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NEAR-RINGS OF COMPATIBLE FUNCTIONS

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1. Summary

In this paper we study near-rings of functions on Ω-groups which are compatible with all congruence relations. Polynomial functions, for instance, are of this type. We employ the structure theory for near-rings to get results for the theory of compatible and polynomial functions (affine completeness, etc.). For notations and results concerning near-rings see e.g. (10). However, we review briefly some terminology from there. $(N, +, .)$ is a near-ring if $(N, +)$ is a group and . is associative and right distributive over $+$. For instance, $M(A) := (A^A, +, \circ)$ is a near-ring for any group $(A, +)$ ($\circ$ is composition). If N is a near-ring then $N_0 := \{n \in N / n0 = 0\}$. A group $(\Gamma, +)$ is an N-group (we write ${}_N\Gamma$) if a "product" $n\gamma$ is defined with $(n + n')\gamma = n\gamma + n'\gamma$ and $(nn')\gamma = n(n'\gamma)$. Ideals of near-rings and N-groups are kernels of (N-) homomorphisms. If Γ is a vector-space, $M_{\text{aff}}(\Gamma)$ is the near-ring of all affine transformations on Γ. N is 2-primitive on ${}_N\Gamma$ if ${}_N\Gamma$ is non-trivial, faithful and without proper N-subgroups. The (2-) radical and (2-) semisimplicity are defined similarly to the ring case.

2. Compatible and polynomial functions

In (13), an N-group ${}_N\Gamma$ (N a near-ring) is called *tame* if $N = N_0$ and every N-subgroup of ${}_N\Gamma$ is an ideal of ${}_N\Gamma$ (= kernel of an N-group homomorphism); N is *tame* if there is some faithful tame N-group. We study a similar concept:

Definition 2.1. Let N be a near-ring and ${}_N\Gamma$ an N-group.

(a) ${}_N\Gamma$ is *gentle* if and only if every normal subgroup of $(\Gamma, +)$ is an ideal of ${}_N\Gamma$.

(b) N is *gentle* if and only if there exists some faithful gentle N-group.

We get a lot of interesting examples in the following way.

Definition 2.2 Let $A = (A, \Omega)$ be a (universal) algebra.

(a) $M(A) := (A^A, \Omega \cup \{\circ\})$, where $\circ$ means the composition of functions; the operations $\omega \in \Omega$ are defined pointwise.

(b) $C(A) := \{f \in M(A) |$ for all congruence relations $\equiv$ of A we have that

$$\forall a, b \in A\colon a \equiv b \Rightarrow f(a) \equiv f(b)\}.$$

The functions in $C(A)$ are called *compatible.*

See (13) for a number of results concerning the structure of compatible near-rings.

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(c) Let $P(A)$ be the subalgebra of $M(A)$ generated by id_A and the constant functions. The elements of $P(A)$ are called *polynomial functions*.

(d) For $n \in \mathbb{N}$ let

$$L_nP(A) := \{f \in M(A) \mid \forall T \subseteq A, |T| \leq n \, \exists p \in P(A): f/T = p/T\}$$

(These functions can be "*interpolated*" by polynomial functions on any n places.)

(e) $LP(A) := \bigcap_{n \in \mathbb{N}} L_nP(A)$. The elements in LP(A) are the "*local polynomial functions*".

It is interesting to find all algebras which enjoy one of the following four properties:

Definition 2.3. An algebra A is

polynomially complete if and only if $P(A) = M(A)$;
locally polynomially complete if and only if $LP(A) = M(A)$;
affine complete if and only if $P(A) = C(A)$;
locally affine complete if and only if $LP(A) = C(A)$.

Remarks 2.4.

(a) Our definitions in 2.3 differ slightly from those in (3)–(9), for these authors consider also functions in several variables.

(b) One has (see e.g. (3))

pol. compl.

loc. pol. compl.

$$P(A) \subseteq LP(A) \subseteq \cdots \subseteq L_3P(A) \subseteq L_2P(A) \subseteq C(A) \subseteq M(A) = L_1P(A)$$

loc. aff. compl.

aff. compl.

(c) If A is an Ω-group (the underlying group written additively) then all guys from $P(A)$ to $M(A)$ are near-rings w.r.t. $+$ and $\circ$.

