HJELMSLEV PLANES DERIVED FROM MODULAR LATTICES

BENNO ARTMANN

In several papers, W. Klingenberg has elaborated the connections between Hjelmslev planes and a class of rings, called H-rings (**4**; **5**; **6**), which are rings of coordinates for the corresponding Hjelmslev planes. Certain homomorphic images of valuation rings are examples of H-rings. In these examples, the lattice of (right) ideals of the ring, say R, is a chain, and the coordinatization of the corresponding Hjelmslev plane yields a natural embedding of the plane in the lattice $L(R^3)$ of (right) submodules of the module R^3. Now, $L(R^3)$ is a modular lattice with a homogeneous basis of order 3 given by the submodules $a_1 = (1, 0, 0)R$, $a_2 = (0, 1, 0)R$, $a_3 = (0, 0, 1)R$, and the sublattices $L(N, a_i)$ of elements less than or equal to a_i are chains. Forgetting about the ring, we find ourselves in the situation of a problem suggested by Skornyakov (**7**, Problem 23, p. 166), namely, to study modular lattices with a homogeneous basis of chains. Baer (**2**) and Inaba (**3**) investigated lattices of this kind with Desarguesian properties and assuming that the chains $L(N, a_i)$ were finite. Representations of the lattices by means of certain rings can be found in both articles.

In this paper, we show that every modular lattice with a homogeneous basis of order 3, consisting of chains, which satisfies two "technical" assumptions (FC) and (S), listed below, leads to a certain Hjelmslev plane. We leave aside all questions about coordinates, since they require assumptions about Desarguesian properties of the lattice (or plane) and a detailed study of the ideal structure of the ring of coordinates. (For the meaning of the properties (FC) and (S) in the ring of coordinates, see Remark 3.) In a subsequent paper,† it will be shown that every uniform Hjelmslev plane can be obtained in the way described here by constructing a lattice from the plane. This gives, also, examples of non-Desarguesian lattices.

Notation. We deal exclusively with modular lattices with a least and a greatest element. The least element of the lattice L is denoted by N, the greatest by U. In order to save brackets, we write $a \cup b \cap c$ instead of $a \cup (b \cap c)$, that is, $\cap$ shall bind closer than $\cup$. The modular law then reads

$$a \leqq c \Rightarrow a \cup b \cap c = (a \cup b) \cap c.$$

$L(a, b)$ is the sublattice of all elements x with $a \leqq x \leqq b$ of L. We have that

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†*Added in proof.* The paper referred to is *Uniforme Hjelmslev-Ebenen und modulare Verbände* (to appear in Math. Z.).

$L(a \cap b, b) \cong L(a, a \cup b)$. The expression $a \cup^{\cdot} b = c$ stands for $a \cup b = c$ and $a \cap b = N$. If $a \cup^{\cdot} c = b \cup^{\cdot} c$, we say that a and b are perspective and c is the centre of perspectivity. The mapping π: $x \rightarrow (c \cup x) \cap b$ for $x \leqq a$ is called a projection with centre c of $L(N, a)$ onto $L(N, b)$. In modular lattices, projections are isomorphisms.

Definition 1. A list $F_3 = (a_1, a_2, a_3, c_{12}, c_{13}, c_{23})$ of elements of a modular lattice L is called a frame of order 3 of L, if the following conditions are satisfied:

(i) $(a_1 \cup^{\cdot} a_2) \cup^{\cdot} a_3 = U$,

(ii) $a_i \cup^{\cdot} c_{ij} = a_i \cup^{\cdot} a_j = a_j \cup^{\cdot} c_{ij}$ for $i, j \in \{1, 2, 3\}, i \neq j$,

(iii) $c_{12} \cup^{\cdot} c_{13} = c_{12} \cup^{\cdot} c_{23} = c_{13} \cup^{\cdot} c_{23}$.

The elements a_1, a_2, and a_3 of the frame are said to form a homogeneous basis of order 3 of L.

