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FURTHER ELEMENTARY TECHNIQUES USING THE MIRACLE OCTAD GENERATOR

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Introduction

In this paper we describe various techniques, some of which are already used by devotees of the art, which relate certain maximal subgroups of the Mathieu group M_{24}, as seen in the MOG, to matrix groups over finite fields. We hope to bring out the wealth of algebraic structure underlying the device and to enable the reader to move freely between these matrices and permutations. Perhaps the MOG was mis-named as simply an "octad generator"; in this paper we intend to show that it is in reality a natural diagram of the binary Golay code.

There are two versions of the MOG in print: the author's original version which appears in [**5, 6, 7, 8**], and what is, in effect, the mirror-image of this which is used in [**1, 3, 4, 10**]. Certain subgroups "look better" in each system and so it is worthwhile having both arrangements available. We shall indicate which we are referring to at any point by appending the subscripts M and M' respectively.

The octads in each case are, of course, those listed in [**9**].

N.B. To go from M to M', or vice versa, take the mirror-image or equivalently (modulo M_{24}) interchange the last two columns.

Mnemonic for obtaining a MOG arrangement of the 24 letters (John Conway)

(1) Insert the first twelve members of the Galois field of order 23 in a 4×6 array as shown in Fig. 1.

(2) Negate the non-squares (Fig. 2).

(3) Fill in the remainder of the 24-point projective line letting the linear fractional map

$$\gamma\colon x\to -1/x \equiv (\infty\ \ 0)(1\ \ 22)(2\ \ 11)(3\ \ 15)(4\ \ 17)(5\ \ 9)$$
$$(6\ \ 19)(7\ \ 13)(8\ \ 20)(10\ \ 16)(12\ \ 21)(14\ \ 18)$$

correspond to the permutation indicated in Fig. 3.

	0		1		2
3		4		5	
6		7		8	
9		10		11	

FIGURE 1

	0		1		2
3		4		18	
6		16		8	
9		13		12	

FIGURE 2

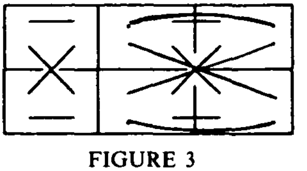

FIGURE 3

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(4) Obtain

∞	0	11	1	22	2
3	19	4	20	18	10
6	15	16	14	8	17
9	5	13	21	12	7

M

0	∞	1	11	2	22
19	3	20	4	10	18
15	6	14	16	17	8
5	9	21	13	7	12

M'

FIGURE 4

A. $M_{21} \cong L_3(4)$

A.1. We take the three fixed points as those indicated by X's in Fig. 5 and the remaining five points of the left-hand brick as the line at infinity, $z=0$. The 16-point affine plane is co-ordinated with x-axis horizontal (left to right) and y-axis vertical (descending) so that the projective point $(001)'$ appears in the top lefthand position.

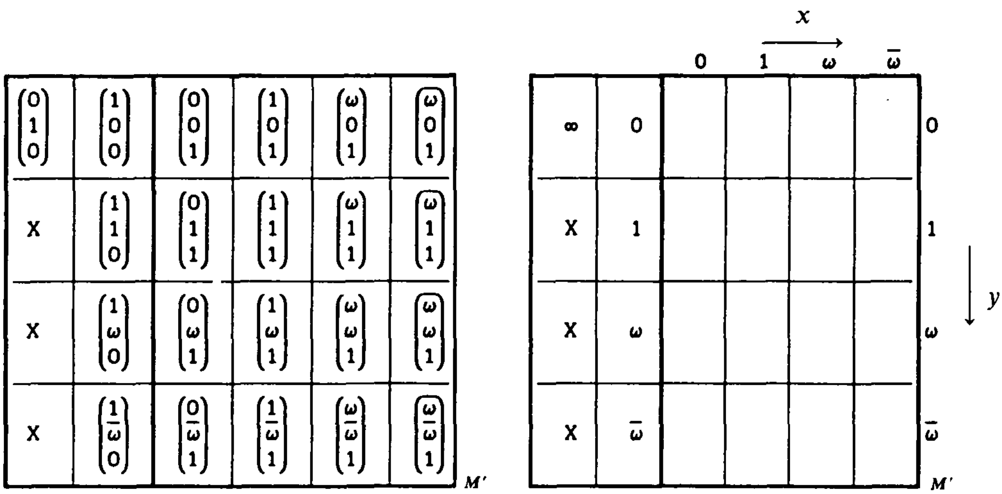
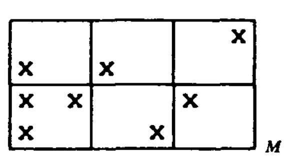