In this case, $C(A)$ can be written as

$$C(A) = \{f \in M(A) \mid \forall I \trianglelefteq A \, \forall i \in I \, \forall a \in A : f(a+i) - f(a) \in I\}$$

Hence all near-rings N contained in $C(A)$ are gentle on A if A is just a group (or, more generally, if every normal subgroup of $(A, +)$ is an ideal of A, as is the case for the rings Z and Z_n).

(d) Every polynomially complete algebra is simple (see e.g. (6)).

Examples 2.5.

(a) Let A be a commutative ring with identity. Then the polynmial functions are as usual.

A is polynomially complete iff A is a finite field (see (6)).

By Lagrange's theorem, $\mathbb{R}$ is locally polynomially complete.

So $P(Z_2) = M(Z_2)$, but (see (14))
$P(Z_4) = C(Z_4)$, and

$$P(Z_8) = \left\{ f \in C(Z_8) \,\middle|\, \begin{matrix} f(4) = 2f(2) - f(0), & f(6) = f(3) + f(4) - f(0) \\ f(5) = 2f(3) - f(1), & f(7) = 6f(1) + 3f(2) \end{matrix} \right\}$$

For further results, see (14) and (7).

(b) Let $(A, +)$ be a group.

Then $P(A) = \{f : a \to a_0 + n_1 a + a_1 + \cdots + n_r a + a_r \mid r \in \mathrm{N}_0, a_i \in A, n_i \in Z\}$
$= I(A) + M_c(A)$ ((6) and (7))

(where $I(A)$ is the d.g. near-ring generated by the inner automorphisms of A).

The polynomially complete groups are Z_2 and the finite non-abelian simple groups, (6). For more, see (7), (9), (3) and (4).

Now we try to apply the structure theory of near-rings to get further results. In order to do this, we need some lemmas.

Lemma 2.6 (9, 12). *Let the universal algebra A be a subdirect product of the algebras A_i $(i \in I)$. Then for all $f \in C(A)$ there exist uniquely determined $f_i \in C(A_i)$ with $f((\ldots, a_i, \ldots)) = (\ldots, f_i(a_i), \ldots)$ for all $(\ldots, a_i, \ldots) \in A$. If $f \in P(A)$ then all f_i are in $P(A_i)$.*

From now on, we will exclusively deal with Ω-groups, the group operation written additively with neutral element 0. First we need some converse of 2.6.

Notation 2.7. (a) Let $P_0(A)$ and $C_0(A)$ be the zero-symmetric parts of $P(A)$ ($C(A)$, respectively), i.e. the subnear-rings consisting of all maps which take 0 into 0.

(b) Let A be the direct sum of the Ω-groups A_i $(i \in I)$ and let f_i be in $C(A_i)$ such that almost all (i.e. all but a finite number) of the f_i's belong to $C_0(A_i)$. Define $f := \bigoplus_{i \in I} f_i$ by $f((\ldots, a_i, \ldots)) := (\ldots, f_i(a_i), \ldots)$ for $(\ldots, a_i, \ldots) \in A$.

Definition 2.8. Let the situation be as in 2.7(b).

(a) The direct sum of the A_i's is *nice* if $f \in C(A)$.

(b) The direct sum of the A_i's is *kind* if $f \in P(A)$ whenever all f_i are in $P(A_i)$.

Proposition 2.8 (cf. (9)). *Let A be the direct sum of the Ω-groups A_i $(i \in I)$, such that A has no "skew congruences", which means that all ideals of A are direct sums of ideals of A_i. Then this sum is nice.*

Proof. Take $f_i \in C(A_i)$ such that almost all f_i are in $C_0(A_i)$. Then $\bigoplus_{i \in I} f_i : A \to A$. We have to show that $f \in C(A)$. Take $J \trianglelefteq A$ and $(\ldots, a_i, \ldots), (\ldots, b_i, \ldots) \in A$ such that $(\ldots, a_i, \ldots) - (\ldots, b_i, \ldots)$ is in J. Then J splits into the direct sum $\bigoplus_{i \in I} J_i$ with $J_i \trianglelefteq A_i$. Hence all $a_i - b_i \in J_i$. Since all f_i are compatible, we get $f((\ldots, a_i, \ldots)) - f((\ldots, b_i, \ldots)) = (\ldots, f_i(a_i) - f_i(b_i), \ldots) \in \bigoplus_{i \in I} J_i = J$ and f is shown to be compatible.