In order to obtain simpler formulas, we shall use the abbreviation $A_i = a_j \cup a_k$ $(\{i, j, k\} = \{1, 2, 3\})$. We say that the lattice L with frame F_3 is frame-complemented, if for every $b \in \{a_1, a_2, a_3, A_1, A_2, A_3\}$ it is true that

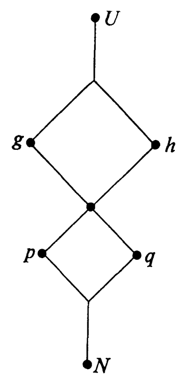
(FC) (a) For every $x \in L$ with $x \cap b = N$ there exists a $y \geqq x$ such that $y \cup^{\cdot} b = U$;

(b) For every $z \in L$ with $z \cup b = U$, there exists a $y \leqq z$ such that $y \cup^{\cdot} b = U$.

Furthermore, we wish that the lattices under consideration have a certain symmetry, as expressed in

(S) (a) Let $p, q \in L$ be complements of A_k and g a complement of a_i with $g \geqq p, q$. Then there exists a complement h of a_i such that $g \cap h = p \cup q$ (for all $i \neq k$, $i, k \in \{1, 2, 3\}$);

(b) Let g and h be complements of a_k and p a complement of A_i with $p \leqq g, h$. Then there exists a complement q of A_i such that $p \cup q = g \cap h$ (for all $i \neq k$, $i, k \in \{1, 2, 3\}$).



We collect all these conditions in the concept of an H-lattice: A modular lattice L with frame of order 3 is called an H-lattice (for Hjelmslev-lattice), if the sublattices $L(N, a_i)$ are chains ($i \in \{1, 2, 3\}$) and L has properties (FC) and (S).

Remark 1. The concept of an H-lattice is self-dual. For, if

$$C_{ij} = c_{ij} \cup a_k \ (\{i, j, k\} = \{1, 2, 3\}),$$

then the list $(A_1, A_2, A_3, C_{12}, C_{13}, C_{23})$ is a frame of order 3 of the lattice $\bar{L}$ dual to L, as simple calculations show. Parts (a) and (b) of (FC) and (S) are duals of each other. Finally, $L(A_i, U)$ is a chain since

$$L(N, a_i) = L(A_i \cap a_i, a_i) \cong L(A_i, A_i \cup a_i) = L(A_i, U).$$

Definition 2. From an H-lattice L we derive an incidence-system

$$H = (P, G, I, \backsim_P, \backsim_G)$$

defining:

$P = \{p \mid \text{there exists } i \in \{1, 2, 3\} \text{ such that } p \,\dot{\cup}\, A_i = U\}$,
$G = \{g \mid \text{there exists } i \in \{1, 2, 3\} \text{ such that } g \,\dot{\cup}\, a_i = U\}$,
$I = \{(p, g) \mid p \in P, g \in G, \text{and } p \leqq g\} \subset P \times G$,
$\backsim_P = \{(p, q) \mid p, q \in P \quad \text{and} \quad p \cap q > N\} \subset P \times P$,
$\backsim_G = \{(g, h) \mid g, h \in G \quad \text{and} \quad g \cup h < U\} \subset G \times G$.

P is called the set of points of H, G the set of lines, I the incidence relation, and $\backsim_P$ and $\backsim_G$ the neighbour relations for points and lines, respectively. As usual, we write $p\ I\ g$ instead of $(p, g) \in I$ and sometimes $p \backsim q$, $g \backsim h$, suppressing the indices P and G, if there is no danger of confusion.

Remark 2. We note that the incidence system $\bar{H}$ defined from the lattice $\bar{L}$ dual to L is nothing other than the "dual" of the system H in the usual geometric meaning, since the definitions of $\backsim_P$ and $\backsim_G$ are duals of each other. Therefore, every statement about H implies, automatically, its dual statement.

THEOREM 1. *The incidence system H derived from an* H-*lattice as in Definition* 2 *is a projective Hjelmslev plane.*

By a projective Hjelmslev plane, we mean a *projektive Inzidenzebene mit Nachbarelementen* as defined by Klingenberg (**4**, pp. 387–88). For the proof we have to show that our relations $\backsim$ are the same as the neighbour relations defined in (**4**) (i.e., two points are neighbours if there are at least two different lines incident with both of them, and dually for lines) and then to check the axioms A1–A6 of (**4**, pp. 387–88), listed at the end of this proof. This is done by a series of lemmas, in which we always assume that $\{i, j, k\} = \{1, 2, 3\}$, and where part (b) will be the dual of (a), if stated. We shall not give the dual explicitly, if it is of no specific interest.

