

Two polarised K3 surfaces associated to the same cubic fourfold

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Abstract

For infinitely many d , Hassett showed that special cubic fourfolds of discriminant d are related to polarised K3 surfaces of degree d via their Hodge structures. For half of the d , each associated K3 surface (S, L) canonically yields another one, (S^τ, L^τ) . We prove that S^τ is isomorphic to the moduli space of stable coherent sheaves on S with Mukai vector $(3, L, d/6)$. We also explain for which d the Hilbert schemes $\text{Hilb}^n(S)$ and $\text{Hilb}^n(S^\tau)$ are birational.

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Special cubic fourfolds were first studied by Hassett [9]. They are distinguished by the property that they carry additional algebraic cycles. They arise in countably many families, parametrised by irreducible divisors \mathcal{C}_d in the moduli space of cubic fourfolds. For infinitely many d , the cubic fourfolds in \mathcal{C}_d are related to polarised K3 surfaces of degree d via their Hodge structures. For half of the d , K3 surfaces associated to generic cubics in \mathcal{C}_d come in pairs. The goal of this paper is to explain how two such K3 surfaces are related.

More precisely, denote by \mathcal{M}_d the moduli space of polarised K3 surfaces of degree d . Hassett constructed, for admissible d , a surjective rational map $\mathcal{M}_d \dashrightarrow \mathcal{C}_d$ sending a K3 surface to a cubic fourfold it is associated to. This map is of degree two when $d \equiv 0 \pmod{6}$ and generically injective otherwise. In the former case, its (regular) covering involution $\tau: \mathcal{M}_d \rightarrow \mathcal{M}_d$ does not depend on the choices made to construct $\mathcal{M}_d \dashrightarrow \mathcal{C}_d$. We prove the following geometric description of τ .

THEOREM 1 (see Theorem 3.2). *Let $(S^\tau, L^\tau) = \tau(S, L)$. Then S^τ is isomorphic to the moduli space $\mathcal{M}_S(v)$ of stable (with respect to a generic polarisation) coherent sheaves on S with Mukai vector $v = (3, L, d/6)$.*

In particular, S and S^τ are Fourier–Mukai partners. For generic $(S, L) \in \mathcal{M}_d$, this also follows from the fact that the bounded derived categories of S and S^τ are both exact equivalent to the Kuznetsov category of the image cubic fourfold [2]. If $\rho(S) = 1$, then S is not isomorphic to S^τ (as unpolarised K3 surfaces). The number of Fourier–Mukai partners of S , which depends on d , can be arbitrarily high [17]. The above gives a natural way of selecting one of them for each $(S, L) \in \mathcal{M}_d$.

We also explain when the Hilbert schemes of n points $\text{Hilb}^n(S)$ and $\text{Hilb}^n(S^\tau)$ are birational. Our main result is the following.

THEOREM 2 (see Proposition 4.5, Corollary 4.9). *Let $d \equiv 0 \pmod 6$ satisfy (**). Consider the following statements:*

- (i) $\text{Hilb}^2(S)$ is isomorphic to $\text{Hilb}^2(S^\tau)$;
- (ii) $\text{Hilb}^2(S)$ is birational to $\text{Hilb}^2(S^\tau)$;
- (iii) there exists an integral solution to $3p^2 - (d/6)q^2 = -1$;
- (iv) $\text{Hilb}^2(S)$ has a line bundle of self-intersection 6.

We have implications (i) \Rightarrow (ii) \Leftarrow (iii) \Rightarrow (iv). If $\rho(S) = 1$, then these are all equivalent.

We will see that the condition in (iii) is satisfied for infinitely many d but not for all of them. As an application, we obtain an example of derived equivalent Hilbert schemes of two points on K3 surfaces which are not birational.

1. Lattices

In this section we set up the notation for the lattice theory that will be needed. See [12, Section 2] for references.

For a lattice Λ with intersection form $(,) : \Lambda \times \Lambda \rightarrow \mathbb{Z}$, we denote by Λ^\vee its dual lattice and by $\text{Disc } \Lambda = \Lambda^\vee / \Lambda$ its discriminant group. Every orthogonal transformation $g \in \text{O}(\Lambda)$ of Λ induces an automorphism on $\text{Disc } \Lambda$, which we denote by \bar{g} . When L is even, the product $(,)$ induces a quadratic form

$$q_L : \text{Disc } L \longrightarrow \mathbb{Q}/2\mathbb{Z}.$$

We denote by $\text{O}(\text{Disc } L)$ the group of automorphisms preserving q_L . We further define

$$\widetilde{\text{O}}(L) := \ker(\text{O}(L) \longrightarrow \text{O}(\text{Disc } L)).$$

We will use a result that is slightly stronger than [11, Proposition 14.2.6], but is proven in the same way. We give the proof here for completeness.

Let Λ_1 be a primitive sublattice of a unimodular lattice Λ and let $\Lambda_2 \subset \Lambda$ be its orthogonal complement. We have an inclusion $\Lambda^\vee \subset (\Lambda_1 \oplus \Lambda_2)^\vee$ which induces

$$\Lambda^\vee / (\Lambda_1 \oplus \Lambda_2) \hookrightarrow \text{Disc}(\Lambda_1 \oplus \Lambda_2) \cong \text{Disc}(\Lambda_1) \oplus \text{Disc}(\Lambda_2).$$

The projection maps $\Lambda^\vee / (\Lambda_1 \oplus \Lambda_2) \rightarrow \text{Disc}(\Lambda_i)$ are isomorphisms, by unimodularity of Λ and primitivity of Λ_1 and Λ_2 . This gives an isomorphism $\varphi : \text{Disc}(\Lambda_1) \rightarrow \text{Disc}(\Lambda_2)$ sending $\bar{x} \in \text{Disc}(\Lambda_1)$ to the unique class $\bar{y} \in \text{Disc}(\Lambda_2)$ such that $x + y \in \Lambda_1^\vee \oplus \Lambda_2^\vee$ is in Λ .

LEMMA 1.1. *If $g_1 \in \text{O}(\Lambda_1)$ and $g_2 \in \text{O}(\Lambda_2)$, then $g_1 \oplus g_2 : \Lambda_1 \oplus \Lambda_2 \rightarrow \Lambda_1 \oplus \Lambda_2$ extends to an automorphism of Λ if and only if $\bar{g}_1 = \bar{g}_2$ under the identification $\text{Disc } \Lambda_1 \cong \text{Disc } \Lambda_2$.*

Proof. The map $g_1 \oplus g_2$ extends to L if and only if for all $x_1 \in \Lambda_1^\vee, x_2 \in \Lambda_2^\vee$ with $x_1 + x_2 \in \Lambda$, the element $g_1^\vee(x_1) + g_2^\vee(x_2)$ also lies in Λ . We have $x_1 + x_2 \in \Lambda$ if and only if $\varphi(\bar{x}_1) = \bar{x}_2$. So $g_1^\vee(x_1) + g_2^\vee(x_2)$ is in Λ if and only if $\varphi(\bar{g}_1(\bar{x}_1))$ equals $\bar{g}_2(\bar{x}_2) = \bar{g}_2(\varphi(\bar{x}_1))$. This holds for all x_1, x_2 if and only if $\varphi \circ \bar{g}_1 = \bar{g}_2 \circ \varphi$.