Examples 2.9. It is shown in (15) that direct sums of rings with identity are kind; it is mentioned in (9) that (finite) direct sums of rings with identity or of finite groups with relatively prime order are nice.

This motivated the following

Conjecture 2.10. A direct sum is nice iff it is kind, which happens iff A has no skew congruences.

Definition 2.11. For an Ω-group A let

$$L_{2,5}P(A) := \{f \in M(A) \mid \forall a_1, a_2 \in A \exists p \in P(A): \quad p(0) = f(0), p(a_1) = f(a_1), p(a_2) = f(a_2)\}.$$

Remark 2.12. $L_{2,5}P(A)$ is then a near-ring between $L_3P(A)$ and $L_2P(A)$. (It is known that for Ω-groups $L_2P(A) = C(A)$.)

From the density theorem for near-rings (see e.g. (10)) we get that "2,5-fold transitivity implies density":

Theorem 2.13. *Let A be an Ω-group such that $P_0(A)$ is no ring. If $M(A) = L_{2,5}P(A)$ then A is locally polynomially complete.*

Proof. If A is as in the statement then $N := P(A)$ is as follows: N_0 is not a ring, $N \neq N_0$ is 2-fold transitive on A-$\{0\}$. By Theorem 4.65 of (10), N is dense in $M(A)$, thereby fulfilling the finite interpolation property, whence $M(A) = LP(A)$.

Examples 2.14. (a) Let A be a group. Then $P_0(A)$ is a ring iff A is an "L-group", which means that all conjugated elements commute (see (2) or (10)).

(b) If A is a ring with identity then $P_0(A)$ is a ring iff A is a Boolean ring. If A is a simple ring then $P_0(A)$ is a ring iff A is either isomorphic to the field Z_2 or if A is a zeroring (see (12)).

The next proposition holds for all $n \in N$ instead of 2,5, but we will need it only in this case.

Proposition 2.15. *Let $A = B \oplus C$ be nice. If $L_{2,5}P(A) = C(A)$ then $L_{2,5}P(B) = C(B)$.*

Proof. Take $b_1, b_2 \in B$ and some $f \in C(B)$. Define $\bar{f}: A \to A$ by $\bar{f}(b + c) := f(b)$. Then $\bar{f}$ is the direct sum of f and the zero map in the sense of 2.7(b), and hence compatible. Therefore $\exists p \in P(A): p(0) = \bar{f}(0)$, $p(b_i) = \bar{f}(b_i)$ $(i = 1, 2)$. If p splits into $p_A \oplus p_B$ as in 2.6 then p_A "interpolates" f at 0 and at b_1, b_2. Therefore $L_{2,5}P(B) = C(B)$.

Corollary 2.16. *Let A be a nice direct sum of Ω-groups A_i $(i \in I)$. If $L_{2,5}P(A) = C(A)$ then $L_{2,5}P(A_i) = C(A_i)$ for all $i \in I$.*

Theorem 2.17. *If A is a kind direct sum of simple Ω-groups A_i $(i \in I)$ such that no $P_0(A_i)$ is a ring and $L_{2,5}P(A) = C(A)$ then A is locally affine complete.*

Proof. If $i \in I$ then, by 2.16, $L_{2,5}P(A_i) = C(A_i) = M(A_i)$ (since A_i is simple), so

$LP(A_i) = M(A_i)$ by 2.13. Now take $a_1 = (\ldots, a_{1i}, \ldots), \ldots, a_n = (\ldots, a_{ni}, \ldots) \in A$ and some $f \in C(A)$ which may split into $\bigoplus_{i \in I} f_i$ as in 2.6. Then

$$\forall i \in I \exists p_i \in P(A_i) \forall k \in \{1, \ldots, n\}: p_i(a_{ki}) = f_i(a_{ki}).$$

Almost all p_i can be taken out of $P_0(A_i) \subseteq C_0(A_i)$. If $p := \bigoplus_{i \in I} p_i$ then $p \in P(A)$ and $\forall k \in \{1, \ldots, n\}: p(a_k) = f(a_k)$. So $LP(A) = C(A)$ and A is locally affine complete.

