energy Δ. In this case the poles of W_Δ approach the cut and the contour in Fig. 6.9 must be deformed to touch the cut at $\varepsilon = \Delta$. As in the semileptonic decay rate, this is usually not considered a problem as long as $\Delta \gg \Lambda_{\rm QCD}$. Here $W_\Delta^{(\infty)}$ is the common choice for the weight function, and we use it for the remainder of this chapter.

To illustrate the utility of Eq. (6.96) we go over to HQET, where the charm and bottom quark masses are taken as infinite, and let $J^\mu = \bar{c}_{v'}\gamma^\mu b_v$ and $a^\mu = v^\mu$.

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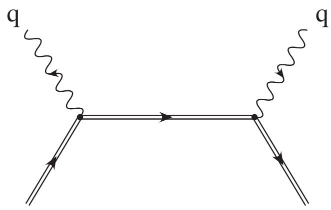

Fig. 6.10. The leading-order diagram for the OPE.

Only the pseudoscalar member of the ground state D, D^* doublet contributes in this case, and

$$\langle D(v')|J \cdot v|\bar{B}(v)\rangle = (1+w)\xi(\omega), \tag{6.98}$$

where $w = v \cdot v'$. For Δ large compared with Λ_{QCD}, the leading contribution to the time-ordered product $T_{\mu\nu}(\varepsilon)$ comes from performing the OPE, evaluating the coefficients to lowest order in α_s, and keeping only the lowest-dimension operators. We work in the $\bar{B}$-meson rest frame $v = v_r$ and define the four velocity of the charm quark by $-\mathbf{q} = m_c \mathbf{v}'$. Then the charm quark's residual momentum is $(k^0 = m_b v_r^0 - q^0 - m_c v'^0, \mathbf{k} = 0)$. In this frame $v_0' = v_r \cdot v' = w$. The leading operator in the OPE is $\bar{b}_{v_r} b_{v_r}$, and its coefficient follows from the Feynman diagram for the b-quark matrix element shown in Fig. 6.10. This yields

$$v_r^\mu T_{\mu\nu}(\varepsilon) v_r^\nu = \frac{(v_r \cdot v' + 1)}{2v_0'(m_b v_{r0} - q_0 - m_c v_0')}. \tag{6.99}$$

The variable ε defined in Eq. (6.93) can be expressed in terms of the heavy quark masses, $\bar{\Lambda}$ and w,

$$\begin{aligned} \varepsilon &= m_b + \bar{\Lambda} - q_0 - \sqrt{(m_c + \bar{\Lambda})^2 + m_c^2(w^2 - 1)} + \cdots \\ &= m_b - q_0 - m_c w + \frac{\bar{\Lambda}\,(w-1)}{w} + \cdots, \end{aligned} \tag{6.100}$$

where the ellipses denote terms suppressed by powers of $\Lambda_{\text{QCD}}/m_{b,c}$. Using this, we find Eq. (6.99) becomes

$$v_r^\mu T_{\mu\nu}(\varepsilon) v_r^\nu = \left(\frac{w+1}{2w}\right) \frac{1}{\varepsilon - \bar{\Lambda}(w-1)/w}. \tag{6.101}$$

Performing the contour integration gives

$$\frac{w+1}{2w} > \frac{|\xi(w)|^2 (1+w)^2}{4w}. \tag{6.102}$$

At zero recoil, $\xi(1) = 1$, and the above bound is saturated. Writing $\rho^2 = -\mathrm{d}\xi/\mathrm{d}w|_{w=1}$, we find the above gives the Bjorken bound on the slope of

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the Isgur-Wise function at zero recoil, $\rho^2 \geq 1/4$. Away from zero recoil the Isgur-Wise function is subtraction-point dependent and consequently ρ^2 depends on the subtraction point. Perturbative QCD corrections add terms of the form $\alpha_s(\mu)(\ln \Delta^2/\mu^2 + C)$ to the bound on ρ^2. Consequently the bound on ρ^2 is more correctly written as

$$\rho^2(\Delta) \geq 1/4 + \mathcal{O}[\alpha_s(\Delta)]. \tag{6.103}$$

6.8 Inclusive nonleptonic decays

The nonleptonic weak decay Hamiltonian for $b \to c\bar{u}d$ decays $H_W^{(\Delta c=1)}$ was given in Eqs. (1.124) and (1.125). The nonleptonic decay rate is related to the imaginary part of the B-meson matrix element of the time-ordered product of this Hamiltonian with its Hermitian conjugate,

$$t = i \int d^4x \; T\big[H_W^{(\Delta c=1)\dagger}(x)\; H_W^{(\Delta c=1)}(0)\big]. \tag{6.104}$$