The middle cohomology $H^2(S, \mathbb{Z})$ of a K3 surface S (with the usual intersection pairing) is isomorphic to the K3 lattice

$$\Lambda_{K3} := E_8(-1)^{\oplus 2} \oplus U^{\oplus 3} = E_8(-1)^{\oplus 2} \oplus U_1 \oplus U_2 \oplus U_3.$$

We denote the standard basis of U_i by e_i, f_i . On the full cohomology $H^*(S, \mathbb{Z})$ of S we consider the Mukai pairing, given by $((x_0, x_2, x_4), (x'_0, x'_2, x'_4)) = x_2x'_2 - x_0x'_4 - x'_0x_4$ for $x_i, x'_i \in H^i(S, \mathbb{Z})$. With this pairing, $H^*(S, \mathbb{Z})$ becomes isomorphic to the Mukai lattice

$$\Lambda_{Muk} := \Lambda_{K3} \oplus U(-1) = E_8(-1)^{\oplus 2} \oplus U_1 \oplus U_2 \oplus U_3 \oplus U_4(-1).$$

As $U \cong U(-1)$, the Mukai lattice is isomorphic to $\Lambda_{K3} \oplus U$. To avoid confusion, we denote the latter by $\tilde{\Lambda}_{K3}$, and fix an isomorphism $\tilde{\Lambda}_{K3} \xrightarrow{\sim} \Lambda_{Muk}$ by sending f_4 to $-f_4$.

Fix $\ell_d = e_3 + (d/2)f_3 \in U_3 \subset \Lambda_{K3}$ and let $\Lambda_d := \ell_d^\perp \subset \Lambda_{K3}$ be its orthogonal complement in Λ_{K3} . Then

$$\Lambda_d = E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus \mathbb{Z}(-d)$$

is isomorphic to the primitive cohomology $L^\perp \subset H^2(S, \mathbb{Z})$ of any polarised K3 surface (S, L) of degree d .

We denote by $H^4(X, \mathbb{Z})(1)$ the middle cohomology of a cubic fourfold X , with the weight of the Hodge structure shifted by two and the intersection product changed by a sign. As a lattice, it is isomorphic to

$$\Lambda_{cub} := E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus \mathbb{Z}(-1)^{\oplus 3}.$$

Let $h = (1, 1, 1) \in \mathbb{Z}(-1)^{\oplus 3} \subset \Lambda_{cub}$. The primitive cohomology $H^4(X, \mathbb{Z})_{prim}(1)$ of X is isomorphic to $\Lambda_{cub}^0 := h^\perp \subset \Lambda_{cub}$. An easy computation shows that

$$\Lambda_{cub}^0 \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus A_2(-1),$$

where A_2 is the lattice $(\mathbb{Z}^{\oplus 2}, \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix})$. There exists a primitive embedding of A_2 into $U_3 \oplus U_4 \subset \tilde{\Lambda}_{K3}$, unique up the action of $O(\tilde{\Lambda}_{K3})$. One can show that $\Lambda_{cub}^0 \cong A_2^\perp \subset \tilde{\Lambda}_{K3}$.

2. Hassett’s construction

We summarise Hassett’s construction, explaining those proofs that we need for our results. For details, see [9].

2.1. Special cubic fourfolds

Inside $H^4(X, \mathbb{Z})(1)$, we consider the negative definite lattice

$$A(X) := H^4(X, \mathbb{Z})(1) \cap H^{2,2}(X).$$

We fix the notation $h_X^2 \in H^4(X, \mathbb{Z})$ for the square of the hyperplane class on X . For X very general, the lattice $A(X)$ has rank one and is generated by h_X^2 . We call X special if $\text{rk } A(X) \geq 2$. By the Hodge conjecture for cubic fourfolds [22], X is special if and only if X contains a surface that is not homologous to a complete intersection.

If X is special, then $A(X)$ contains a primitive sublattice K of rank two. Hassett proved that fixing the discriminant disc K of such K gives divisors in the moduli space \mathcal{C} of smooth cubic fourfolds. Namely, define

$$\mathcal{C}_d := \{X \in \mathcal{C} \mid \exists K \subset A(X) \text{ primitive, } h_X^2 \in K, \text{rk } K = 2, \text{disc } K = d\}.$$

Then the set of special cubic fourfolds in \mathcal{C} is the union of all \mathcal{C}_d .

THEOREM 2.1 ([9, Theorem 1.0.1]). *The set \mathcal{C}_d is either empty or an irreducible divisor in \mathcal{C} . It is non-empty if and only if d satisfies*

$$d > 6 \text{ and } d \equiv 0, 2 \pmod{6}. \tag{*}$$

2.2. *Periods of special cubic fourfolds*

Recall the definition of the period domain for a lattice Λ of signature (n_+, n_-) with $n_+ \geq 2$:

$$\mathcal{D}(\Lambda) = \{x \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid (x)^2 = 0, (x, \bar{x}) > 0\}.$$

This is a complex manifold of dimension $\text{rk } \Lambda - 2$, which is connected when $n_+ > 2$ and has two connected components when $n_+ = 2$. It has a natural action by the group $\tilde{\mathcal{O}}(\Lambda)$. This is an arithmetic group; if $n_+ = 2$, then by [20, Section 6 of appendix] and [4], the quotient

$$\mathcal{QD}(\Lambda) := \mathcal{D}(\Lambda) / \tilde{\mathcal{O}}(\Lambda)$$

is a (connected) quasi-projective variety with at most finite quotient singularities.

In particular, this holds for $\Lambda = \Lambda_d$, yielding a 19-dimensional variety $\mathcal{QD}(\Lambda_d)$. The period map induces an open embedding $\mathcal{M}_d \hookrightarrow \mathcal{QD}(\Lambda_d)$ (see e.g. [11, Corollary 6.4.3]).

The lattice Λ_{cub}^0 also satisfies $n_+ = 2$; the variety $\mathcal{QD}(\Lambda_{\text{cub}}^0)$ has dimension 20. The period map gives an open embedding $\mathcal{C} \hookrightarrow \mathcal{QD}(\Lambda_{\text{cub}}^0)$ [9, Section 2.2].

