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ON LOCAL FIELDS GENERATED BY DIVISION VALUES OF FORMAL DRINFELD MODULES

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Abstract. In this paper, we study some aspects of the local fields generated by division values of formal Drinfeld modules.

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1. Introduction. Let K be a local field of positive characteristic p. This means, in particular, that K is complete and locally compact with respect to a normalized discrete valuation v_K. Let us denote by $\mathcal{O}_K = \{x \in K| \quad v_K(x) \geq 0\}$ the valuation ring of v_K, and let $\mathfrak{p}_K$ be its maximal ideal. The residue field $\mathbb{F} := \mathcal{O}_K/\mathfrak{p}_K$ is finite of order some power q of p. Let Ω be a fixed algebraic closure of K and let v_Ω the unique extension of v_K to Ω. We denote by $(\overline{\Omega}, v_{\overline{\Omega}})$ the completion of (Ω, v_Ω). For any extension $F \subset \overline{\Omega}$ of K, we denote by v_F the restriction of $v_{\overline{\Omega}}$ to F. We also denote by $\mathcal{O}_F \subset F$ the valuation ring of v_F, and by $\mathfrak{p}_F$ the maximal ideal of $\mathcal{O}_F$. The completion of F in $\overline{\Omega}$ will be noted $\overline{F}$. In all this paper, π is a fixed prime of K. In other words $\pi \in K$, and we have $v_K(\pi) = 1$.

Let B be a commutative ring with unity. A one-dimensional formal group over B is, by definition, a formal series $F \in B[[X, Y]]$ such that $F(X, Y) = X + Y + \textit{terms of degree} \geq 2$, and $F(F(X, Y), Z) = F(X, F(Y, Z))$. A homomorphism from a formal group F into a formal group G is a formal power series $\beta \in XB[[X]]$ such that $\beta(F(X, Y)) = G(\beta(X), \beta(Y))$. We denote by $Hom(F, G)$ the set of all homomorphisms from F into G. The set $M = XB[[X]]$ forms an abelian group with respect to the addition $f +_G g = G(f(X), g(X))$, and $Hom(F, G)$ is a subgroup of M. If $G = F$, we usually write $End(F)$ instead of $Hom(F, F)$ and call it the set of endomorphisms of F. We recall that $End(F)$ is a ring with respect to the addition $f +_F g = F(f(X), g(X))$ and the composition $f \circ g$. Let $D : End(F) \longrightarrow B$ be the ring homomorphism that sends the endomorphism Φ to $\Phi'(0)$.

When B is an $\mathcal{O}_K$-algebra, with $\gamma : \mathcal{O}_K \longrightarrow B$ being the structure map, Drinfeld defined in [5, §1] a formal $\mathcal{O}_K$-module over B to be a pair (F, f) where F is a formal group over B and f is a homomorphism from $\mathcal{O}_K$ into $End(F)$ such that $D \circ f = \gamma$. If $G_a(X, Y) = X + Y$ is the additive group, then every element of $End(G_a)$ has the form $\sum_{i=0}^{\infty} b_i X^{p^i}$. Hence, We may identify $End(G_a)$ with the twisted power series ring $B\{\{\tau_p\}\}$ where τ_p is the p-Frobenius, that is the element satisfying the rule $\tau_p^n b = b^{p^n} \tau_p^n$, for any $b \in B$ and $n \in \mathbb{N}$. In this case, we view D as the map $B\{\{\tau_p\}\} \longrightarrow B$ that assigns to a power series $f = \sum_{n=0}^{\infty} b_n \tau_p^n$ its constant term b_0. Let us fix, once for all, $\mathbb{F}_0$ a subfield of $\mathcal{O}_K/\mathfrak{p}_K$ of order , say $q_0 := p^e$ and let $\tau := \tau_p^e$. Following Michael Rosen, cf. [8], we define a formal Drinfeld $\mathcal{O}_K$-module over B to be a ring homomorphism

$$\rho : \mathcal{O}_K \longrightarrow B\{\{\tau\}\}, \quad a \longmapsto \rho_a$$

such that

(1) for all $a \in \mathcal{O}_K$, $D(\rho_a) = \gamma(a)$.
(2) $\rho(\mathcal{O}_K) \not\subset B$ and $\rho_\pi \neq 0$.