Taking the matrix element of t between B-meson states at rest and inserting a complete set of states between the two Hamiltonian densities yields

$$\begin{aligned}\Gamma^{(\Delta c=1)} &= \sum_X (2\pi)^4\, \delta^4(p_B - p_X)\, \frac{\left|\langle X(p_X)|H_W^{(\Delta c=1)}(0)|\bar{B}(p_B)\rangle\right|^2}{2m_B} \\ &= \frac{\mathrm{Im}\,\langle \bar{B}|t|\bar{B}\rangle}{m_B},\end{aligned} \tag{6.105}$$

where the first line is the definition of $\Gamma^{(\Delta c=1)}$.

Inclusive nonleptonic decays can also be studied by using the OPE. In the case of semileptonic decays, one can smear the decay distributions over the leptonic kinematic variables q^2 and $q \cdot v$. The corresponding smearing variables do not exist for nonleptonic decay, since all the final-state particles are hadrons. For nonleptonic decays, one needs the additional assumption that the OPE answer is correct even without averaging over the hadron invariant mass, which is fixed to be the B-meson mass. This assumption is reasonable because m_B is much greater than Λ_{QCD}. The leading term in the OPE is computed from the diagram in Fig. 6.11. Its imaginary part gives the total nonleptonic decay width. The

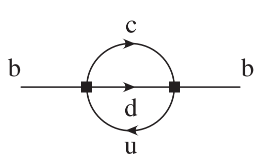

Fig. 6.11. OPE diagram for inclusive nonleptonic decay.

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situation in the case of nonleptonic decays is not very different from the case of semileptonic decays, since there the contour of $v \cdot q$ integration cannot be deformed so that it is always far from the physical cut; see Fig. 6.3.

One can compare the OPE computation of the nonleptonic decay width with that of the semileptonic decay distribution. Imagine evaluating Fig. 6.11 with $\bar{u}d$ replaced by $\bar{\nu}e$. Computing the imaginary part of the diagram is equivalent to evaluating the phase space integral for the final-state fermions. Thus performing an OPE from the imaginary part of Fig. 6.11 is equivalent to integrating the decay distributions to obtain the total decay width in Eq. 6.60. In the case of nonleptonic decays, only the total width can be computed. Decay distributions are not accessible using this method. Another difference between the semileptonic and nonleptonic decays is that the weak Hamiltonian $H_W^{(\Delta c=1)}$ contains two terms with coefficients $C_1(m_b)$ and $C_2(m_b)$ due to summing radiative corrections, using the renormalization group equations.

Including $\Lambda_{\rm QCD}^2/m_b^2$ terms, we find the final result for the nonleptonic decay width $\Gamma^{(\Delta c=1)}$ computed using the OPE together with a transition to HQET is

$$\Gamma^{(\Delta c=1)} = 3\frac{G_F^2 m_b^5}{192\pi^3}|V_{cb}V_{ud}|^2 \left\{\left(C_1^2 + \frac{2}{3}C_1C_2 + C_2^2\right)\left[\left(1 + \frac{\lambda_1}{2m_b^2}\right) + \frac{3\lambda_2}{2m_b^2}\left(2\rho\frac{d}{d\rho} - 3\right)\right] f(\rho) - 16C_1C_2\frac{\lambda_2}{m_b^2}(1-\rho)^3\right\}, \quad (6.106)$$

where $f(\rho)$ was defined in Eq. (6.61) and $C_{1,2}$ are evaluated at $\mu = m_b$.

The form of the leading-order term was computed in Problem 8 in Chapter 1. The order $\Lambda_{\rm QCD}^2/m_b^2$ part of Eq. (6.106) proportional to λ_1 can be deduced using the techniques of Sec. (6.4). Equation (6.78) holds for both the semileptonic and nonleptonic decay widths. However, the correction proportional to λ_2 cannot be deduced as simply. Like the semileptonic decay case, it arises from two sources. One is from a b-quark matrix element of the time-ordered product t, where the b quarks have momentum $p_b = m_b v + k$. Expanding in the residual momentum k gives, at quadratic order in k, dependence on λ_2 through the transition from full QCD to HQET. This part of the λ_2 dependence is the same for the nonleptonic and semileptonic decays. There is also λ_2 dependence that is identified from the $b \to b +$ gluon matrix element. It is different in the nonleptonic decay case because of the possibility that the gluon is emitted off the d or $\bar{u}$ quarks, as shown in Fig. 6.12. This contribution depends on the color structure of the operators O_1 and O_2, and we consider the pieces in $\Gamma^{(\Delta c=1)}$ proportional to C_1^2, C_2^2, and C_1C_2 successively.