Inside $\mathcal{D}(\Lambda_{\text{cub}}^0)$, we can identify those periods coming from special cubic fourfolds. Note that a cubic fourfold X is special if and only if there exists a negative definite sublattice $K \subset H^4(X, \mathbb{Z})(1)$ of rank two with $h_X^2 \in K$, such that $K \otimes \mathbb{C}$ is contained in $H^{3,1}(X)^\perp \subset H^4(X, \mathbb{C})(1)$. On the level of the period domain, this means the following: After choosing a marking $H^4(X, \mathbb{Z})_{\text{prim}}(1) \xrightarrow{\sim} \Lambda_{\text{cub}}^0$, the period of X lands in

$$\{x \in \mathcal{D}(\Lambda_{\text{cub}}^0) \mid (K \cap \Lambda_{\text{cub}}^0)_{\mathbb{C}} \subset x^\perp\}$$

for some primitive, negative definite sublattice $K \subset \Lambda_{\text{cub}}^0$ of rank two containing h . Let K^\perp be its orthogonal complement; note that $K^\perp \subset \Lambda_{\text{cub}}^0$. The set above is the divisor $\mathcal{D}(K^\perp) \subset \mathcal{D}(\Lambda_{\text{cub}}^0)$.

We fix a primitive sublattice $K_d \subset \Lambda_{\text{cub}}^0$ as above, of discriminant d . Let $\bar{\mathcal{C}}_d \subset \mathcal{QD}(\Lambda_{\text{cub}}^0)$ be the image of $\mathcal{D}(K_d^\perp) \subset \mathcal{D}(\Lambda_{\text{cub}}^0)$ under the quotient map $\mathcal{D}(\Lambda_{\text{cub}}^0) \rightarrow \mathcal{QD}(\Lambda_{\text{cub}}^0)$. The following shows that $\bar{\mathcal{C}}_d$ does not depend on the choice of K_d .

PROPOSITION 2.2 ([9, Proposition 3.2.4]). *Let $K, K' \subset \Lambda_{\text{cub}}^0$ be primitive sublattices of rank two containing h . Then $K = f(K')$ for some $f \in \tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0)$ if and only if $\text{disc } K = \text{disc } K'$.*

Note that the immersion $\mathcal{C} \hookrightarrow \mathcal{QD}(\Lambda_{\text{cub}}^0)$ maps \mathcal{C}_d into $\bar{\mathcal{C}}_d$. In fact, we have $\mathcal{C}_d = \mathcal{C} \cap \bar{\mathcal{C}}_d$.

2.3. *Associated K3 surfaces*

Consider the following condition on $d \in \mathbb{N}$:

$$d \text{ is even and not divisible by } 4, 9, \text{ or any odd prime } p \equiv 2 \pmod{3}. \tag{**}$$

This implies that $d \equiv 0, 2 \pmod{6}$. Hassett proved the following statement:

PROPOSITION 2.3 ([9, Proposition 5.1.4]). *The number d satisfies (**) if and only if there is an isomorphism $\Lambda_d \cong K_d^\perp$.*

So when d satisfies (**), there is an isomorphism of period domains $\mathcal{D}(\Lambda_d) \cong \mathcal{D}(K_d^\perp)$. Proposition 2.6 below tells us that under the identification $\Lambda_d \cong K_d^\perp$, the group $\tilde{\mathcal{O}}(\Lambda_d)$ forms a subgroup of $\tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0)$. Hence, we also get a map from $\mathcal{QD}(\Lambda_d) = \mathcal{O}(\Lambda_d) \backslash \mathcal{D}(\Lambda_d)$ to $\bar{\mathcal{C}}_d$. This gives us the following commutative diagram:

$$\begin{array}{ccccc}
 \mathcal{D}(\Lambda_d) & \xrightarrow{\sim} & \mathcal{D}(K_d^\perp) & \hookrightarrow & \mathcal{D}(\Lambda_{\text{cub}}^0) \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{QD}(\Lambda_d) & \twoheadrightarrow & \bar{\mathcal{C}}_d & \hookrightarrow & \mathcal{QD}(\Lambda_{\text{cub}}^0) \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathcal{M}_d & \xrightarrow{\varphi} & \mathcal{C}_d & \hookrightarrow & \mathcal{C}
 \end{array}$$

It can be shown that the rational map $\varphi: \mathcal{M}_d \dashrightarrow \mathcal{C}_d$ is regular on an open subset which maps surjectively to \mathcal{C}_d , see [9, Section 5.1]. Note that φ depends on the choice of an isomorphism $\Lambda_d \cong K_d^\perp$, thus it is only unique up to $\mathcal{O}(\Lambda_d) / \tilde{\mathcal{O}}(\Lambda_d)$.

If φ sends $(S, L) \in \mathcal{M}_d$ to X , then there exists an isometry of Hodge structures

$$\mathbb{H}^4(X, \mathbb{Z})(1) \supset K^\perp \cong L^\perp \subset \mathbb{H}^2(S, \mathbb{Z})$$

for some primitive sublattice $K \subset A(X)$ of rank two and discriminant d containing h_X^2 . Conversely, if such a Hodge isometry exists, it induces a lattice isomorphism

$$\Lambda_d \cong L^\perp \cong K^\perp \cong K_d^\perp$$

such that the induced map $\varphi: \mathcal{M}_d \dashrightarrow \mathcal{C}_d$ sends (S, L) to X . This motivates the following definition.

Definition 2.4. Let $X \in \mathcal{C}_d$. A polarised K3 surface $(S, L) \in \mathcal{M}_d$ is associated to X if there exists a Hodge isometry

$$\mathbb{H}^4(X, \mathbb{Z})(1) \supset K^\perp \cong L^\perp \subset \mathbb{H}^2(S, \mathbb{Z})$$

for some primitive sublattice $K \subset A(X)$ of rank two and discriminant d containing h_X^2 .

For the rest of this section, we fix one choice of the rational map φ .

Remark 2.5. The complement of the image of the inclusion $\mathcal{C} \hookrightarrow \mathcal{QD}(\Lambda_{\text{cub}}^0)$ is $\bar{\mathcal{C}}_2 \cup \bar{\mathcal{C}}_6$ [13, 14]. Therefore, $\varphi(S, L)$ is defined if and only if the image of (S, L) under $\mathcal{QD}(\Lambda_d) \rightarrow \bar{\mathcal{C}}_d$ lies in $\bar{\mathcal{C}}_d \setminus (\bar{\mathcal{C}}_2 \cup \bar{\mathcal{C}}_6)$. In particular, this holds when $\rho(S) = 1$.