LEMMA 1. *For all $p \in P$, $L(N, p)$ is a chain.*

Proof. There exists an i such that $p \,\dot{\cup}\, A_i = U = a_i \,\dot{\cup}\, A_i$; hence $L(N, p) \cong L(N, a_i)$.

LEMMA 2. *$\backsim_P$ and $\backsim_G$ are equivalence relations.*

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We give the proof for $\backsim_P$: $p \backsim p$ and $p \backsim q \Rightarrow q \backsim p$ are obvious. Let $p, q, r \in P$, $p \backsim q$, and $q \backsim r$; e.g., $p \cap q = a > N$ and $q \cap r = b > N$. a and b are comparable, since both are less than or equal to q and $L(N, q)$ is a chain. We may assume that $b \leqq a$. Then $p \cap r \geqq p \cap q \cap r = a \cap b = b > N$; hence $p \backsim r$.

LEMMA 3. (a) *$p, q \in P$ and $q \leqq p$ implies $p = q$.*
(b) *$g, h \in G$ and $g \leqq h$ implies $g = h$.*

Proof. (a) Let $p \dot{\cup} A_i = U = q \dot{\cup} A_j$. From $q \leqq p$ we have that $p \cup A_j = U$. If $p \cap A_j = x$, then x and q are comparable, as $L(N, p)$ is a chain. Now, $q \leqq x$ is impossible, since it would imply that

$$q \cap A_j = q \cap x \cap A_j = q \cap x = q.$$

Hence, we have that $x \leqq q$ and $p \cap A_j = x \cap A_j \leqq q \cap A_j = N$, and therefore $p \dot{\cup} A_j = U$. This yields $p = q$ by the incomparability of complements in modular lattices. (b) is the dual of (a).

LEMMA 4. *Let $p, q \in P$. There exists an i such that $(p \cup q) \cap a_i = N$.*

Proof. We assume that $p \dot{\cup} A_k = U$. If $(p \cup q) \cap a_j = N$, then there is nothing to prove. Thus, we may assume that $(p \cup q) \cap a_j > N$. If we put $z = (p \cup q) \cap A_k$, then we have that $z \cup p = (p \cup q) \cap (A_k \cup p) = p \cup q$. Now, from the isomorphisms

$$L(p \cap q, q) \cong L(p, p \cup q) = L(p, p \cup z) \cong L(p \cap z, z) = L(N, z),$$

we know that $L(N, z)$ is a chain. Furthermore,

$$(p \cup q) \cap a_j = (p \cup q) \cap A_k \cap a_j = z \cap a_j > N$$

(by assumption), and as $L(N, z)$ is a chain, this implies that $(p \cup q) \cap a_i = z \cap a_i = N$, since otherwise $a_j \cap a_i > N$.

Up to now we have not made use of the properties (FC) and (S) of L; however, for the following lemmas we need property (FC).

LEMMA 5. (i) *If $p \dot{\cup} A_k = U$ and $p \cap A_j = N$, then $p \dot{\cup} A_j = U$;*
(ii) *If $p \dot{\cup} A_k = U$ and $p \cup A_j = U$, then $p \dot{\cup} A_j = U$.*

Proof. (i) By (FC) (a) there exists a complement q of A_j with $p \leqq q$. But then $p, q \in P$, and, by Lemma 3, $p = q$.

(ii) By (FC) (b), there exists a complement r of A_j with $r \leqq p$. Again we have that $r, p \in P$, and therefore $r = p$ by Lemma 3.