Corollary 2.18. *Let A be the direct sum of simple rings with identity and let $L_{2,5}P(A) = C(A)$. Then A is locally affine complete.*

Proof. By 2.9, direct sums of rings with identity are kind. $P_0(A_i)$ is no ring unless A_i is a simple Boolean ring; in this case, A_i is isomorphic to Z_2. But then we also get that $LP(A_i) = P(A_i) = C(A_i) = M(A_i)$ which shows that in this case 2.17 also works if $P_0(A_i)$ is a ring.

Problems 2.19. Is 2.10 true?

Since simple non-abelian groups are not L-groups, it makes sense to ask: Let A be the direct sum of simple non-abelian groups. Does $L_{2,5}P(A) = C(A)$ imply that A is locally affine complete?

When is the direct sum of polynomially complete (locally polynomially complete) groups, rings, etc. locally affine complete?

Can one get similar results concerning chain conditions for gentle near-rings as for tame near-rings?

3. Structure theorems

Again, let A be an Ω-group, written additively with zero element 0. Polynomials are excellent for describing generated ideals:

Theorem 3.1. *For $a \in A$, let $\langle a \rangle$ be the principal ideal of A generated by a. Then $\langle a \rangle = \{p(a) \mid p \in P_0(A)\}$ (see 2.8).*

Proof. (a) Let $N := P_0(A)$. Since $\langle a \rangle \leqslant A$, $\langle a \rangle \leqslant_N A$ by 2.4 (c), hence $\langle a \rangle \trianglelefteq_N A$, whence $\{p(a) \mid p \in P_0(A)\} = Na \subseteq \langle a \rangle$.

(b) Conversely, we will show that this Na is an ideal of A containing a, from which we get $\langle a \rangle \subseteq Na$.

(i) Because of $\mathrm{id}_A \in N$, $a = \mathrm{id}(a) \in Na$.

(ii) Clearly, Na is a subgroup of $(A, +)$.

(iii) $\forall b \in A \forall p(a) \in Na: b + p(a) - b = (\underline{b} + p - \underline{b})(a)$, where $\underline{b}$ is the constant polynomial function with value b. Now $\underline{b} + p - \underline{b} \in P(A)$ and $(\underline{b} + p - \underline{b})(0) = b + p(0) - b = b - b = 0$, so $\underline{b} + p - \underline{b} \in N$ and $b + p(a) - b \in Na$.

Hence Na is normal in $(A, +)$.

(iv) Let ω be an n-ary operation on A and $b_1, \ldots, b_n \in A$, $p_1(a), \ldots, p_n(a) \in Na$. Consider

$$b := \omega(b_1 + p_1(a), \ldots, b_n + p_n(a)) - \omega(b_1, \ldots, b_n).$$

Let q be the polynomial $\omega(\underline{b}_1+p_1,\ldots,\underline{b}_n+p_n)-\omega(\underline{b}_1,\ldots,\underline{b}_n)$. Then $q(a)=b$ and $q(0)=0$, therefore $b=q(a)\in Na$.

Theorem 3.2. *Let N be a near-ring between $P_0(A)$ and $C(A)$. Then every minimal ideal S of A is an N_0-group of type 2.*

Proof. (i) Since $S\trianglelefteq A$, $S\trianglelefteq_N A$ by 2.4(c), so $S\leqslant_{N_0}A$ which shows that S is an N_0-group.

(ii) Because of $\mathrm{id}_A\in N_0$ we get $N_0S=S\neq\{0\}$.

(iii) By 3.1, the ideal $\langle s\rangle$ generated by $s\in S$ is given by $\{p(s)\mid p\in P_0(A)\}$.

(iv) Let $T\leqslant_{N_0}S$. Then $\forall n\in N_0\forall t\in T:n(t)\in T$. By (iii), $\forall t\in T:\langle t\rangle\subseteq T$, so $T\leqslant A$ and $T\subseteq S$.

(v) Since S is minimal, we see from (iv) that S is N_0-simple.

(vi) Let s_0 be in $S-\{0\}$. Then $N_0s_0=\langle s_0\rangle=S$. So ${}_{N_0}S$ is strictly monogenic and of type 2.

From (iv) of the preceding proof we see that every N_0-subgroup of A is an ideal of A, hence an ideal of ${}_NA$. So we get

Corollary 3.3. *Every zero-symmetric near-ring between $P_0(A)$ and $C(A)$ is tame (on A).*

So in this case, "N-subgroup", "ideal of ${}_NA$" and "ideal of A" all mean the same.

Theorem 3.4.