To describe the map $\mathcal{QD}(\Lambda_d) \rightarrow \bar{\mathcal{C}}_d$, we define two subgroups of $\tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0)$. Let v_d be a generator of $K_d \cap \Lambda_{\text{cub}}^0$, which is unique up to a sign. Define $\tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0, v_d) \subset \tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0)$ by

$$\begin{aligned}
 \tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0, v_d) &:= \{f \in \mathcal{O}(\Lambda_{\text{cub}}) \mid f|_{K_d} = \text{id}_{K_d}\} \\
 &= \{f \in \tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0) \mid f(v_d) = v_d\}.
 \end{aligned}$$

The next statement is part of [9, Theorem 5.2.2]. It follows directly from Lemma 1.1.

PROPOSITION 2.6. *Suppose that d satisfies $(**)$. Under the isomorphism $\Lambda_d \cong K_d^\perp$, the group $\tilde{\mathcal{O}}(\Lambda_d)$ is identified with $\tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0, v_d)$.*

In particular, there is an isomorphism $\mathcal{QD}(\Lambda_d) \cong \tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0, v_d) \backslash \mathcal{D}(K_d^\perp)$.

The second group we consider is

$$\begin{aligned} \tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0, \langle v_d \rangle) &= \{f \in \mathcal{O}(\Lambda_{\text{cub}}) \mid f(h) = h \text{ and } f(K_d) = K_d\} \\ &= \{f \in \tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0) \mid f(v_d) = \pm v_d\}. \end{aligned}$$

By the Baily–Borel theorem [4], the quotient $\tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0, \langle v_d \rangle) \backslash \mathcal{D}(K_d^\perp)$ is a normal quasi-projective variety. In fact, $\tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0, \langle v_d \rangle) \backslash \mathcal{D}(K_d^\perp) \rightarrow \bar{\mathcal{C}}_d$ is the normalisation of $\bar{\mathcal{C}}_d$. Namely, a very general special cubic fourfold X satisfies $\text{rk } A(X) = 2$ [21, Section 5.1], so there is only one sublattice $K_d \subset A(X)$. It follows that the map $\tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0, \langle v_d \rangle) \backslash \mathcal{D}(K_d^\perp) \rightarrow \bar{\mathcal{C}}_d$ is generically injective. To see that it is proper, note that the action of $\tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0)$ on $\mathcal{D}(\Lambda_{\text{cub}}^0)$ is properly discontinuous [11, Remark 6.1.10]. Hence the map $\mathcal{D}(\Lambda_{\text{cub}}^0) \rightarrow \mathcal{QD}(\Lambda_{\text{cub}}^0)$ is closed, as is its restriction $\mathcal{D}(K_d^\perp) \rightarrow \bar{\mathcal{C}}_d$ to the closed subset $\mathcal{D}(K_d^\perp) \subset \mathcal{D}(\Lambda_{\text{cub}}^0)$. Since this factors as

$$\mathcal{D}(K_d^\perp) \rightarrow \tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0, \langle v_d \rangle) \backslash \mathcal{D}(K_d^\perp) \rightarrow \bar{\mathcal{C}}_d,$$

the map $\tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0, \langle v_d \rangle) \backslash \mathcal{D}(K_d^\perp) \rightarrow \bar{\mathcal{C}}_d$ is closed as well. Moreover, it has finite fibres, so it is proper.

Summarising, we have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{QD}(\Lambda_d) \cong \tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0, v_d) \backslash \mathcal{D}(K_d^\perp) & \xrightarrow{\bar{\gamma}} & \tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0, \langle v_d \rangle) \backslash \mathcal{D}(K_d^\perp) & \cong & \bar{\mathcal{C}}_d^{\text{norm}} \twoheadrightarrow \bar{\mathcal{C}}_d \\ \uparrow & & & & \uparrow \\ \mathcal{M}_d & \dashrightarrow & \mathcal{C}_d^{\text{norm}} & \dashrightarrow & \mathcal{C}_d \end{array}$$

The spaces $\tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0, v_d) \backslash \mathcal{D}(K_d^\perp)$ and $\tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0, \langle v_d \rangle) \backslash \mathcal{D}(K_d^\perp)$ can be seen as period domains of *marked* and *labelled* cubic fourfolds, respectively [9, Section 3.1, 5.2].

The following describes the generic fibre of the quotient map

$$\bar{\gamma}: \tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0, v_d) \backslash \mathcal{D}(K_d^\perp) \longrightarrow \tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0, \langle v_d \rangle) \backslash \mathcal{D}(K_d^\perp).$$

PROPOSITION 2.7 ([9, proposition 5.2.1]). *There is an isomorphism*

$$\tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0, \langle v_d \rangle) \backslash \tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0, v_d) \cong \begin{cases} \{0\} & \text{if } d \equiv 2 \pmod{6} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } d \equiv 0 \pmod{6}. \end{cases}$$

As a consequence, $\bar{\gamma}$ is an isomorphism when $d \equiv 2 \pmod{6}$ and has degree two when $d \equiv 0 \pmod{6}$. In the latter case, the covering involution of $\bar{\gamma}$ is induced by an automorphism $g \in \tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0, \langle v_d \rangle)$ whose class modulo $\tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0, v_d)$ generates $\mathbb{Z}/2\mathbb{Z}$. We will explain the construction of an explicit such g .

LEMMA 2.8. *Let $x \in \Lambda_{\text{cub}}^0$ be primitive with $(x, x) \neq 0$ and $3 \nmid (x, x)$. There exists an $f \in \tilde{\mathcal{O}}(\Lambda_{\text{cub}}^0)$ such that $f(x) = e_2 + f_2 \cdot (x, x)/2$.*

Proof. As $\text{Disc } \Lambda_{\text{cub}}^0 = \mathbb{Z}/3\mathbb{Z}$, the divisibility $\text{div}(x)$ of x in Λ_{cub}^0 is either 1 or 3. Since $3 \nmid (x, x)$, it must be 1. Therefore the class of $x/\text{div}(x)$ in $\text{Disc } \Lambda_{\text{cub}}^0$ is trivial. By Eichler’s criterion [8, Proposition 3.3], there is an $f \in \widetilde{\text{O}}(\Lambda_{\text{cub}}^0)$ sending x to $e_2 + f_2 \cdot (x, x)/2$.

Suppose $d \equiv 0 \pmod 6$. By Lemma 2.8, we can assume

$$v_d = e_2 - \frac{d}{6} f_2 \in \Lambda_{\text{cub}} = E_8(-1)^{\oplus 2} \oplus U_1 \oplus U_2 \oplus \mathbb{Z}(-1)^{\oplus 3}.$$

Let $g \in \text{O}(\Lambda_{\text{cub}})$ be given by

$$g|_{E_8(-1)^{\oplus 2} \oplus \mathbb{Z}(-1)^{\oplus 3}} = \text{id}; \quad g|_{U_1 \oplus U_2} = -\text{id}.$$

Then we have $g(v_d) = -v_d$, so $g|_{\Lambda_{\text{cub}}^0}$ generates $\widetilde{\text{O}}(\Lambda_{\text{cub}}^0, \langle v_d \rangle) / \widetilde{\text{O}}(\Lambda_{\text{cub}}^0, v_d)$.