If $f \in B\{\{\tau\}\}$ has the expansion $f = \sum_{i=0}^{\infty} c_n \tau^n$, then we denote by $ord_\tau(f)$ the least nonnegative integer n such that $c_n \neq 0$. We know by [8, Lemma 1.2] that for any formal Drinfeld $\mathcal{O}_K$-module ρ and any prime element of K, the integer $ord_\tau(\rho_\pi)$ (which is obviously independent of π) is divisible by the degree

$$d := [\mathbb{F} : \mathbb{F}_0].$$

By definition, the height of ρ is the integer $ht(\rho) = ord_\tau(\rho_\pi)/d$. Let $L \subset \overline{\Omega}$ be an extension of K. Then $\mathcal{O}_L$ and $\mathcal{O}_L/\mathfrak{p}_L$ are naturally commutative $\mathcal{O}_K$ algebras with unity. When $B = \mathcal{O}_L$, the structure map $\gamma : \mathcal{O}_K \longrightarrow \mathcal{O}_L$ will always be the inclusion map. For such a formal Drinfeld $\mathcal{O}_K$-module, we say that ρ has stable reduction if the ring homomorphism $\bar{\rho} : \mathcal{O}_K \longrightarrow \mathcal{O}_L/\mathfrak{p}_L\{\{\tau\}\}$ is a formal Drinfeld $\mathcal{O}_K$-module over $\mathcal{O}_L/\mathfrak{p}_L$.

Let $K_u^\infty \subset \Omega$ be the maximal unramified extension of K in Ω. For any positive integer m, we denote by $K_u^m \subset K_u^\infty$ the unramified extension of K of degree $[K_u^m : K] = m$. Let R^m (resp. R) be the set of all formal Drinfeld $\mathcal{O}_K$-modules over $\mathcal{O}_{K_u^m}$ (resp. $\mathcal{O}_{\overline{K_u^\infty}}$) for some positive integer m having stable reduction and such that $ht(\bar{\rho}) = 1$. We denote by R^∞ the union of R^m for all the integers $m > 0$. The purpose of this short article is to continue the exploration of the properties of formal Drinfeld modules begun by Michael Rosen in [8]. In particular, we focus on the questions raised at the end of section 3 of [8], concerning the Galois module structure of the torsion of formal Drinfeld modules belonging to R. Thus we want to study the fields generated by the sets

$$W_\rho^n = \{\alpha \in \mathfrak{p}_{\overline{\Omega}} \textit{ such that } \rho_a(\alpha) = 0, \textit{ for all } a \in \mathfrak{p}_K^n\}, \tag{1}$$

where $\rho \in \mathsf{R}$, and $n \geq 1$ is any positive integer. Our first result is the following:

THEOREM 1.1. *(Theorems 3.4 and 3.5) The field $K_u^\infty(W_\rho^n)$ does not depend on the formal $\mathcal{O}_K$-module $\rho \in \mathsf{R}^\infty$. Let W_ρ be the union of W_ρ^n for all $n \geq 1$. Then $K_u^\infty(W_\rho)$ is the maximal abelian extension K^{ab} of K in Ω.*

Proposition 3.1, which is a generalization of [7, Lemma 1], is an important ingredient in the proof of the above theorem. Let $\rho \in \mathsf{R}^m$ and let $H = K_u^m$. In Section 2, we used the Weierstrass preparation theorem [8, Theorem 3.2] to study the extensions $H_\rho^n = K_u^m(W_\rho^n)$. It is worthwhile to give further details about these local fields as abelian extensions of K in terms of the local class field theory. It is interesting to know how these fields vary as a function of ρ. What can we say about the group $N_{H_\rho^n/K}\big((H_\rho^n)^*\big)$?