FIGURE 5
The 21-point projective plane as it appears in the MOG

Notes:

(i) The 21 octads containing the three fixed points have convenient names as lines $y=mx+b$.

e.g. $y=\omega x+1 \equiv$



(ii) A pleasing construction of the Golay code and, hence, of M_{24} can be furnished straight from here.

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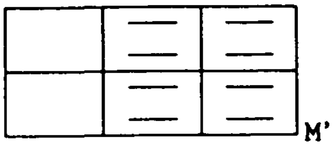
A.2. Identification of permutations of M_{21} with matrices (modulo the centre of $SL_3(4)$)

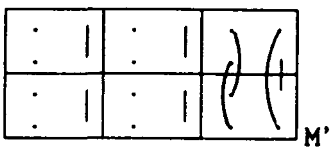
A matrix is determined (up to multiplication by a scalar matrix) by its action on the triangle of reference $((100)^t, (010)^t, (001)^t)$ and the unit point $(111)^t$. Thus



$$\sim \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & 1 \\ \cdot & \cdot & 1 \end{bmatrix}, \text{ and}$$

$$\left\{ \begin{bmatrix} 1 & \cdot & \alpha \\ \cdot & 1 & \beta \\ \cdot & \cdot & 1 \end{bmatrix} ; \alpha, \beta \in GF_4 \right\} = \text{the elementary abelian } 2^4 \text{ fixing each point on the line at infinity.}$$

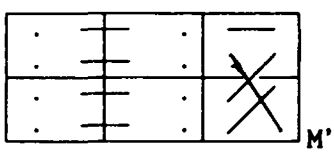
Again



$$\sim \begin{bmatrix} 1 & \cdot & \cdot \\ 1 & 1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix}, \text{ and}$$

$$\left\{ \left[\begin{array}{c|c} A & \begin{matrix} \cdot \\ \cdot \end{matrix} \\ \hline \cdot \quad \cdot & 1 \end{array} \right] ; A \in L_2(4) \right\} = \text{the } A_5 \text{ acting on the line at infinity fixing } (001)^t.$$

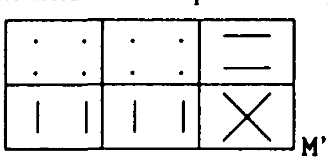
Adjoining



$$\sim \begin{bmatrix} \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \end{bmatrix}, \text{ we obtain } M_{21} \cong L_3(4).$$

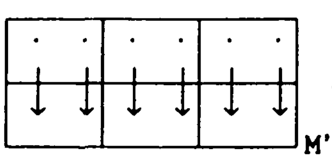
A.3. The automorphisms of M_{21}, $M_{21}: S_3$

(i) The field automorphism of $L_3(4)$ corresponds to



in M_{24}.

(ii) The diagonal automorphism of $L_3(4)$ may be taken as



$$\sim \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & \omega & \cdot \\ \cdot & \cdot & 1 \end{bmatrix}.$$

These, together with M_{21}, generate the maximal subgroup M_{21}: S_3 of M_{24}.

Note that the graph automorphism of $L_3(4)$, which interchanges points and lines, cannot be realised here.