3. The involution on \mathcal{M}_d

From now on, we will assume that d satisfies (**). In the previous section, we explained that the map

$$\bar{\gamma}: \mathcal{QD}(\Lambda_d) \longrightarrow \widetilde{\text{O}}(\Lambda_{\text{cub}}^0, \langle v_d \rangle) \backslash \mathcal{D}(K_d^\perp)$$

is an isomorphism if $d \equiv 2 \pmod 6$ and has degree two if $d \equiv 0 \pmod 6$. In the second case, define $\tau: \mathcal{QD}(\Lambda_d) \rightarrow \mathcal{QD}(\Lambda_d)$ to be the covering involution of $\bar{\gamma}$. It is induced by the map $g|_{K_d^\perp}$, seen as an element of $\widetilde{\text{O}}(\Lambda_d)$. Note that τ maps \mathcal{M}_d to itself: As explained in e.g. [11, remark 6.3.7], the complement of \mathcal{M}_d in $\mathcal{QD}(\Lambda_d)$ is

$$\bigcup_{\delta \in \Lambda_d, \delta^2 = -2} \delta^\perp$$

and this set is clearly preserved under g .

PROPOSITION 3.1. *The morphism τ does not depend on the choice of isomorphism $\Lambda_d \cong K/d$.*

Proof. Precomposing the isomorphism $\Lambda_d \cong K_d^\perp$ with $f \in \text{O}(\Lambda_d)$ changes g on Λ_d to $f^{-1} \circ g \circ f$. Note that this has the same action on the abelian group $\text{Disc}(\Lambda_d)$ as g , thus it induces the same action on $\mathcal{QD}(\Lambda_d)$.

For a K3 surface S , we denote by $\widetilde{\text{H}}(S, \mathbb{Z})$ the full cohomology of S with the Mukai pairing and the Hodge structure of weight two defined by $\widetilde{\text{H}}^{2,0}(S) := \text{H}^{2,0}(S)$.

Let H be a generic polarisation on S . For a primitive vector $v = (r, \ell, s) \in \widetilde{\text{H}}(S, \mathbb{Z})$, denote by $M_S(v)$ the moduli space of H -stable coherent sheaves on S with Mukai vector v . Recall [11, Chapter 10] that if there exists a w in $\widetilde{\text{H}}^{1,1}(S, \mathbb{Z}) = \text{H}^0(S, \mathbb{Z}) \oplus \text{H}^{1,1}(S, \mathbb{Z}) \oplus \text{H}^4(S, \mathbb{Z})$ with $(v, w) = 1$, then this is a fine moduli space. If $r > 0$ (or $(\ell)^2 \geq 2$ and $(\ell, H) > 0$) and $(v)^2 = 0$, then $M_S(v)$ is a K3 surface.

THEOREM 3.2. *Let $(S^\tau, L^\tau) = \tau(S, L)$. Then S^τ is isomorphic to the moduli space $M_S(v)$ of stable coherent sheaves on S with Mukai vector $v = (3, L, d/6)$. Under the natural identification $\text{H}^2(S^\tau, \mathbb{Z}) \cong v^\perp/\mathbb{Z}v$, we have $L^\tau = (d, (d/3 - 1)L, d/3(d/6 - 1))$.*

Let us describe the strategy of the proof. Since the map g does not induce the identity on $\text{Disc } \Lambda_d$, it does not extend to an orthogonal transformation of Λ_{K3} . However, we will show that g extends to $\tilde{g} \in \text{O}(\tilde{\Lambda}_{K3})$. Using \tilde{g} , we find S^τ as follows. Let x be a representative of the period of (S, L) in $\Lambda_{d,C} \subset \tilde{\Lambda}_{K3,C}$. The K3 surface S^τ is the one whose period can be represented by $g(x) \in \Lambda_{d,C}$. The map \tilde{g} induces a Hodge isometry $\tilde{H}(S, \mathbb{Z}) \cong \tilde{H}(S^\tau, \mathbb{Z})$, so by the derived Torelli theorem ([18], see also [11, Proposition 16.3.5]), S and S^τ are Fourier–Mukai partners.

More precisely, denote by \tilde{g}_{Muk} the morphism \tilde{g} , seen as an orthogonal transformation of Λ_{Muk} . Let $v = (r, \ell, s) := \tilde{g}_{\text{Muk}}^{-1}(0, 0, 1)$. There exists a universal sheaf \mathcal{E} on $S \times M_S(v)$. Let $\Phi_{\mathcal{E}}^H : \tilde{H}(M_S(v), \mathbb{Z}) \rightarrow \tilde{H}(S, \mathbb{Z})$ be the induced cohomological Fourier–Mukai transform, which satisfies $\Phi_{\mathcal{E}}^H(0, 0, 1) = v$. Then $\tilde{g}_{\text{Muk}} \circ \Phi_{\mathcal{E}}^H$ sends $H^2(M_S(v), \mathbb{Z})$ to $H^2(S^\tau, \mathbb{Z})$, which shows that S^τ is isomorphic to $M_S(v)$.

To describe L^τ , note that $\Phi_{\mathcal{E}}^H$ induces an isomorphism $H^2(M_S(v), \mathbb{Z}) \cong v^\perp / \mathbb{Z}v$, where $v^\perp \subset \tilde{H}(S, \mathbb{Z})$ (this is a result by Mukai, see [11, Remark 10.3.7]). Thus, $\tilde{g}_{\text{Muk}}^{-1}$ restricts to an isomorphism $H^2(S^\tau, \mathbb{Z}) \cong v^\perp / \mathbb{Z}v$. Under this identification, the polarisation L^τ is mapped to $\tilde{g}_{\text{Muk}}^{-1}(\ell_d)$.

Remark 3.3. The extension \tilde{g} of g is not unique. But if \tilde{g}' is another extension, then $\tilde{g}_{\text{Muk}}^{-1} \circ \tilde{g}'_{\text{Muk}}$ is an orthogonal transformation of Λ_{Muk} sending $v' = (\tilde{g}'_{\text{Muk}})^{-1}(0, 0, 1)$ to v . This induces a Hodge isometry $H^2(M_S(v'), \mathbb{Z}) \cong H^2(M_S(v), \mathbb{Z})$, so $M_S(v')$ and $M_S(v)$ are isomorphic.

Remark 3.4. The space $\mathcal{QD}(\Lambda_d)$ can be interpreted as the moduli space of *quasi-polarised* K3 surfaces, i.e. pairs (S, L) with L the class of a big and nef line bundle [10, Section 5]. For such pairs the theorem is still valid.