For the piece of the λ_2 term proportional to C_1^2, the contribution where a gluon attaches to a d or $\bar{u}$ quark vanishes by color conservation because these diagrams are proportional to ${\rm Tr}\, T^A = 0$. Consequently, the λ_2 dependence proportional to C_1^2 is the same for nonleptonic and semileptonic decays. For the contribution

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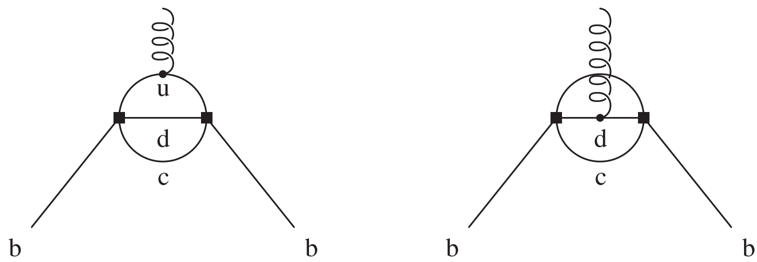

Fig. 6.12. OPE diagram for inclusive nonleptonic B decay with a gluon emitted from one of the light quark lines.

proportional to C_2^2, the $b \to b +$ gluon matrix element is the same as semileptonic $b \to u$ decay, provided the electron is not massless but rather has a mass equal to that of the c quark. This is easily seen after making a Fierz rearrangement of the quark fields in O_2. Note that the c-quark mass only enters the calculation of $T_{\mu\nu}$ through Δ_0. The left-handed projectors P_L remove the c-quark mass term in the numerator of its propagator. After taking the imaginary part, the m_c dependence in Δ_0 goes into setting the correct three-body phase space. However, the phase space is the same for $b \to c$ decay with massless leptons and $b \to u$ decay with a massless neutrino and the electron having the same mass as the c quark. Consequently, the $\lambda_2 C_2^2$ term is also the same as in semileptonic decay. For the $\lambda_2 C_1 C_2$ term there is the usual part that is the same as for semileptonic decays, as well as an additional contribution from the piece of the $b \to b +$ gluon matrix element of t where the gluon attaches to either the d or $\bar{u}$ quarks. This additional part is the last term in Eq. (6.106), and the remainder of this section is devoted to computing it.

The part of $\text{Im}\langle bg|t|b\rangle$ coming from Fig. 6.12 is

$$16\pi i G_F^2 |V_{cb}|^2 |V_{ud}|^2 \int \frac{d^4q}{(2\pi)^4} \delta\left[(m_b v - q)^2 - m_c^2\right] \times \bar{u}\gamma^\mu P_L (m_b \not{v} - \not{q} + m_c)\gamma^\nu P_L u \, \text{Im}\,\Pi_{\mu\nu}. \tag{6.107}$$

In Eq. (6.107) the δ function comes from the imaginary part of the c-quark propagator and

$$\Pi_{\mu\nu} = g T^A \varepsilon^{A\lambda *} \int \frac{d^4k}{(2\pi)^4} \times \text{Tr}\left[\gamma^\mu P_L \frac{\not{k} - \not{p}/2}{(k - p/2)^2 + i\varepsilon}\gamma_\lambda \frac{\not{k} + \not{p}/2}{(k + p/2)^2 + i\varepsilon}\gamma_\nu P_L \frac{\not{k} - \not{q}}{(k - q)^2 + i\varepsilon} + \gamma_\mu P_L \frac{\not{k} + \not{q}}{(k + q)^2 + i\varepsilon}\gamma_\nu P_L \frac{\not{k} - \not{p}/2}{(k - p/2)^2 + i\varepsilon}\gamma_\lambda \frac{\not{k} + \not{p}/2}{(k + p/2)^2 + i\varepsilon}\right]. \tag{6.108}$$

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As in the semileptonic decay case, the gluon has outgoing momentum p and the initial and final b quarks have residual momentum $p/2$ and $-p/2$, respectively. Expanding in p, keeping only the linear term, combining denominators by using the Feynman trick, performing the k integration, taking the imaginary part, and performing the Feynman parameter integration gives

$$\text{Im}\,\Pi_{\mu\nu} = \frac{igp^{\beta}T^{A}\varepsilon^{A\lambda*}}{32\pi}\delta(q^2)\text{Tr}[\gamma_\mu(\gamma_\beta\gamma_\lambda \not{q} - \not{q}\gamma_\lambda\gamma_\beta)\gamma_\nu \not{q} P_R + (\mu \leftrightarrow \nu)]. \tag{6.109}$$

