BULL. AUSTRAL. MATH. SOC. 47A25
VOL. 24 (1981), 349-355.

NECESSARY AND SUFFICIENT CONDITIONS FOR SPECTRAL SETS

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We shall show necessary and sufficient conditions for which a closed set X in the complex plane is a spectral set of an operator T on a complex Hilbert space.

1. Introduction

In this paper an operator T means a bounded linear operator on a complex Hilbert space with the spectrum $\sigma(T)$. The notion of a spectral set for an operator T was introduced by von Neumann [2] as follows: a closed set X in the complex plane is a spectral set of T if $X \supset \sigma(T)$ and if

$$\|f(T)\| \leq \sup\{|f(z)| : z \in X\}$$

for any rational function $f(z)$ with poles off X , *cf.* [1] and [3] for details.

For sake of the subsequent discussion we shall define two classes of rational functions.

DEFINITION 1. For two closed sets X and Y in the complex plane, two classes R_X, R_X^Y of rational functions are defined as follows:

$R_X = \{f(z)$: rational function $f(z)$ with poles off $X\}$,

$R_X^Y = \{f(z)$: rational function $f(z)$ with zeros off Y and poles off $X\}$.

Received 14 July 1981.

https://doi.org/10.1017/S0004972700004883 Published online by Cambridge University Press

Let R_σ, R_σ^σ, R_X^σ and R_σ^X denote R_X with $X = \sigma(T)$, R_X^X with $X = \sigma(T)$, R_X^Y with $Y = \sigma(T)$ and R_Y^X with $Y = \sigma(T)$ respectively.

2. Statement of the result

We shall show necessary and sufficient conditions for which a closed set X in the complex plane is a spectral set of T as follows.

THEOREM 1. *The following twelve conditions on* T *and a closed set* X *are equivalent:*

$(\sup\text{-}R_\sigma)$ $X \supset \sigma(T)$ *and the following (*) holds for any* $f(z) \in R_\sigma$,

(*) $$\|f(T)\| \le \sup\{|f(z)| : z \in X\} ;$$

$(\sup\text{-}R_X)$ X *is a spectral set of* T , *that is,* $X \supset \sigma(T)$ *and (*) holds for any* $f(z) \in R_X$,

$(\sup\text{-}R_\sigma^\sigma)$ $X \supset \sigma(T)$ *and (*) holds for any* $f(z) \in R_\sigma^\sigma$,

$(\sup\text{-}R_X^\sigma)$ $X \supset \sigma(T)$ *and (*) holds for any* $f(z) \in R_X^\sigma$,

$(\sup\text{-}R_\sigma^X)$ $X \supset \sigma(T)$ *and (*) holds for any* $f(z) \in R_\sigma^X$,

$(\sup\text{-}R_X^X)$ $X \supset \sigma(T)$ *and (*) holds for any* $f(z) \in R_X^X$,

$(\inf\text{-}R_\sigma)$ $X \supset \sigma(T)$ *and the following (**) holds for any* $g(z) \in R_\sigma$ *for any unit vector* x ,

(**) $$\inf\{|g(z)| : z \in X\} \le \|g(T)x\| ;$$

$(\inf\text{-}R_X)$ $X \supset \sigma(T)$ *and (**) holds for any* $g(z) \in R_X$ *and for any unit vector* x ,

$(\inf\text{-}R_\sigma^\sigma)$ $X \supset \sigma(T)$ *and (**) holds for any* $g(z) \in R_\sigma^\sigma$ *and for any unit vector* x ,

$(\inf\text{-}R_\sigma^X)$ $X \supset \sigma(T)$ *and (**) holds for any* $g(z) \in R_\sigma^X$ *and for* *any unit vector* x ,

$\left(\inf\text{-}R_X^\sigma\right)$ $X \supset \sigma(T)$ *and* (**) *holds for any* $g(z) \in R_X^\sigma$ *and for any unit vector* x ,

$\left(\inf\text{-}R_X^X\right)$ $X \supset \sigma(T)$ *and* (**) *holds for any* $g(z) \in R_X^X$ *and for any unit vector* x .