Remark 3.5. For $d \equiv 0 \pmod 6$, the ramification locus of $\bar{\gamma}$ over \mathcal{M}_d consists of those (S, L) which are polarised isomorphic to (S^τ, L^τ) . It follows from [10, Section 8] that $\bar{\gamma}$ is unramified over $\{(S, L) \in \mathcal{M}_d \mid \rho(S) = 1\}$.

3.1. Proof of Theorem 3.2

We first compute the action of g on $\text{Disc } K_d^\perp \cong \text{Disc } K_d$. Suppose $T \in K_d$ is a primitive element such that h and T generate K_d . Then (h, T) is divisible by 3. We can write $v_d = (h, T)h/3$, which has square $-d/3$. It follows that $K_d = \mathbb{Z}h \oplus \mathbb{Z}v_d$, so

$$\text{Disc } K_d \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/\left(\frac{d}{3}\right)\mathbb{Z},$$

with the action of g given by $\text{id} \oplus -\text{id}$.

LEMMA 3.6. *The induced action of g on $\text{Disc } \Lambda_d \cong \mathbb{Z}/d\mathbb{Z}$ is given by $x \mapsto (d/3 - 1)x$.*

Proof. If we reduce modulo $d/3$, multiplication by $d/3 - 1$ equals $-\text{id}$. If we reduce modulo 3, it is the identity, as $d/3 - 1 \equiv 1 \pmod 3$.

To extend g on Λ_d to $\tilde{\Lambda}_{K3}$ it thus suffices, by Lemma 1.1, to find an orthogonal transformation of $\Lambda_d^\perp = \mathbb{Z}\ell_d \oplus U_4$ acting on the discriminant group by $x \mapsto (d/3 - 1)x$. Consider $u \in \text{O}(\mathbb{Z}\ell_d \oplus U_4)$ defined by (see Remark 3.7)

$$\begin{aligned} e_4 &\longmapsto -\frac{d}{6}e_4 - \frac{1}{3}\left(\frac{d}{6} - 1\right)\ell_d + \frac{1}{3}\left(\frac{d}{6} - 1\right)^2 f_4 \\ f_4 &\longmapsto 3e_4 + \ell_d - \frac{d}{6}f_4 \\ \ell_d &\longmapsto de_4 + \left(\frac{d}{3} - 1\right)\ell_d - \frac{d}{3}\left(\frac{d}{6} - 1\right)f_4 \end{aligned}$$

One computes that this is an involution. The discriminant group $\text{Disc}(\mathbb{Z}\ell_d \oplus U_4) \cong \mathbb{Z}/d\mathbb{Z}$ is generated by the class $\overline{\ell_d/d}$, which \bar{u} multiplies by $d/3 - 1$.

It follows that $g \oplus u \in \text{O}(\Lambda_d \oplus \mathbb{Z}\ell_d \oplus U_4)$ extends to $\tilde{g} \in \text{O}(\tilde{\Lambda}_{K3})$. Since \tilde{g} is an involution, we have $\tilde{g}^{-1}(f_4) = \tilde{g}(f_4) = 3e_4 + \ell_d - (d/6)f_4$. As an element of the Mukai lattice, this is

$$v = (3, \ell_d, d/6) \in \Lambda_{\text{Muk}} = \mathbb{Z}e_4 \oplus \Lambda_{K3} \oplus \mathbb{Z}(-f_4).$$

The polarisation $L^\tau = \tilde{g}^{-1}(\ell_d)$, seen as an element of $v^\perp/\mathbb{Z}v \subset \tilde{H}(S, \mathbb{Z})$, is

$$L^\tau = \left(d, \left(\frac{d}{3} - 1 \right) L, \frac{d}{3} \left(\frac{d}{6} - 1 \right) \right).$$

This completes the proof of Theorem 3.2.

Remark 3.7. The map u can be found as follows. One extends $g|_{K_d^\perp}$ to $\tilde{\Lambda}_{K3}$ by taking $\text{id} \oplus -\text{id}$ on the orthogonal complement $A_2(-1) \oplus \mathbb{Z}(d/3)$ of K_d^\perp in Λ_{K3} . In order to find the Mukai vector v , one writes down an explicit isomorphism $A_2(-1) \oplus \mathbb{Z}(d/3) \cong \mathbb{Z}\ell'_d \oplus U_4$ and computes the corresponding orthogonal transformation u of $\mathbb{Z}\ell'_d \oplus U_4$. We chose not to spell out these tedious computations.

Remark 3.8. As shown in the proof of Proposition 3.1, we can replace g by any element of $\text{O}(\Lambda_d)$ with the same action on $\text{Disc } \Lambda_d$. For instance, we can take the automorphism given by the identity on $E_8(-1)^{\oplus 2} \oplus U_1$ and u on $U_2 \oplus \mathbb{Z}(e_3 - (d/2)f_3) \cong (\mathbb{Z}\ell_d \oplus U_4)(-1)$. This allows us to define τ on \mathcal{M}_d for all $d \equiv 0 \pmod 6$ with $d/6 \equiv 1 \pmod 3$.

4. Birationality of Hilbert schemes

In this section we study the Hilbert schemes of n points $\text{Hilb}^n(S)$ and $\text{Hilb}^n(S^\tau)$ of our K3 surfaces S and S^τ . Corollary 4.9 and the results in Section 4.2 hold for all d such that $d/6 \equiv 1 \pmod 3$, using Remark 3.8.

4.1. Hilbert schemes of two points

For a cubic fourfold X we denote by $F(X)$ the Fano variety of lines on X , a four-dimensional hyperkähler variety of $K3^{[2]}$ type. Hassett proved the following:

THEOREM 4.1 ([9, Theorem 6.1.4]). *Assume that d satisfies*

$$d = 2(n^2 + n + 1)$$

for some integer $n \geq 2$. Let X be a generic cubic fourfold in \mathcal{C}_d . Then $F(X)$ is isomorphic to $\text{Hilb}^2(S)$, where $(S, L) \in \mathcal{M}_d$ is associated to X .

If also $d \equiv 0 \pmod 6$, then $F(X)$ is isomorphic to both $\text{Hilb}^2(S)$ and $\text{Hilb}^2(S^\tau)$ (Hassett calls $F(X)$ *ambiguous*). Since birationality specialises in families of hyperkähler manifolds, it follows that $\text{Hilb}^2(S)$ is birational to $\text{Hilb}^2(S^\tau)$ for all K3 surfaces S of degree d . We can generalise this using the following result by Addington. See also Remark 4.6.