Only the part antisymmetric in β and λ gives a contribution of the type we are interested in. Performing the trace yields

$$\text{Im}\,\Pi_{\mu\nu} = \frac{gp^{\beta}T^{A}\varepsilon^{A\lambda*}}{4\pi}\delta(q^2)[\epsilon_{\beta\nu\lambda\alpha}q^{\alpha}q_\mu + (\mu \leftrightarrow \nu)]. \tag{6.110}$$

Putting this into Eq. (6.107), identifying the spinors with HQET b-quark fields, and using $p^{\beta}T^{A}\epsilon^{A\lambda*} \to -iG^{\beta\lambda}/2$ and Eq. (6.50) for the resulting B-meson matrix element, Fig. 6.12 gives the following contribution to the nonleptonic width:

$$\begin{aligned}\delta\Gamma^{(\Delta c=1)} &= -32C_1C_2|V_{cb}|^2|V_{ud}|^2G_F^2\lambda_2\\ &\quad\times\int\frac{d^4q}{(2\pi)^4}\delta\left[(m_b v - q)^2 - m_c^2\right]\delta(q^2)m_b(v\cdot q)^2.\end{aligned} \tag{6.111}$$

Performing the q^0 and $\boldsymbol{q}$ integrations with the δ functions yields

$$\delta\Gamma^{(\Delta c=1)} = -\frac{C_1C_2|V_{cb}|^2|V_{ud}|^2G_F^2\lambda_2 m_b^3}{4\pi^3}\left(1-\frac{m_c^2}{m_b^2}\right)^3, \tag{6.112}$$

which is the last term in Eq. (6.106). The contribution of dimension-six four-quark operators to the nonleptonic width is thought to be more important than the dimension-five operators considered in this section, because their coefficients are enhanced by a factor of $16\pi^2$. The influence of similar four-quark operators in the case of $B_s - \bar{B}_s$ mixing will be considered in the next section.

6.9 $B_s - \bar{B}_s$ mixing

The light antiquark in a $\bar{B}$ or $\bar{B}_s$ meson is usually called the spectator quark, because at leading order in the OPE, its field does not occur in the operators whose matrix elements give the inclusive decay rate. This persists at order $1/m_b^2$ since $\lambda_{1,2}$ are defined as the matrix elements of operators constructed from b-quark and gluon fields. At order $1/m_b^3$, the spectator quark fields first appear because dimension-six four-quark operators of the form $\bar{b}_v b_v \bar{q}q$ occur in the OPE. These operators play a very important role in $B_s - \bar{B}_s$ width mixing.

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Recall that $CP|B_s\rangle = -|\bar{B}_s\rangle$, so the *CP* eigenstates are

$$\begin{aligned} |B_{s1}\rangle &= \frac{1}{\sqrt{2}}(|B_s\rangle + |\bar{B}_s\rangle) \\ |B_{s2}\rangle &= \frac{1}{\sqrt{2}}(|B_s\rangle - |\bar{B}_s\rangle), \end{aligned} \tag{6.113}$$

with $CP|B_{sj}\rangle = (-1)^j |B_{sj}\rangle$. At second order in the weak interactions there are $|\Delta b| = 2$, $|\Delta s| = 2$ processes that cause mass and width mixing between the $|B_s\rangle$ and $|\bar{B}_s\rangle$ states. In the limit that *CP* is conserved, it is the states $|B_{sj}\rangle$ rather than $|B_s\rangle$ and $|\bar{B}_s\rangle$ that are eigenstates of the effective Hamiltonian $H_{\rm eff} = \mathbb{M} + i\mathbb{W}/2$, where $\mathbb{M}$ and $\mathbb{W}$ are the 2×2 mass and width matrices for this system. For simplicity, we will neglect *CP* violation in the remainder of this section; it is straightforward to extend the arguments to include *CP* violation. In the $B_s - \bar{B}_s$ basis, the width matrix $\mathbb{W}$ is

$$\mathbb{W} = \begin{pmatrix} \Gamma_{B_s} & \Delta\Gamma \\ \Delta\Gamma & \Gamma_{\bar{B}_s} \end{pmatrix}. \tag{6.114}$$

CPT invariance implies that $\Gamma_{B_s} = \Gamma_{\bar{B}_s}$, so the widths of the eigenstates of $H_{\rm eff}$ are

$$\Gamma_j = \Gamma_{B_s} - (-1)^j \Delta\Gamma. \tag{6.115}$$

The difference between the widths of the two eigenstates $|B_{s1}\rangle$ and $|B_{s2}\rangle$ is $\Gamma_1 - \Gamma_2 = 2\Delta\Gamma$.