LEMMA 6. *Let $p \dot{\cup} A_k = q \dot{\cup} A_k = U$, $g = p \cup a_j$, $h = q \cup a_j$, $q \dot{\cup} g = U = p \dot{\cup} h$. Then $z = (p \cup q) \cap A_k$ is a common complement of g and h, and the projection from g onto h with centre z maps points onto points.*

Proof. First we note that $g, h \in G$. Now

$$z \cup g = z \cup p \cup a_j = (p \cup q) \cap (A_k \cup p) \cup a_j$$
$$= p \cup q \cup a_j = p \cup h = U,$$

and similarly $z \cup h = U$.

$$z \cap g = (p \cup q) \cap A_k \cap g = (p \cup q \cap g) \cap A_k = N,$$

and also $z \cap h = N$. Let π be the projection of $L(N, g)$ onto $L(N, h)$ with centre z. We have that $p^\pi = (p \cup z) \cap h = (p \cup q) \cap h = q$. We look at $r \in P$, $r \leqq g$, and claim that $r^\pi \in P$. In order to establish this, we distinguish the cases (i) $r \cap A_k = N$ and (ii) $r \cap A_k > N$. (i) $r \cap A_k = N$ implies, by Lemma 5, that $r \cup A_k = U$, and

$$\begin{aligned} r^\pi \cap A_k = (r \cup z) \cap h \cap A_k &= (r \cup z) \cap a_j \\ &= (r \cup z) \cap g \cap a_j = (r \cup z \cap g) \cap a_j = r \cap a_j = N, \end{aligned}$$

$$\begin{aligned} r^\pi \cup A_k = (r \cup z) \cap h \cup A_k \cup z \\ = (r \cup z) \cap (h \cup z) \cup A_k = r \cup z \cup A_k = U. \end{aligned}$$

(ii) $r \cap A_k = r \cap g \cap A_k = r \cap a_j = x > N$. From $r \cap a_j > N$ and $a_j \cap A_j = N$, we see that $r \cap A_j = N$, as in the proof of Lemma 3, and hence $r \dot{\cup} A_j = U$, since $r \in P$.

Now, $r^\pi \cap a_j = r \cap a_j = x > N$, as all elements less than or equal to a_j are fixed by π. This implies that $r^\pi \cap A_j = N$, as $L(N, r^\pi)$ is a chain. Therefore, by property (FC), there exists an $s \geqq r^\pi$ such that $s \dot{\cup} A_j = U$. Hence, $L(N, s)$ is a chain, and $s^{\pi^{-1}} \geqq r$ implies that $s^{\pi^{-1}} \cap A_j = N$, since otherwise $r \cap A_j > N$. Therefore, there exists a $t \geqq s^{\pi^{-1}}$ with $t \dot{\cup} A_j = U$, and by Lemma 3 we have that $t = r = s^{\pi^{-1}}$; therefore, $s = r^\pi \in P$.

LEMMA 7. *If* $p \dot{\cup} A_k = U$ *and* $p \cap A_i > N$, *then* $p \dot{\cup} (c_{ik} \cup a_j) = U$.

Proof. Let $p \cap A_i = x > N$. As $L(N, p)$ is a chain, $p \cap (c_{ik} \cup a_j) = y$ is comparable with x, and $y \cap x = p \cap (c_{ik} \cup a_j) \cap A_i = p \cap a_j = N$, therefore $y = N$. Also, $L(p \cup a_j, U)$ is a chain, since

$$(p \cup a_j) \cap a_i = (p \cup a_j) \cap A_k \cap a_i = N,$$

and hence $L(N, a_i) = L((p \cup a_j) \cap a_i, a_i) \cong L(p \cup a_j, p \cup a_i \cup a_j) = L(p \cup a_j, U)$. By Lemma 5 (ii), we know that $p \cup A_i < U$ since $p \cup A_i = U$ would imply that $p \cap A_i = N$. Therefore, $p \cup a_j \cup c_{ik} = z$ is comparable with $p \cup a_j \cup a_k = w < U$, and $z \cup w = p \cup a_j \cup a_k \cup c_{ik} = U$; hence $z = U$.

LEMMA 8. (a) *Let* $p, r \in P$. *Then* $p \cap r = N$ *implies* $p \cup r \in G$.
(b) *Let* $g, h \in G$. *Then* $g \cup h = U$ *implies* $g \cap h \in P$.

Proof. (a) By Lemma 4, we may assume that $(p \cup r) \cap a_i = N$, and it remains to show that $p \cup r \cup a_i = U$.