THEOREM 4.2 ([1, Theorem 2]). *A cubic fourfold X lies in \mathcal{C}_d for some d satisfying*

$$a^2d = 2(n^2 + n + 1) \tag{***}$$

if and only if $F(X)$ is birational to $\text{Hilb}^2(S)$ for some K3 surface S .

Note that (***) implies (**).

LEMMA 4.3. *Suppose that d satisfies (***). Then there exists a choice of the rational map $\varphi: \mathcal{M}_d \dashrightarrow \mathcal{C}_d$ such that if $(S, L) \in \mathcal{M}_d$ is associated to $X \in \mathcal{C}_d$ via φ , then $\text{Hilb}^2(S)$ and $F(X)$ are birational.*

Proof. Consider the sublattices

$$K_d^\perp \oplus T \subset \widetilde{\Lambda}_{K3} \supset \Lambda_d \oplus \mathbb{Z}\ell_d \oplus U_4,$$

where $T \supset A_2 = \langle \lambda_1, \lambda_2 \rangle$ is the orthogonal complement of K/d in $\widetilde{\Lambda}_{K3}$. Then d satisfies (**) if and only if $T \cong \mathbb{Z}\ell_d \oplus U_4$. Addington showed that (***) holds if and only if $\psi: T \rightarrow \mathbb{Z}\ell_d \oplus U_4$ can be chosen such that $\psi(\lambda_1) = e_4 + f_4$. Extend ψ to an element of $O(\widetilde{\Lambda}_{K3})$ (use Lemma 1.1 and [11, Theorem 14.2.4]) and let φ be the induced map $\mathcal{M}_d \dashrightarrow \mathcal{C}_d$.

Assume that $(S, L) \in \mathcal{M}_d$ is associated to $X \in \mathcal{C}_d$ via φ . Choose an isomorphism $H^2(S, \mathbb{Z}) \cong U_4^\perp \subset \widetilde{\Lambda}_{K3}$ sending L to ℓ_d , and consider the induced Hodge structure on $\widetilde{\Lambda}_{K3}$. There are isometries of sub-Hodge structures

$$H^2(F(X), \mathbb{Z}) \cong \lambda_1^\perp \cong \psi(\lambda_1)^\perp = (e_4 + f_4)^\perp \cong H^2(M_S(1, 0, -1), \mathbb{Z}),$$

where the sign in $M_S(1, 0, -1)$ appears because we view the Mukai vector as an element of Λ_{Muk} . By Markman’s birational Torelli theorem for manifolds of $K3^{[n]}$ type [15, Corollary 9.9], $F(X)$ is birational to $M_S(1, 0, -1) \cong \text{Hilb}^2(S)$.

COROLLARY 4.4. *When $d \equiv 0 \pmod 6$ satisfies (***), then $\text{Hilb}^2(S) \sim_{\text{bir}} \text{Hilb}^2(S^\tau)$ for any K3 surface $(S, L) \in \mathcal{M}_d$.*

The following proposition shows that we have more than just birationality: if d is such that $\text{Hilb}^2(S) \sim_{\text{bir}} \text{Hilb}^2(S^\tau)$, then for S generic, $\text{Hilb}^2(S)$ and $\text{Hilb}^2(S^\tau)$ are isomorphic.

PROPOSITION 4.5. *Let (S, L) be a polarised K3 surface of degree d with $\text{Pic}(S) = \mathbb{Z}L$ and $3|d$. Then $\text{Hilb}^2(S)$ has only one birational model.*

Proof. By [6, Theorem 5.1] the walls of the ample cone of $\text{Hilb}^2(S)$ in the interior of the movable cone are given by the hypersurfaces $x^\perp \subset \text{NS}(\text{Hilb}^2(S)) \otimes \mathbb{R}$ for all $x \in \text{NS}(\text{Hilb}^2(S))$ of square -10 and divisibility two. We will show that there are no such x .

There is an isomorphism

$$\text{NS}(\text{Hilb}^2(S)) \cong \text{NS}(S) \oplus \mathbb{Z}\delta = \mathbb{Z}L \oplus \mathbb{Z}\delta,$$

where δ is a (-2) -class orthogonal to L [5]. So any class in $\text{NS}(\text{Hilb}^2(S))$ is given by $aL + b\delta$ for some $a, b \in \mathbb{Z}$, and its square is $a^2d - 2b^2$. Setting this equal to -10 gives the Pell equation $b^2 - a^2d/2 = 5$ which, after reducing modulo 3, gives $b^2 \equiv 2 \pmod 3$. This is not possible.

It follows that under any birational map $\text{Hilb}^2(S) \dashrightarrow Y$ the pullback of an ample class is ample, thus the map is an isomorphism [7].

Remark 4.6. This also implies that when d satisfies (***) and $3|d$, then for a generic cubic fourfold X of discriminant d , $F(X)$ is actually isomorphic to $\text{Hilb}^2(S)$ for a K3 surface S associated to X . When $3 \nmid d$, Proposition 4.5 does not hold. For example, consider $d = 38$, which satisfies (***) but not the condition in Theorem 4.1. The element $2L + 9\delta$ has square -10 and divisibility $\text{gcd}(2, 18) = 2$. Hence, $\text{Hilb}^2(S)$ has more than one birational model.

It is natural to ask for the exact conditions on d for $\text{Hilb}^2(S)$ to be birational to $\text{Hilb}^2(S^\tau)$, for all S of degree d . It turns out that (***) is too strong. We use the following two results by [16].

PROPOSITION 4.7 ([16, Proposition 2.1]). *Let (S, L) be a polarised K3 surface satisfying $\text{Pic}(S) = \mathbb{Z}L$. Let $v = (x, cL, y)$ be a primitive isotropic Mukai vector such that $M_S(v)$ is a fine moduli space. Then $v = (p^2r, pqL, q^2s)$ for some integers p, r, q, s with $\text{gcd}(pr, qs) = 1$ and $(L)^2 = 2rs$, and there is an isomorphism $M_S(v) \cong M_S(r, L, s)$. Moreover, $M_S(r, L, s)$ is isomorphic to $M_S(r', L, s')$ if and only if $\{r, s\} = \{r', s'\}$.*

PROPOSITION 4.8 ([16, Theorem 2.2]). *Let S_1 and S_2 be derived equivalent K3 surfaces of Picard number one. Then $\text{Hilb}^n(S_1)$ and $\text{Hilb}^n(S_2)$ are birational if and only if S_2 is isomorphic to $M_{S_1}(p^2r, pqL, q^2s)$ for some p, q with $p^2r(n - 1) - q^2s = \pm 1$. Moreover, $\{r, s\}$ is uniquely determined by S_2 .*

Note that $p^2r(n - 1) - q^2s = \pm 1$ is equivalent to $((1, 0, 1 - n), (p^2r, pqL, q^2s)) = \pm 1$. So when (p^2r, pqL, q^2s) is primitive then $M_{S_1}(p^2r, pqL, q^2s)$ is a fine moduli space, isomorphic (also when $\rho(S_1) > 1$) to $M_{S_1}(r, L, s)$ by Proposition 4.7.