The width mixing element $\Delta\Gamma$ in Eq. (6.114) is defined by

$$\begin{aligned} \Delta\Gamma &\equiv \sum_X (2\pi)^4 \delta^4(p_B - p_X) \frac{\langle B_s | H_W^{(\Delta c = 0)} | X \rangle \langle X | H_W^{(\Delta c = 0)} | \bar{B}_s \rangle}{2m_{B_s}} \\ &= \mathrm{Im}\, \frac{\langle B_s | i \int d^4x\; T\big[H_W^{(\Delta c = 0)}(x)\; H_W^{(\Delta c = 0)}(0)\big] | \bar{B}_s \rangle}{2m_{B_s}}. \end{aligned} \tag{6.116}$$

The first line is the definition of $\Delta\Gamma$, and the second line can be verified by inserting a complete set of states. There is a difference of a factor of 2 when compared with Eq. (6.105), because now both time orderings contribute. The width transition matrix element $\Delta\Gamma$ comes from final states that are common in B_s and $\bar{B}_s$ decay. For this reason it involves only the $\Delta c = 0$ part of the weak Hamiltonian; the $\Delta c = 1$ part does not contribute. The $\Delta c = 0$ part of the weak Hamiltonian gives at tree level the quark decay $b \to c\bar{c}s$. In the leading logarithmic approximation,

$$H_W^{(\Delta c=0)} = \frac{4G_F}{\sqrt{2}} V_{cb} V_{cs}^* \sum_i C_i(\mu) Q_i(\mu), \tag{6.117}$$

where the operators $Q_i(\mu)$ that occur are

$$\begin{aligned}
Q_1 &= (\bar{c}^\alpha \gamma_\mu P_L b_\alpha)(\bar{s}^\beta \gamma^\mu P_L c_\beta), \\
Q_2 &= (\bar{c}^\beta \gamma_\mu P_L b_\alpha)(\bar{s}^\alpha \gamma^\mu P_L c_\beta), \\
Q_3 &= (\bar{s}^\alpha \gamma_\mu P_L b_\alpha) \sum_{q=u,d,s,c,b} \bar{q}^\beta \gamma^\mu P_L q_\beta, \\
Q_4 &= (\bar{s}^\beta \gamma_\mu P_L b_\alpha) \sum_{q=u,d,s,c,b} \bar{q}^\alpha \gamma^\mu P_L q_\beta, \\
Q_5 &= (\bar{s}^\alpha \gamma_\mu P_L b_\alpha) \sum_{q=u,d,s,c,b} \bar{q}^\beta \gamma^\mu P_R q_\beta, \\
Q_6 &= (\bar{s}^\beta \gamma_\mu P_L b_\alpha) \sum_{q=u,d,s,c,b} \bar{q}^\alpha \gamma^\mu P_R q_\beta.
\end{aligned} \tag{6.118}$$

At the subtraction point $\mu = M_W$, the coefficients are

$$C_1(M_W) = 1 + \mathcal{O}\left[\alpha_s\left(M_W\right)\right], \qquad C_{j\neq 1}(M_W) = 0 + \mathcal{O}\left[\alpha_s\left(M_W\right)\right]. \tag{6.119}$$

The operators Q_1 and Q_2 are analogous to O_1 and O_2 in the $\Delta c = 1$ nonleptonic Hamiltonian. The new operators $Q_3 - Q_6$ occur because new "penguin" diagrams shown in Fig. 6.13 occur in the renormalization of Q_1. The sum of diagrams in Fig. 6.13 is proportional to the tree-level matrix element of the operator

$$g(\bar{s} T^A \gamma_\mu P_L b) D_\nu G^{A\nu\mu}, \tag{6.120}$$

which after using the equation of motion $D_\nu G^{A\nu\mu} = g \sum_q \bar{q} \gamma^\mu T^A q$ becomes

$$g^2(\bar{s} T^A \gamma_\mu P_L b) \sum_{q=u,d,s,c,b} \bar{q} \gamma^\mu T^A q. \tag{6.121}$$

This is a linear combination of $Q_3 - Q_6$. Penguin-type diagrams with more gluons attached to the loop are finite and do not contribute to the operator renormalization.

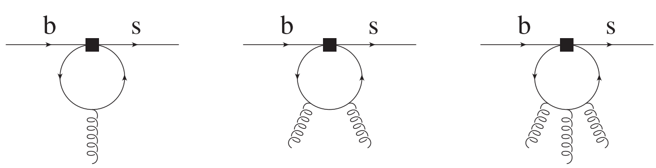

Fig. 6.13. Penguin diagrams that renormalize the weak Hamiltonian.