(i) Let $p \cap A_j = x > N$ and $p \cap A_k = y$. Then

$$N = p \cap a_i = p \cap A_j \cap p \cap A_k = x \cap y,$$

and therefore $y = N$, as $L(N, p)$ is a chain. Thus, we have that $p \cap A_j = N$ or $p \cap A_k = N$. Let us assume that $p \cap A_k = N$.

(ii) $p \cup (r \cup a_i) \cap (p \cup a_j) \cup a_i = p \cup (r \cup a_i) \cap (p \cup a_j \cup a_i) =$

$p \cup r \cup a_i$, so we may also assume that $r \leqq p \cup a_j$ without loss of generality.

(iii) By Lemma 7, we have that either $p \cap A_i = N$ or that $p \cap (a_j \cup c_{ik}) = N$. When $p \cap A_i = N$, we apply Lemma 6 with $q = a_k$ and have that $g = p \cup a_j$, $h = a_k \cup a_j = A_i$, and the isomorphism π as defined in Lemma 6 (hence $p^\pi = a_k$). Now, $(r \cap p)^\pi = r^\pi \cap a_k = N$, $r^\pi \leqq A_i$ implies $r^\pi \cap A_j = N$, and, by Lemmas 5 and 6, we know that $r^\pi \cup^\cdot A_j = U$ which yields $r^\pi \cup a_k = A_i = a_k \cup a_j$. From this equation we derive $r \cup p = p \cup a_j \in G$ by applying π^{-1}, so that $r \cup p \in G$.

Similarly, if $p \cap A_i > N$, then by Lemma 7 we have that $p \cup^\cdot (c_{ik} \cup a_j) = U$ and by $c_{ik} \cap (p \cup a_j) = c_{ik} \cap (c_{ik} \cup a_j) \cap (p \cup a_j) = N$, we also have that $c_{ik} \cup^\cdot (p \cup a_j) = U$. Thus, we may apply Lemma 6 with $q = c_{ik}$, $g = p \cup a_j$, $h = c_{ik} \cup a_j$, and $p^\pi = c_{ik}$. Here, we find that $r^\pi \cap c_{ik} = N$, $r^\pi \leqq c_{ik} \cup a_j$; hence, $r^\pi \cap A_j = N$ and, as above, $r^\pi \cup^\cdot A_j = U$. This implies that $r^\pi \cup c_{ik} = c_{ik} \cup a_j$. By applying π^{-1} as before we obtain $r \cup p = p \cup a_j \in G$.

LEMMA 9. (a) *Let $p, r \in P$. Then $p \cup r \in G$ implies $p \cap r = N$.* (b) *Let $g, h \in G$. Then $g \cap h \in P$ implies $g \cup h = U$.*

Proof. Let $(p \cup r) \cup^\cdot a_i = U$. Then we have that $p \cap a_i = r \cap a_i = N$ and by Lemma 8 (a), $p \cup a_i, r \cup a_i \in G$. On the other hand,

$$p \cup a_i \cup r \cup a_i = U$$

by assumption, therefore by Lemma 8 (b), $(p \cup a_i) \cap (r \cup a_i) \in P$. However, $(p \cup a_i) \cap (r \cup a_i) \geqq a_i$; hence, we have that

$$\begin{aligned} a_i &= (p \cup a_i) \cap (r \cup a_i) \quad \text{(by Lemma 3)} \\ &= (p \cup a_i) \cap r \cup a_i. \end{aligned}$$

This implies that $(p \cup a_i) \cap r \leqq a_i$, $p \cap r \leqq (p \cup a_i) \cap r = (p \cup a_i) \cap r \cap a_i = N$.

PROPOSITION 1. *Without assuming the axiom* (S) *in L, we have the following:*

(a) *For any $p, q \in P$, there exists a $g \in G$ with $p, q \leqq g$. g is unique if $p \cap q = N$;*

(b) *For any $g, h \in G$, there exists a $p \in P$ with $p \leqq g, h$. p is unique if $g \cup h = U$.*

Proof. (a) By Lemma 4, there exists an i such that $(p \cup q) \cap a_i = N$. Hence, by property (FC), there exists a $g \geqq p \cup q$ with $g \cup^\cdot a_i = U$. If $p \cap q = N$, then by Lemma 8 we have that $p \cup q \in G$, and therefore $g = p \cup q$ by Lemma 3 (b). (b) is the dual of (a).