(a) Let A in 3.2 be a simple group. Then $P(A)$ is dense in $M_{\mathrm{aff}}(\Gamma)$ (if A is abelian) or otherwise in $M(A)$.

(b) If A is a simple ring with identity then $P(A)$ is dense in $M(A)$.

In the second case of (a) and in (b), A is locally polynomially complete (5).

Proof. (a) follows from the density theorem 4.52 of (10) and the fact that $P_0(A)$ is a ring iff A is an L-group; but a simple L-group is abelian. $P(A)$ is 2-primitive on A.

We only have to show that $G:=\mathrm{Aut}_{P_0(A)}(A)=\{\mathrm{id}\}$ (10, 4.54) if A is non-abelian:

Let $h\in G$. Suppose that $h\neq\mathrm{id}$. Then $\exists a\in A:-a+h(a)\notin\{0\}=Z(A,+)$ (the centre of A); so $\exists b\in A:-a+h(a)+b\neq b+(-a+h(a))$.

Since h is $1-1$, $\exists c\in A:b=h(c)$.

Hence $(-a+h(a))+h(c)-(-a+h(a))\neq h(c)$, so $h(a)+h(c)-h(a)\neq a+h(c)-a$.

Take $p_0:A\to A:x\to a+x-a$.

Then $p_0\in P_0(A)$.

$$h(a)+h(c)-h(a)=h(p_0(c)) \text{ and}$$
$$a+h(c)-a=p_0(h(c)),$$

whence $h\notin\mathrm{Aut}_{P_0(A)}(A)$, a contradiction.

In this second case, A is locally polynomially complete—a result due to H. Kaiser (5).

(b) Let A be a field.

$P_0(A)$ is a ring iff A is Boolean (by **12**).

If A is Boolean then $A \cong Z_2$ and $M_{\mathrm{aff}}(Z_2) = M(Z_2)$.

Anyhow, we compute (cf. (**11**) $\mathrm{Aut}_{P_0(A)}(A, +)$; in order to do this, we take some h from it. Let $a \in A \backslash \{0\}$.

Since $h(x^2 \circ a) = x^2 \circ h(a)$, $h(a^2) = h(a)^2$.

Also, $h(ax \circ a) = ax \circ h(a)$, so $h(a^2) = ah(a)$.

Hence $(a - h(a))(h(a)) = 0$. Now take $a := h^{-1}(1)$. Then $(a-1)\cdot 1 = 0$, whence $1 = a = h^{-1}(1)$. Since $\mathrm{Aut}_{P_0(A)}(A)$ is fixed-point-free by (**1**) or (**10**), $h^{-1} = h = \mathrm{id}$.

By the density theorem, $P(A)$ is dense in $M(A)$.

So in any case, $P(A)$ is dense in $M(A)$ and A is locally polynomially complete.

Theorem 3.5. *Let A either be a group or a ring with identity and the direct sum of simple (or minimal) A_i's of the same kind.*

Then $P(A)$ is a subdirect product of 2-primitive near-rings $N_i = C(A_i)$ $(i \in I)$ which are dense in

(α) *$M_{\mathrm{aff}}(A_i)$ (where D_i is a skew-field which makes A_i into a vector space) iff A_i is an abelian group*

or in

(β) *$M(A_i)$ iff A_i is a non-abelian group.*

In the ring case, we always arrive at (β).

Proof. (i) Since A is completely reducible, the A_i's are minimal iff they are simple.

By 3.2, the A_i's are $P_0(A)$-groups of type 2.

(ii) The annihilators $(0:A_i)$ are primitive ideals of $P_0(A)$ and $N_i := P_0(A)/(0:A_i)$ is 2-primitive on $(A_i, +)$ for all $i \in I$ with $(p + (0:A_i))(a_i) = p(a_i) + (0:A_i)$ for all $p \in P_0(A)$ and $a_i \in A_i$ (see (**10**), 4.3b, 4.2b, and 3.14a).

(iii) As a homomorphic image of $P_0(A)$, N_i is a zerosymmetric near-ring with identity.

(iv) We consider the decomposition of $f \in P_0(A)$ into a family $(f_i)_{i \in I}$ according to 2.6. Then the map $\pi_i : P_0(A) \to P_0(A_i) : f \to f_i$ is a homomorphism with kernel $\{g \in P_0(A) \mid \forall j \neq i : f_j = \emptyset\} = (0:A_i)$.