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The coefficients of Q_{1-6} at $\mu = m_b$ are computed by using the renormalization group equation Eq. (1.134), where the anomalous dimension matrix is

$$\gamma = \frac{g^2}{8\pi^2}\begin{pmatrix} -1 & 3 & -\frac{1}{9} & \frac{1}{3} & -\frac{1}{9} & \frac{1}{3} \\ 3 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{11}{9} & \frac{11}{3} & -\frac{2}{9} & \frac{2}{3} \\ 0 & 0 & \frac{22}{9} & \frac{2}{3} & -\frac{5}{9} & \frac{5}{3} \\ 0 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & -\frac{5}{9} & \frac{5}{3} & -\frac{5}{9} & -\frac{19}{3} \end{pmatrix}. \tag{6.122}$$

Solving Eq. (1.134) for the coefficients at the scale $\mu = m_b$, it is easy to see that C_1 and C_2 have the same value as in the $\Delta c = 1$ case, whereas $C_3 - C_6$ are quite small.

The operators in the OPE for the time-ordered product of weak Hamiltonians that gives $\Delta\Gamma$ must be both $\Delta s = 2$ and $\Delta b = 2$. Consequently, the lowest-dimension operators are four-quark operators, and $\Delta\Gamma$ is suppressed by $\Lambda^3_{\rm QCD}/m_b^3$ in comparison with Γ.

Neglecting the operators $Q_3 - Q_6$, we calculate the operator product for the time-ordered product in Eq. (6.116) from the imaginary part of the one-loop Feynman diagram in Fig. 6.14. This gives

$$\begin{aligned}\Delta\Gamma = \Big[&C_1^2\langle B_s(v)|(\bar{s}^\beta\gamma^\mu P_L b_{v\alpha})(\bar{s}^\alpha\gamma^\nu P_L b_{v\beta})|\bar{B}_s(v)\rangle + \left(3C_2^2 + 2C_1C_2\right) \\ &\times \langle B_s(v)|(\bar{s}^\alpha\gamma^\mu P_L b_{v\alpha})(\bar{s}^\beta\gamma^\nu P_L b_{v\beta})|\bar{B}_s(v)\rangle\Big]\,\mathrm{Im}\,\Pi_{\mu\nu}(p_b). \end{aligned} \tag{6.123}$$

Taking the imaginary part converts the loop integration into a phase space integration for the intermediate c and $\bar{c}$ quarks:

$$\begin{aligned}\mathrm{Im}\,\Pi_{\mu\nu}(p_b) = 4G_F^2\left(V_{cb}V_{cs}^*\right)^2 \int \frac{d^3p_c}{(2\pi)^3 2E_c}\frac{d^3p_{\bar{c}}}{(2\pi)^3 2E_{\bar{c}}}(2\pi)^4\,\delta^4\,(p_b - p_c - p_{\bar{c}}) \\ \times \mathrm{Tr}[\gamma_\mu P_L(\not{p}_c + m_c)\gamma_\nu P_L(\not{p}_{\bar{c}} - m_c)]. \end{aligned} \tag{6.124}$$

Performing the phase space integration above yields

$$\mathrm{Im}\,\Pi_{\mu\nu}(p_b) = 4G_F^2\left(V_{cb}V_{cs}^*\right)^2 m_b^2(E v_\mu v_\nu + F g_{\mu\nu}), \tag{6.125}$$

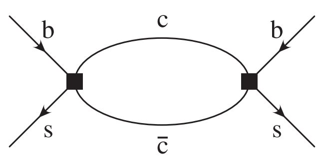

Fig. 6.14. One-loop diagram for $B_s - \bar{B}_s$ mixing.

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where

$$E = \frac{1+2\rho}{24\pi}\sqrt{1-4\rho},$$
$$F = -\frac{1-\rho}{24\pi}\sqrt{1-4\rho}, \tag{6.126}$$