In Section 4, we prove that the logarithm λ_ρ of ρ is such that $\lambda_\rho(X)$ is the limit in $H[[X]]_1$ of the sequence $\frac{\rho_{\pi^n}(X)}{\pi^n}$. Such a result was first proved for the Lubin–Tate logarithm by Wiles in [10, Lemma 3] in the case of local fields of characteristic zero. It was then generalized by Coleman to all characteristics in [4, Lemma 21].

In the last section, we define a trace operator $\mathcal{S}_{\rho,\pi}$ and a norm operator $\mathcal{N}_{\rho,\pi}$ similar to those defined by Coleman. It would be interesting to explore the connection between the image of the logarithm of ρ and the eigenspaces of $\mathcal{S}_\rho$, in the spirit of [4, Section VI].

When ρ satisfies a certain congruence, described in the theorem below, we are able to prove a theorem that may be considered as a generalization of [4, Theorem A]. It also generalizes [3, Theorem 11]. In the sequel, if R is a ring, then we note R^* the multiplicative group of R. We also let $\mathcal{O}_H((X)) = \mathcal{O}_H[[X]][\frac{1}{X}]$ be the ring of Laurent power series f such that $X^s f \in \mathcal{O}_H[[X]]$, for some non-negative integer s.

THEOREM 1.2. *(Theorem 5.8) Suppose we have*

(i) $\rho \in \mathsf{R}^{m_0}$ *for some rational integer* $m_0 > 0$ *dividing* m.
(ii) *There exists* $\eta \in \mathcal{O}_K$ *such that* $v_K(\eta) = m_0$ *and* $\rho_\eta \equiv \tau^{dm_0}$ *modulo* $\pi \mathcal{O}_{K_u^{m_0}}\{\{\tau\}\}$.

Let $u \in \mathcal{O}_K^*$ *be such that* $u\eta = \pi^{m_0}$. *Then define the operator* $\widetilde{\mathcal{N}}$ *by* $\widetilde{\mathcal{N}}(f) = \mathcal{N}_{\rho,\pi}^{m_0}(f) \circ \rho_u(X)$ *and set* $\widetilde{\varphi} = \varphi^{m_0}$, *where* φ *is the frobenius automorphism of* K_u^∞/K. *For any* $n \in \mathbb{N}^*$ *let* $E_n = K_u^m(W_\rho^{nm_0})$, *and define* $X_\infty = \varprojlim_n E_n^*$ *to be the projective limit of the multiplicative groups* E_n^* *with respect to the norm maps. Let* $\mathcal{M}_\infty = \{f \in \mathcal{O}_H((X))^*,\ \widetilde{\mathcal{N}}(f) = f^{\widetilde{\varphi}}\}$. *There exists a topological isomorphism*

$$ev_{\widetilde{\mathcal{N}}} : \mathcal{M}_\infty \longrightarrow X_\infty,$$

given by $ev_{\widetilde{\mathcal{N}}}(f) = \left(f^{\widetilde{\varphi}^{-n}}(\widetilde{v}_n)\right)_n$, *where the system* $(\widetilde{v}_n)_n$ *is defined in subsection 5.3.1*

The formal Drinfeld modules coming from rank 1 Drinfeld modules all satisfy the hypotheses of the above Theorem 1.2. See, for instance, [**3**]. Francesc Bars and Ignazio Longhi proved a reciprocity law for rank 1 Drinfeld modules, cf. [**3**, Theorem 24]. It is then natural to ask if any reciprocity laws can be formulated for the formal Drinfeld $\mathcal{O}_K$-modules satisfying the hypotheses of the above theorem.