Remark 3. For the last lemma, we are now going to use the symmetry axiom (S) of an H-lattice. Actually, a somewhat weaker form of (S) would be sufficient for our purposes; however, we prefer (S) because of its meaning for the ring of coordinates which sometimes can be constructed. Essentially, it states that there exists a dual isomorphism of the chain of principal right ideals onto the chain of principal left ideals given by the construction of the annihilator ideal. From property (FC), we have that the principal right (and left) ideals form a chain (see the factorization property in **6**, p. 198).

LEMMA 10. *Let* $p, q \in P$ *and* $g \in G, p \cup q \leqq g$. *If* g *is unique, then* $p \cap q = N$.

Proof. Let us assume that $p \cap q > N$. If $p \cup^{\cdot} A_k = U$, then also $q \cup^{\cdot} A_k = U$ since $p \cap q > N$ implies $q \cap A_k = N$ in this case. As in the proof of Lemma 4, we have that either $(p \cup q) \cap a_i = N$ or that $(p \cup q) \cap a_j = N$. Hence, we may assume that $(p \cup q) \cap a_i = N$ and $g \cup^{\cdot} a_i = U$, without loss of generality. Now by (S), there exists an $h \in G$ such that $h \cap g = p \cup q$. From $p \cap q > N$, we obtain $p \cup q < g$ since $p \cup q = g$ implies $p \cap q = N$ by Lemma 9; therefore, we have that $h \cap g < g$, i.e. $h \neq g$, hence two different lines incident with p, q.

We are now ready for the following proof.

Proof of Theorem 1. (i) By Proposition 1 and Lemma 10 we have that $p \backsim q$ if and only if there exist $g, h \in G$ such that $g \neq h$ and $p, q \leqq g, h$, and by duality, $g \backsim h$ if and only if there exist $p, q \in P$ such that $p \neq q$ and $p, q \leqq g, h$. Therefore, the relations $\backsim_P$ and $\backsim_G$ are the same as the neighbour relations defined in (**4**, p. 387).

(ii) We consider the axioms of (**4**, pp. 387–88) separately.

(A1) Let $p, q \in P$. There exists $g \in G$ with $p, q\, I\, g$. This was proved in Proposition 1.

(A2) Let $g, h \in G$. There exists $p \in P$ with $p\, I\, g, h$. This is the dual of (A1).

(A3) There exist $p_1, p_2, p_3, p_4 \in P$ such that $p_i \not\backsim p_k$ and $p_i \cup p_k \not\backsim p_i \cup p_j$ for $i \neq j \neq k \neq i$, $i, j, k \in \{1, 2, 3, 4\}$. For the proof we may choose $p_1 = a_1$, $p_2 = a_2$, $p_3 = a_3$, $p_4 = (a_3 \cup c_{12}) \cap (a_2 \cup c_{13})$, and the desired properties hold because of the properties of the normalized frame $(a_1, a_2, a_3, c_{12}, c_{13}, c_{23})$.

(A4) Let $p \in P, f, g, h \in G$ and $p\, I f, g, h$. If $f \backsim g$ and $g \not\backsim h$, then $f \not\backsim h$. This is true since $\backsim$ is an equivalence relation (Lemma 2).

(A5) Let $f, g, h \in G$. If $f \backsim g$ and $g \not\backsim h$, then $f \cap h \backsim g \cap h$. For the proof we first note that we have $p = f \cap h \in P, q = g \cap h \in P$, and $f \cup g < U$ by hypothesis. We have to show that $p \cap q > N$.

$$p \cup q = f \cap h \cup g \cap h = (f \cap h \cup g) \cap h \leqq (f \cup g) \cap h < h$$

since $f \cup g < U$, $L(h, U)$ is a chain, and $f \cup g \geqq h$ would imply that $g \cup h \leqq f \cup g \cup h < U$, in contradiction to $g \not\backsim h$. This shows that $p \cap q > N$ by Lemma 8.

(A6) Let $p, q, r \in P$. If $p \backsim q$ and $q \not\backsim r$, then $p \cup r \backsim q \cup r$. This is the dual of (A5).

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Universitaet,
Giessen, Germany