and $\rho = m_c^2/m_b^2$. Putting the above results together gives

$$\begin{aligned}\Delta\Gamma = &\frac{G_F^2 \left(V_{cb}V_{cs}^*\right)^2 m_b^2}{6\pi}\sqrt{1-4\rho} \\ &\times \Big\{\Big[C_1^2 \langle B_s(v)|(\bar{s}^\beta P_R b_{v\alpha})\,(\bar{s}^\alpha P_R b_{v\beta})|\bar{B}_s(v)\rangle \\ &+ \left(3C_2^2 + 2C_1C_2\right) \\ &\times \langle B_s(v)|(\bar{s}^\beta P_R b_{v\beta})\,(\bar{s}^\alpha P_R b_{v\alpha})|\bar{B}_s(v)\rangle\Big](1+2\rho) \\ &- \left(C_1^2 + 3C_2^2 + 2C_1C_2\right) \\ &\times \langle B_s(v)|(\bar{s}^\beta \gamma^\mu P_L b_{v\beta})\,(\bar{s}^\alpha \gamma_\mu P_L b_{v\alpha})|\bar{B}_s(v)\rangle(1-\rho)\Big\}.\end{aligned} \tag{6.127}$$

One of the four-quark operators can be eliminated by using the Fierz identity:

$$\begin{aligned}&(\bar{s}^\alpha \gamma^\mu P_L b_\alpha)(\bar{s}^\beta \gamma^\nu P_L b_\beta) + (\bar{s}^\beta \gamma^\mu P_L b_\alpha)(\bar{s}^\alpha \gamma^\nu P_L b_\beta) \\ &\quad = \frac{1}{2} g^{\mu\nu} (\bar{s}^\alpha \gamma^\lambda P_L b_\alpha)(\bar{s}^\beta \gamma_\lambda P_L b_\beta).\end{aligned} \tag{6.128}$$

Making the transition to HQET and contracting with $v_\mu v_\nu$, we find this Fierz identity gives

$$\begin{aligned}&(\bar{s}^\alpha P_R b_{v\alpha})(\bar{s}^\beta P_R b_{v\beta}) + (\bar{s}^\beta P_R b_{v\alpha})(\bar{s}^\alpha P_R b_{v\beta}) \\ &\quad = \frac{1}{2}(\bar{s}^\alpha \gamma^\lambda P_L b_{v\alpha})(\bar{s}^\beta \gamma_\lambda P_L b_{v\beta}).\end{aligned} \tag{6.129}$$

Using this, Eq. (6.127) becomes

$$\begin{aligned}\Delta\Gamma = &-\frac{G_F^2 \left(V_{cb}V_{cs}^*\right)^2 m_b^2}{6\pi}\sqrt{1-4\rho} \times \Big\{\left(-C_1^2 + 2C_1C_2 + 3C_2^2\right)(1+2\rho) \\ &\times \langle B_s(v)|(\bar{s}^\beta P_R b_{v\beta})\,(\bar{s}^\alpha P_R b_{v\alpha})|\bar{B}_s(v)\rangle \\ &+ \left[\frac{1}{2}C_1^2(1-4\rho) + \left(3C_2^2 + 2C_1C_2\right)(1-\rho)\right] \\ &\times \langle B_s(v)|(\bar{s}^\beta \gamma^\mu P_L b_{v\beta})\,(\bar{s}^\alpha \gamma_\mu P_L b_{v\alpha})|\bar{B}_s(v)\rangle\Big\}.\end{aligned} \tag{6.130}$$

Estimates of the matrix elements in this equation suggest that $|\Delta\Gamma/\Gamma_{B_s}|$ is ~ 0.1.

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6.10 Problems

1. At fixed q^2, show that the structure functions $F_{1,2}(\omega, q^2)$ defined in Sec. 1.8 have cuts on the real ω axis for $|\omega| \geq 1$. Also show that the discontinuity across the positive ω cut is given by Eq. (1.165).
2. Derive Eqs. (6.10) and (6.11).
3. Define the parton-level dimensionless energy and invariant mass variables $\hat{E}_0$ and $\hat{s}_0$ by

$$\hat{E}_0 = v \cdot (p_b - q)/m_b = 1 - v \cdot \hat{q},$$
$$\hat{s}_0 = (p_b - q)^2/m_b^2 = 1 - 2v \cdot \hat{q} + \hat{q}^2.$$

The hadronic energy E_H and invariant mass s_H are given by

$$E_H = v \cdot (p_B - q) = m_B - v \cdot q,$$
$$s_H = (p_B - q)^2 = m_B^2 - 2m_B v \cdot q + q^2.$$

(a) Show that E_H and s_H are related to the parton-level quantities by

$$E_H = \bar{\Lambda} - \frac{\lambda_1 + 3\lambda_2}{2m_B} + \left(m_B - \bar{\Lambda} + \frac{\lambda_1 + 3\lambda_2}{2m_B}\right)\hat{E}_0 + \cdots$$
$$s_H = m_c^2 + \bar{\Lambda}^2 + \left(m_B^2 - 2\bar{\Lambda}m_B + \bar{\Lambda}^2 + \lambda_1 + 3\lambda_2\right)(\hat{s}_0 - \rho) + (2\bar{\Lambda}m_B - 2\bar{\Lambda}^2 - \lambda_1 - 3\lambda_2)\hat{E}_0 + \cdots,$$

where the ellipses denote terms of higher order in $1/m_B$.