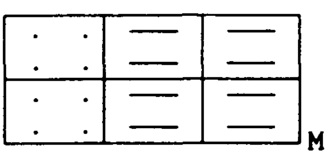
B. The stabilizer of an octad (2^4: $A_8 = 2^4$: $L_4(2)$)

		e_1	e_2	e_1+e_2
	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$
e_3	$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$
e_4	$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\\1\end{pmatrix}$
e_3+e_4	$\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

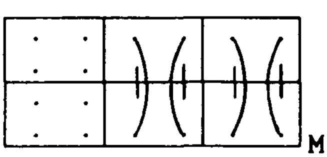
FIGURE 6

Label points in the complementary 16-ad as 4-dimensional vectors over GF_2 to give the 16-point affine plane. The stabilizer of an octad is now the semi-direct product of the elementary abelian group of translations generated by addition of a vector by the matrix group $L_4(2)$. This semi-direct product may be written as 5×5 matrices in the standard way. We have the following identification:

B.2. Identification of permutations with matrices in $L_5(2)$



$$\sim (e_1, I) \sim \left[\begin{array}{cccc|c} 1&.&.&.&1\\ .&1&.&.&.\\ .&.&1&.&.\\ .&.&.&1&.\\ \hline .&.&.&.&1\end{array}\right];$$



$$\sim (e_3+e_4, I) \sim \left[\begin{array}{cccc|c} 1&.&.&.&.\\ .&1&.&.&.\\ .&.&1&.&1\\ .&.&.&1&1\\ \hline .&.&.&.&1\end{array}\right];$$

and similarly

$$\left\{\left[\begin{array}{c|c} I_4 & \underline{v} \\ \hline & 1\end{array}\right];\ \underline{v} \in V_4\right\} = \text{the elementary abelian group of order 16 fixing every point of the octad.}$$

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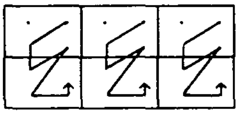
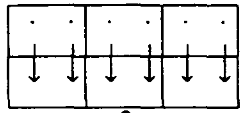
The subgroup fixing the zero vector acts as A_8 on the 8 points of the octad and as $L_4(2)$ on the 15 non-zero vectors. Thus the MOG furnishes the isomorphism:



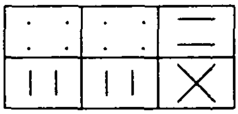
$_M$ (α)

$$\sim \left(\underline{0}, \begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix}\right);$$



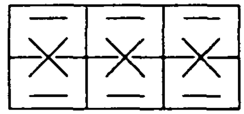
$_M$ (β)

$$\sim \left(\underline{0}, \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & 1 \end{bmatrix}\right);$$



$_M$

$$\sim \left(\underline{0}, \begin{bmatrix} 1 & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}\right);$$



$_M$ (γ)

$$\sim \left(\underline{e}_1 \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix}\right);$$

Plainly a Sylow 2-subgroup of M_{24} may be taken as the non-singular upper triangular matrices in this identification.

B.3. A related maximal subgroup—the trio group

It will be noted that the elements α, β, γ act identically on the three *bricks* and, with the labelling

∞	0
1	3
2	6
4	5

,

may be seen to generate $PSL_2(7)$ with α: $x \to x+1$, β: $x \to 2x$, γ: $x \to -1/x$. As was pointed out in [2] the permutation $(\infty\ \ 0)(1\ \ 3)(2\ \ 6)(4\ \ 5)$ and its 7 images under $L_2(7)$, together with the identity permutation, form an elementary abelian group of order 2^3. If we place an element of this group in each of the three bricks but with the restriction that the product of the three be the identity, we obtain an elementary abelian group of order 2^6. This, together with the above-mentioned $L_2(7)$ and the S_3 bodily permuting the bricks, gives the maximal *trio group* of shape 2^6: $(S_3 \times L_2(7))$.