Our description of τ gave us $S^\tau = M_S(3, L, d/6)$, so $r = 3$ and $s = d/6$. Thus, Proposition 4.8 tells us that when $\rho(S) = 1$, then we have $\text{Hilb}^n(S) \sim_{\text{bir}} \text{Hilb}^n(S^\tau)$ if and only if there exist non-zero integers p, q such that $3p^2(n - 1) - (d/6)q^2 = \pm 1$. Note that $3p^2(n - 1) - (d/6)q^2 = 1$ does not happen in our case: since $d/6 \equiv 1 \pmod 3$, reducing modulo 3 gives $q^2 \equiv 2 \pmod 3$ which is not possible.

COROLLARY 4.9. *Suppose that $\rho(S) = 1$. Then $\text{Hilb}^2(S)$ and $\text{Hilb}^2(S^\tau)$ are birational if and only if there exists an integral solution to the equation*

$$3p^2 - (d/6)q^2 = -1. \tag{4.1}$$

Equivalently, $\text{Hilb}^2(S)$ admits a line bundle of degree 6.

Proof. A class $aL + b\delta$ in $\text{NS}(\text{Hilb}^2(S)) = \mathbb{Z}L \oplus \mathbb{Z}\delta$ has square $a^2d - 2b^2 = 6$, in particular $b = 3b_0$ for some b_0 , if and only if $3b_0^2 - (d/6)a^2 = -1$.

Note that when $\rho(S) > 1$, the existence of a solution to (4.1) still implies that $\text{Hilb}^2(S)$ and $\text{Hilb}^2(S^\tau)$ are birational and that $\text{Hilb}^2(S)$ admits a line bundle of degree 6.

Table I. Conditions (**), (***) and (4.1) for small values of d

$d/6$	d	(**)	(***)	(4.1)	$d/6$	d	(**)	(***)	(4.1)
1	6	x	x	x	52	312			
4	24			x	55	330			
7	42	x	x	x	58	348			
10	60				61	366	x	x	x
13	78	x		x	64	384			
16	96				67	402	x	x	x
19	114	x	x	x	70	420			
22	132				73	438	x		
25	150				76	456			x
28	168			x	79	474	x	x	x
31	186	x	x	x	82	492			
34	204				85	510			
37	222	x	x	x	88	528			
40	240				91	546	x	x	x
43	258	x	x	x	94	564			
46	276				97	582	x		
49	294	x		x	100	600			

Condition (***) implies that (4.1) is solvable. Namely, assume we have $a^2d/2 = n^2 + n + 1$. Multiplying with 4 gives $(2a)^2d/2 = (2n + 1)^2 + 3$. As d is divisible by 3, so is $2n + 1$, and we find that $3((2n + 1)/3)^2 - (2a)^2d/6 = -1$.

In fact, (***) is equivalent to the existence of a solution to (4.1) with p odd and q even. One can show that such a solution always exists when $d/6$ is a prime $m \equiv 3 \pmod{4}$. The following example shows that there exist d for which (4.1) is solvable but (***) does not hold.

Example 4.10. Let $d = 78$, which satisfies (**) but not (***) (see [1]). Equation (4.1) holds with $p = 2$ and $q = 1$. It follows that $\text{Hilb}^2(S) \sim_{\text{bir}} \text{Hilb}^2(S^\tau)$ for any S of degree 78. Moreover, when $\rho(S) = 1$, Proposition 4.5 tells us that $\text{Hilb}^2(S)$ and $\text{Hilb}^2(S^\tau)$ are isomorphic (even though S and S^τ are not).

Next, we give an example where (4.1) is not solvable. As S and S^τ are derived equivalent, so are $\text{Hilb}^n(S)$ and $\text{Hilb}^n(S^\tau)$ for all $n \geq 1$ [19, Proposition. 8]. Therefore, we obtain two derived equivalent Hilbert schemes of two points on K3 surfaces which are not birational. The first example of this phenomenon was given in [16, Example 2.5]. Note the similarity between S^τ and the K3 surface Y in [16, Proposition 1.2].

Example 4.11. Consider $d = 6 \cdot 73$. This again satisfies (**) but not (***). Note that (4.1) holds if and only if $(3p)^2 - (d/2)q^2 = -3$, which is equivalent to $x^2 - (d/2)y^2 = -3$ when 3 divides d . This is a usual Pell type equation and one can easily check (using e.g. [3, theorem 4.2.7]) that it has no solution for $d = 6 \cdot 73$. So for S generic of degree $6 \cdot 73$, $\text{Hilb}^2(S)$ is not birational to $\text{Hilb}^2(S^\tau)$.

In Table I we give an overview of which $d \leq 600$ with $d/6 \equiv 1 \pmod{3}$ satisfy the numerical conditions (**), (***) and (4.1) (see also [1]).

4.2. Higher-dimensional Hilbert schemes

We have seen that $\text{Hilb}^n(S)$ and $\text{Hilb}^n(S^\tau)$ are birational if and only if there is a solution to

$$3p^2(n-1) - (d/6)q^2 = -1. \quad (4.2)$$

We give some examples for low n .

$n = 3$. The lowest d satisfying $(**)$ and $6 \mid d$ is $d = 42$. Equation (4.2) with $n = 3$ reads $6p^2 - 7q^2 = -1$, which is solved by $p = q = 1$. In general, one can show that if $d/6$ is a prime $m \equiv 5, 7 \pmod{8}$, then $\text{Hilb}^3(S) \sim_{\text{bir}} \text{Hilb}^3(S^\tau)$.

$n = 4$. In this case, equation (4.2) reads $(3p)^2 - (d/6)q^2 = -1$. This is always solvable when $d/6$ is a prime $m \equiv 1 \pmod{4}$. Namely, note that when $m > 2$ is prime, then $x^2 - my^2 = -1$ has a solution if and only if $m \equiv 1 \pmod{4}$. Reducing this modulo 3 gives $x^2 - y^2 \equiv -1 \pmod{3}$. This implies that $x \equiv 0 \pmod{3}$. Writing $x = 3x'$ gives $9(x')^2 - (d/6)y^2 = -1$, i.e. (4.2) with $n = 4$.

$n = 5$. Equation (4.2) is given by $3(2p)^2 - (d/6)q^2 = -1$, which is a solution for (4.1). This shows that $\text{Hilb}^5(S) \sim_{\text{bir}} \text{Hilb}^5(S^\tau)$ implies $\text{Hilb}^2(S) \sim_{\text{bir}} \text{Hilb}^2(S^\tau)$.

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