3. Proof of the result

Proof of Theorem 1. (i) $\left(\sup\text{-}R_X^\sigma\right) \leftrightarrow \left(\inf\text{-}R_\sigma^X\right)$. Suppose $\left(\sup\text{-}R_X^\sigma\right)$ holds. For any $g(z) \in R_\sigma^X$, there exists $g(T)$ and $f(z) = 1/g(z) \in R_X^\sigma$ since zeros and poles of $f(z)$ are interchanged with poles and zeros of $g(z)$. Therefore there exists $f(T)$ since $X \supset \sigma(T)$ holds and $f(T) = \left(g(T)\right)^{-1}$. As $\left(\sup\text{-}R_X^\sigma\right)$ holds, for any unit vector x and for this $f(z) \in R_X^\sigma$,

$$\|f(T)x\| \le \sup\{|f(z)| : z \in X\}$$

that is, for any vector x ,

$$\frac{\|x\|}{\|f(T)x\|} \ge \frac{1}{\sup\{|f(z)| : z \in X\}}$$

equivalently,

$$\frac{\|g(T)y\|}{\|y\|} \ge \inf\{|g(z)| : z \in X\}$$

for any vector y ; that is,

$$\inf\{|g(z)| : z \in X\} \le \|g(T)x\|$$

for any unit vector x and for any $g(z) \in R_\sigma^X$, namely $\left(\inf\text{-}R_\sigma^X\right)$ holds.

Conversely suppose $\left(\inf\text{-}R_\sigma^X\right)$ holds. For any $f(z) \in R_X^\sigma$, there exists $f(T)$ since $X \supset \sigma(T)$ holds and $g(z) = 1/f(z) \in R_\sigma^X$ since zeros and poles of $g(z)$ are interchanged with poles and zeros of $f(z)$. Hence there exists $g(T)$ such that $g(T) = \left(f(T)\right)^{-1}$. As $\left(\inf\text{-}R_\sigma^X\right)$ holds, for

https://doi.org/10.1017/S0004972700004883 Published online by Cambridge University Press

any unit vector x and for this $g(z) \in R_\sigma^X$,

$$\inf\{|g(z)| : z \in X\} \leq \|g(T)x\|$$

that is,

$$\frac{1}{\inf\{|g(z)| : z \in X\}} \geq \frac{\|x\|}{\|g(T)x\|}$$

for any vector x, that is,

$$\frac{\|f(T)y\|}{\|y\|} \leq \sup\{|f(z)| : z \in X\}$$

for any vector y, equivalently

$$\|f(T)x\| \leq \sup\{|f(z)| : z \in X\}$$

for any unit vector x, namely $\left[\text{sup-}R_X^\sigma\right]$ holds.

Similarly $\left[\text{sup-}R_\sigma^X\right] \leftrightarrow \left[\text{inf-}R_X^\sigma\right]$, $\left[\text{sup-}R_X^X\right] \leftrightarrow \left[\text{inf-}R_X^X\right]$ and $\left[\text{sup-}R_\sigma^\sigma\right] \leftrightarrow \left[\text{inf-}R_\sigma^\sigma\right]$ are obtained.

(ii) $\left(\text{sup-}R_\sigma\right) \leftrightarrow \left(\text{sup-}R_X\right)$. As $X \supset \sigma(T)$ we have $\left(\text{sup-}R_\sigma\right) \rightarrow \left(\text{sup-}R_X\right)$.

Conversely suppose $\left(\text{sup-}R_X\right)$ holds. If $\sup\{|f(z)| : z \in X\} < \infty$ for any $f(z) \in R_\sigma$, then the poles of $f(z)$ lie off X and $f(z) \in R_X$ and there exists $f(T)$ since $X \supset \sigma(T)$, namely, $\left(\text{sup-}R_\sigma\right)$ substantially coincides with $\left(\text{sup-}R_X\right)$ in this case. On the other hand if $\sup\{|f(z)| : z \in X\} = \infty$ for $f(z) \in R_\sigma$, then there exists $f(T)$ and $\left(\text{sup-}R_\sigma\right)$ obviously holds.

(iii) $\left(\text{inf-}R_\sigma\right) \leftrightarrow \left[\text{inf-}R_\sigma^\sigma\right] \leftrightarrow \left[\text{inf-}R_\sigma^X\right]$. $X \supset \sigma(T)$ easily implies $\left(\text{inf-}R_\sigma\right) \rightarrow \left[\text{inf-}R_\sigma^\sigma\right] \rightarrow \left[\text{inf-}R_\sigma^X\right]$ and we have only to show $\left[\text{inf-}R_\sigma^X\right] \rightarrow \left(\text{inf-}R_\sigma\right)$. Suppose $\left[\text{inf-}R_\sigma^X\right]$ holds. If $\inf\{|g(z)| : z \in X\} = 0$ for $g(z) \in R_\sigma$, then there exists $g(T)$ and

$(\text{inf-}R_\sigma)$ automatically holds. On the other hand $\inf\{|g(z)| : z \in X\} \neq 0$ for any $g(z) \in R_\sigma$, then zeros of $g(z)$ lie off X, so that $g(z) \in R_\sigma^X$ and there exists $g(T)$, whence $(\text{inf-}R_\sigma)$ substantially coincides with $\left(\text{inf-}R_\sigma^X\right)$ in this case. Whence we have $(\text{inf-}R_\sigma) \leftrightarrow \left(\text{inf-}R_\sigma^\sigma\right) \leftrightarrow \left(\text{inf-}R_\sigma^X\right)$. Similarly we have $(\text{inf-}R_X) \leftrightarrow \left(\text{inf-}R_X^\sigma\right) \leftrightarrow \left(\text{inf-}R_X^X\right)$.