B.4. The centralizer of an involution

The centralizer of a central $(1^8.2^8)$ involution, which we may take to be $(e_1, 1)$, will consist of all non-singular matrices of the form

$$\left[\begin{array}{c|cccc} 1 & & & & \\ \cdot & & A & & \\ \cdot & & & & \\ \cdot & & & & \\ \hline \cdot & \cdot & \cdot & \cdot & 1 \end{array}\right]$$

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It has shape $2.2^{3+3}\colon L_3(2)$ where the two elementary abelian groups of order 16 (2^{1+3}) consist of matrices of the form

$$\left[\begin{array}{cccc|c} & & & & \\ & I_4 & & & u \\ & & & & \\ \hline \cdot & \cdot & \cdot & \cdot & 1 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|cccc} 1 & & v^t & & \\ \hline \cdot & & & & \\ \cdot & & I_4 & & \\ \cdot & & & & \\ \cdot & & & & \end{array}\right] \quad \text{respectively.}$$

The $L_3(2)$ may be taken to be those matrices of the form

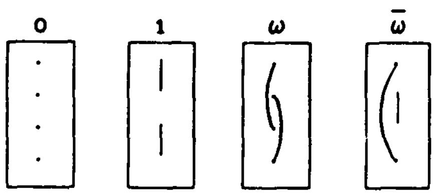
$$\left\{ \left[\begin{array}{c|ccc|c} 1 & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & & & & \cdot \\ \cdot & & A & & \cdot \\ \cdot & & & & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & 1 \end{array}\right] ; A \in L_3(2) \right\}.$$

Clearly the upper uni-triangular matrices, give a Sylow 2-subgroup of M_{24}.

C. The sextet stabilizer, the hexacode and the Sylow 2-subgroup of M_{24}

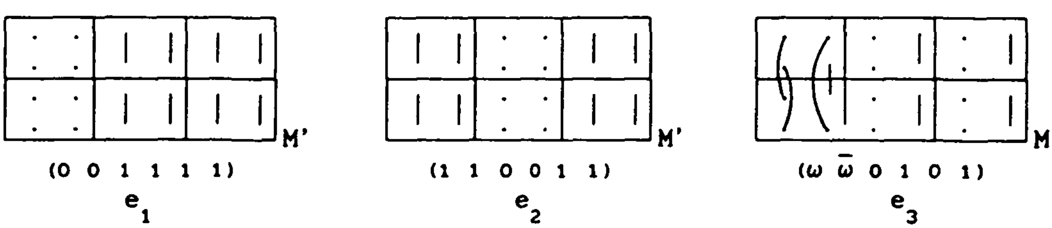
(M or M' are equally good here; we choose M' to be consistent with [4]).

C.1. The stabilizer of a decomposition into 6 mutually complementary tetrads [9] or a *sextet* [2, 6] is a maximal subgroup of shape $2^6\colon 3\cdot S_6$, and an example is the subgroup preserving the columns of the MOG. The permutations in the normal subgroup 2^6 fix all the columns and have a four-group action on them. We denote these actions:



(and note that $1+\omega+\bar{\omega}=0$, in the obvious sense).

The elements of the 2^6 now become 6-dimensional vectors over GF_4 and clearly form a 3-dimensional subspace, We take as basis:



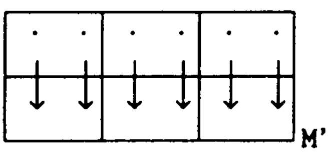
As can be readily verified from their expressions as 6-dimensional vectors over GF_4, 45 of the involutions of the elementary abelian 2^6 have cycle shape $1^8 \cdot 2^8$ while the remaining 18 have cycle shape 2^{12}. These 18 fall into 6 blocks of size three under multiplication by ω; that is they consist of the non-trivial elements of 6 disjoint four-groups which, of course, correspond to 1-dimensional subspaces over GF_4. A set of generators for the six 1-spaces is:

$$\left\{\begin{pmatrix}1\\ \omega\\ \omega\end{pmatrix}, \begin{pmatrix}1\\ \bar\omega\\ \bar\omega\end{pmatrix}, \begin{pmatrix}\omega\\ \bar\omega\\ 0\end{pmatrix}, \begin{pmatrix}\omega\\ \omega\\ 0\end{pmatrix}, \begin{pmatrix}\omega\\ 0\\ \bar\omega\end{pmatrix}, \begin{pmatrix}\omega\\ 0\\ \omega\end{pmatrix}\right\}.$$

As usual the most pleasing outcome occurs as our S_6 realizes both its 6-point actions: one on the six tetrads of the sextet, the other on these six four-groups. Thus the stabilizer of a four-group remains transitive on the tetrads and vice versa.