So $N_i = P_0(A)/(0:A_i) \cong \mathrm{Im}\ \pi_i \subseteq P_0(A_i)$ and we can replace the N_i's (if necessary) by their isomorphic copies $\mathrm{Im}\ \pi_i$ to get them into $P_0(A_i)$.

(v) In the case of rings with identity we know from 2.9 that $\mathrm{Im}\ \pi_i = P_0(A_i)$. So N_i is dense in $M_0(A_i)$ by 3.4(b). For groups, we seemingly have no guarantee for equality. So we still have to work out the group case.

(vi) We have to ask: when is N_i a ring?

If A_i is abelian, $P_0(A_i)$ is a ring, and so is $\mathrm{Im}\ \pi_i$, hence also N_i.

Conversely, if N_i is a ring then $\forall f \in P_0(A) : f/A_i \in \mathrm{Hom}(A_i, A_i)$. In particular, we take $f : a \to a + a = 2a$. $f \in P_0(A)$, and for all $c, d \in A_i$ we get $c + d + c + d = f(c+d) = f(c) + f(d) = c + c + d + d$, whence $d + c = c + d$ and A_i is shown to be abelian.

Hence N_i is a ring iff A_i is abelian.

(vii) If A_i is non abelian then one sees as in the proof of 3.4(a) that $\mathrm{Aut}_{N_i}(A_i)$ is $= \{\mathrm{id}\}$ (the p_0 in this proof appears in $\mathrm{Im}\ \pi_i$!).

(viii) We can now apply the density theorem 4.52 in (**10**) (due to G. Betsch (**1**)) to get N_i dense in $\mathrm{Hom}_{D_i}(A_i, A_i)$ iff A_i is abelian or dense in $M_0(A_i)$ in the other case.

(ix) Since $\bigcap_{i\in I}(0:A_i) = \{0\}$ by 2.6, $P_0(A)$ is isomorphic to the subdirect product of the near-rings N_i.

(x) Finally, we can add the constant polynomials in $P(A)$, which can be decomposed into constant polynomials in $P(A_i)$ by 2.6. In this way we get $\mathrm{Hom}_{D_i}(A_i, A_i) + M_c(A_i) = M_{\mathrm{aff}}(A_i)$ and $M_0(A_i) + M_c(A_i) = M(A_i)$, respectively.

Corollary 3.6. *Let A be as in* 3.5. *If A is finite or, more generally, if $P_0(A)$ has the descending chain condition for left ideals, then $P(A)$ is isomorphic to a finite direct sum of $M_{\mathrm{aff}}(A_i)$'s (where A_i is a finite dimensional vector space) or $M_0(A_i)$'s (according to the cases in* 3.5).

Proof. If $P_0(A)$ fulfills the descending chain condition for left ideals then the same applies to all N_i's of the last proof. Hence each N_i is isomorphic to $\mathrm{Hom}_{D_i}(A_i, A_i)$ or to $M_0(A_i)$, both being simple near-rings. Hence all $(0:A_i)$ turn out to be maximal, and the index set can be chosen to be finite by 2.52(b) of (**10**).

Finally, we again add the constant maps as in the preceding proof.

Corollary 3.7. *Let A be as in* 3.5. *Then $P_0(A)$ is* 2-*semisimple.*

Problems 3.8. Which classes of algebras have the property that direct sums of (locally) polynomially complete algebras are locally affine complete?

Can one conclude some kinds of completeness out of certain subdirect representations?

When is S in 3.2 faithful (then N_0 would be 2-primitive on S with identity, so the density theorem is at hand)?

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Addendum. Private communications from Dr. Kaiser (Vienna, Austria) and Dr. Kaarli (Tartu, USSR) and the paper "On on the concept of length in the sense of Lausch-Nöbauer" by G. Eigenthaler (J. Austr. Math. Soc. **24** (1977), 162–169) have revealed the answer to Conjecture 2.10:

A is nice $\Leftrightarrow A$ has no skew congruences $\Leftarrow A$ is kind. $S_3 \oplus Z_3$ is nice, but not kind. For abelian groups, "nice" and "kind" are equivalent. See also the forthcoming paper "On near-rings generated by the endomorphisms of some groups" by K. Kaarli. This paper implies that $L_{2,5}P(A) = C(A) \not\Rightarrow A$ is locally affine complete (2.19).

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