(iv) $(\text{sup-}R_X) \rightarrow \left(\text{sup-}R_X^\sigma\right) \rightarrow \left(\text{sup-}R_X^X\right)$. These implications are easily obtained since $X \supset \sigma(T)$ holds.

(v) $(\text{inf-}R_\sigma) \rightarrow (\text{inf-}R_X)$. This implication is also easily obtained since $X \supset \sigma(T)$ holds.

(vi) $\left(\text{sup-}R_X^X\right) \leftrightarrow (\text{sup-}R_X)$. We have only to show $\left(\text{sup-}R_X^X\right) \rightarrow (\text{sup-}R_X)$ since the reverse relation is obtained in (iv). Suppose $\left(\text{sup-}R_X^X\right)$ holds. If $X \supset \sigma(T)$ and $\sup\{|f(z)| : z \in X\} \leq 1$ for any $f(z) \in R_X$, then we have only to show $\|f(T)\| \leq 1$. For all complex z, we define $\phi(z)$ as follows:

$$\phi(z) = \frac{\beta-\bar{\alpha}z}{z-\alpha\beta}$$

where $1 < |\alpha| < |\beta|$. This $\phi(z)$ maps D into D (where D denotes the unit disk of the complex plane) since

$$1 - |\phi(z)|^2 = \frac{\left(|z|^2-|\beta|^2\right)\left(1-|\alpha|^2\right)}{|z-\alpha\beta|^2} > 0 .$$

For all complex z, also we define $g(z)$ as follows:

$$g(z) = \phi\big(f(z)\big) = \frac{\beta-\bar{\alpha}f(z)}{f(z)-\alpha\beta} .$$

We have $|g(z)| < 1$ since $|f(z)| \leq 1$ and $\phi(z)$ maps D into D. As $|\beta/\bar{\alpha}| > 1$, $|\alpha\beta| > 1$ and $f(z) \in R_X$, so that zeros and poles of $g(z)$ lie off X, therefore we have $g(z) \in R_X^X$. Whence there exists $g(T)$ and the hypothesis of $\left(\text{sup-}R_X^X\right)$ implies

https://doi.org/10.1017/S0004972700004883 Published online by Cambridge University Press

$$\|g(T)\| \leq \sup\{|g(z)| : z \in X\} \leq 1$$

that is,

$$\|(\beta-\bar{\alpha}f(T))(f(T)-\alpha\beta)^{-1}x\| \leq \|x\|$$

for any vector x , namely

$$\|(\beta-\bar{\alpha}f(T))y\|^2 - \|(f(T)-\alpha\beta)y\|^2 = (|\alpha|^2-1)(\|f(T)y\|^2-|\beta|^2\|y\|^2) \leq 0$$

for any vector y . As $|\alpha| > 1$, we have

$$\|f(T)\| \leq |\beta| ,$$

as $|\beta|$ tends to 1 , we have

$$\|f(T)\| \leq 1$$

which is the desired relation.

Similarly we have $\left(\text{sup-}R_\sigma^X\right) \leftrightarrow \left(\text{sup-}R_\sigma\right)$.

Hence we have finished the proof of Theorem 1 by (i), (ii), (iii), (iv), (v) and (vi) obtained above.

REMARK 1. At a glance $\left(\text{sup-}R_\sigma\right)$ seems to be more general than $\left(\text{sup-}R_X\right)$ (this is just the definition of a spectral set) and $\left(\text{sup-}R_X^X\right)$ seems to be more restrictive than $\left(\text{sup-}R_X\right)$, but it turns out to be that these completely coincide with the original $\left(\text{sup-}R_X\right)$ by Theorem 1. It is somewhat surprising that $\left(\text{inf-}R_X^X\right)$ completely coincides with the original $\left(\text{sup-}R_X\right)$ by Theorem 1.

References

[1] Arnold Lebow, "On von Neumann's theory of spectral sets", *J. Math. Anal. Appl.* 7 (1963), 64-90.

[2] Johann von Neumann, "Eine Spektraltheorie für allegemeine Operatoren eines unitären Raumes", *Math. Nachr.* 4 (1950/51), 258-281.

[3] Frédéric Riesz et Béla Sz.-Nagy, *Leçons d'analyse fonctionelle* (Akadémiai Kiadó, Budapest, 1952).

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