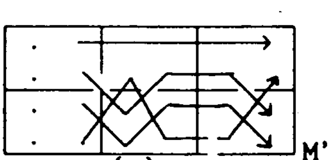
C.2. Identification of permutations with elements of $P\Sigma L_3(4)$

Now the subgroup $3 \cdot S_6$, which consists of permutations preserving the columns and fixing the top row of the MOG, acts as linear transformations (together with the field automorphism) of this space. In particular we see:



$$\sim \begin{bmatrix}1 & \cdot & \cdot\\ \cdot & \omega & \cdot\\ \cdot & \cdot & 1\end{bmatrix}.$$

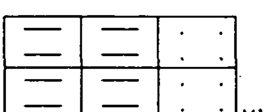
and



(a)

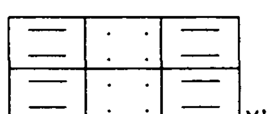
$$\sim \begin{bmatrix}0 & \omega & 1\\ 1 & \omega & 0\\ \omega & \omega & 1\end{bmatrix} \left\{\begin{array}{l} e_1 \to (0 \quad \bar\omega \quad 0 \quad \bar\omega \quad 1 \quad \omega) = e_2 + \bar\omega e_3\\ e_2 \to (1 \quad \bar\omega \quad \omega \quad 0 \quad 0 \quad \omega) = \omega e_1 + \omega e_2 + \omega e_3\\ e_3 \to (\omega \quad \bar\omega \quad 1 \quad 0 \quad 1 \quad 0) = e_1 + e_3 \end{array}\right\}$$

Further the elements:



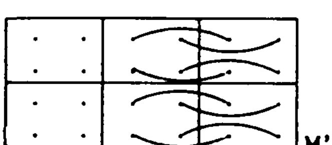
(b)

$$\sim \begin{bmatrix}1 & \cdot & 1\\ \cdot & 1 & 1\\ \cdot & \cdot & 1\end{bmatrix};$$



(c)

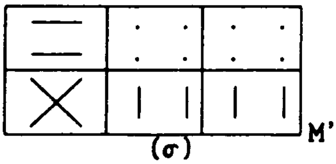
$$\sim \begin{bmatrix}1 & \cdot & \cdot\\ \cdot & 1 & 1\\ \cdot & \cdot & 1\end{bmatrix};$$



(d)

$$\sim \begin{bmatrix}1 & 1 & \cdot\\ \cdot & 1 & \cdot\\ \cdot & \cdot & 1\end{bmatrix}$$ clearly generate a copy of the dihedral group D_8.

We also observe that



$\sim$ $(\omega \leftrightarrow \bar{\omega})$, the field automorphism of GF_4.

It should be noted that $\langle a, b\rangle \cong 3\cdot A_6$, the triple cover of A_6.

C.3. The Sylow 2-subgroup of M_{24}

Now the vector space together with the upper uni-triangular matrices over GF_2 ($\langle b, c, d\rangle$ above) and the field automorphism (σ), clearly generate a subgroup of shape $2^6\colon(D_8\times C_2)$ which must be a Sylow 2-subgroup of M_{24}. Indeed we see from the above that

$$\left\langle \left[\begin{array}{ccc|c} 1 & a & b & \alpha \\ . & 1 & c & \beta \\ . & . & 1 & \gamma \\ \hline . & . & . & 1 \end{array}\right], \sigma \;\middle|\; \begin{array}{l} a, b, c \in GF_2 \\ \alpha, \beta, \gamma \in GF_4 \end{array} \right\rangle \cong Syl_2(M_{24}),$$

and the correspondence with permutations can be read off.

It should be noted that a Sylow 2-subgroup of M_{24} consists precisely of those permutations preserving the octad, trio, sextet and pairing indicated in Fig. 7.

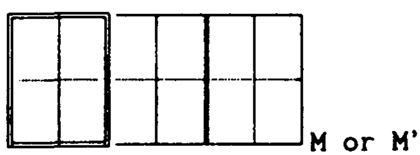


FIGURE 7
The Sylow 2-subgroup of M_{24}.

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