(b) For the $b \to u$ case, set $m_c = 0$ in the above and show that

$$\langle \hat{s}_0 \rangle = \frac{13\lambda_1}{20m_b^2} + \frac{3\lambda_2}{4m_b^2},$$
$$\langle \hat{E}_0 \rangle = \frac{13\lambda_1}{40m_b^2} + \frac{63\lambda_2}{40m_b^2},$$

where the symbol $\langle \cdot \rangle$ denotes an average over the decay phase space.

(c) Use the previous results to show that

$$\langle s_H \rangle = m_B^2 \left[\frac{7\bar{\Lambda}}{10m_B} + \frac{3}{10m_B^2}(\bar{\Lambda}^2 + \lambda_1 - \lambda_2)\right].$$

4. Define

$$T_{\mu\nu} = -i \int d^4x\, e^{-iq\cdot x} \frac{\langle \bar{B}|T\left[J_\mu^\dagger(x) J_\nu(0)\right]|\bar{B}\rangle}{2m_B},$$

where J is a $b \to c$ vector or axial current. An operator product expansion of $T_{\mu\nu}$ in the zero-recoil case $\mathbf{q} = 0$ yields

$$\frac{1}{3}T_{ii}^{AA} = \frac{1}{\varepsilon} - \frac{(\lambda_1 + 3\lambda_2)(m_b - 3m_c)}{6m_b^2\varepsilon(2m_c + \varepsilon)} + \frac{4\lambda_2 m_b - (\lambda_1 + 3\lambda_2)(m_b - m_c - \varepsilon)}{m_b\varepsilon^2(2m_c + \varepsilon)},$$
$$\frac{1}{3}T_{ii}^{VV} = \frac{1}{2m_c + \varepsilon} - \frac{(\lambda_1 + 3\lambda_2)(m_b + 3m_c)}{6m_b^2\varepsilon(2m_c + \varepsilon)} + \frac{4\lambda_2 m_b - (\lambda_1 + 3\lambda_2)(m_b - m_c - \varepsilon)}{m_b\varepsilon(2m_c + \varepsilon)^2},$$

where $\varepsilon = m_b - m_c - q_0$.

(a) Use these results to deduce the sum rules

$$\frac{1}{6m_B}\sum_X (2\pi)^3\,\delta^3(\mathbf{p}_X)\,|\langle X|A_i|\bar{B}\rangle|^2 = 1-\frac{\lambda_2}{m_c^2}+\frac{\lambda_1+3\lambda_2}{4}\left(\frac{1}{m_c^2}+\frac{1}{m_b^2}+\frac{2}{3m_bm_c}\right),$$

$$\frac{1}{6m_B}\sum_X (2\pi)^3\,\delta^3(\mathbf{p}_X)\,|\langle X|V_i|\bar{B}\rangle|^2 = \frac{\lambda_2}{m_c^2}-\frac{\lambda_1+3\lambda_2}{4}\left(\frac{1}{m_c^2}+\frac{1}{m_b^2}-\frac{2}{3m_bm_c}\right).$$

(b) Use part (a) to deduce the bounds

$$h_{A_1}^2(1)\le 1-\frac{\lambda_2}{m_c^2}+\frac{\lambda_1+3\lambda_2}{4}\left(\frac{1}{m_c^2}+\frac{1}{m_b^2}+\frac{2}{3m_bm_c}\right),$$

$$0\le \frac{\lambda_2}{m_c^2}-\frac{\lambda_1+3\lambda_2}{4}\left(\frac{1}{m_c^2}+\frac{1}{m_b^2}-\frac{2}{3m_bm_c}\right).$$

5. Use the results of Sec. 6.2 to derive the double differential decay rate in Eq. (6.57).
6. Calculate the renormalization of $Q_1 - Q_6$ and verify the anomalous dimension matrix in Eq. (6.122).
7. Suppose the effective Hamiltonian for semileptonic weak B decay is

$$H_W = \frac{G_F}{\sqrt{2}} V_{cb}(\bar{c}\gamma_\mu b)(\bar{e}\gamma^\mu \nu_e).$$

Perform an OPE on the time-ordered product of vector currents and deduce the nonperturbative $1/m_b^2$ corrections to $d\Gamma/d\hat{q}^2\,dy$.

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