2. The $\mathcal{O}_K$-modules W_ρ^n. Let $H = \overline{K_u^\infty}$ or $H = K_u^m$, for some m, so that H is complete for v_H and unramified above K. Let ρ be a formal Drinfeld $\mathcal{O}_K$-module over $\mathcal{O}_H$, having stable reduction and such that $ht(\bar{\rho}) = 1$. It is clear that W_ρ^n is a subgroup of $(\mathfrak{p}_{\overline{\Omega}}, +)$. It is even an $\mathcal{O}_K$-submodule of $\mathfrak{p}_{\overline{\Omega}}$ for the action

$$a \cdot_\rho \alpha = \rho_a(\alpha),\ \textit{for } a \in \mathcal{O}_K \textit{ and } \alpha \in \mathfrak{p}_{\overline{\Omega}}.$$

We also have

$$W_\rho^n = \{\alpha \in \mathfrak{p}_{\overline{\Omega}} \textit{ such that } \rho_{\pi^n}(\alpha) = 0\}.$$

LEMMA 2.1. *There exists* $(U_n)_{n\geq 1}$ *and* $(Q_n)_{n\geq 1}$ *two sequences of elements of* $\mathcal{O}_H\{\{\tau\}\}$ *uniquely determined by the following conditions:*

(1) Q_n *is a distinguished polynomial of degree* d *and* U_n *is a unit.*
(2) $\rho_\pi = U_1 Q_1$ *and* $Q_1 U_1 = U_2 Q_2$.
(3) $Q_1 U_1 U_{n-1} = U_n Q_n$, *for all* $n > 2$. *Moreover, we have*

$$\rho_{\pi^n} = U_1 U_n Q_n Q_{n-1} \cdots Q_2 Q_1.$$

Proof. We simply apply the Weierstrass preparation theorem [**8**, Theorem 3.2] first to ρ_π, then to $Q_1 U_1$ and inductively to $Q_1 U_1 U_{n-1}$. The degree of Q_1 is d because by hypothesis $ht(\bar{\rho}) = 1$. This gives the degree of the polynomials Q_i. The last equality is easy to check. □

By the above lemma 2.1, we see that

$$Q_n = \tau^d + c_{d-1}^{(n)} \tau^{d-1} + \cdots + c_1^{(n)} \tau + \pi_n,$$

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where π_n is a prime of H, and the other coefficients $c_i^{(n)}$ are in $\mathfrak{p}_H$. Hence, $P_n = Q_nQ_{n-1}\cdots Q_2Q_1$ is a distinguished polynomial of degree nd. Therefore, $P_n(X)$ is a monic separable polynomial of degree q^n, with coefficients in $\mathcal{O}_H[X]$ and such that $P_n(0)=0$. The $\mathcal{O}_K$-module W_ρ^n is the set of all roots of $P_n(X)$ in Ω. In particular, the order of W_ρ^n is q^n and $H(W_\rho^n)$ is a finite Galois extension of H. Let $P_0=\tau^0$, so that $P_0(X)=X$ and

$$h_n(X)=\frac{P_n(X)}{P_{n-1}(X)}=\frac{Q_n(P_{n-1}(X))}{P_{n-1}(X)}, \quad \textit{for all } n\geq 1, \tag{2}$$

is an Eisenstein polynomial in $\mathcal{O}_H[X]$ of degree $(q-1)q^{n-1}$. Moreover, we have the equality

$$P_n(X)=h_n(X)h_{n-1}(X)\cdots h_2(X)h_1(X)X. \tag{3}$$

COROLLARY 2.2. *The sequences of polynomials $(P_n(X))_{n\geq 0}$ and $(h_n(X))_{n\geq 0}$, and, in particular, the sequence $(\pi_n)_{n\geq 0}$ of prime elements of H depend only on ρ (and not on π). We will denote them P_n^ρ, h_n^ρ and π_n^ρ, respectively.*

Proof. On one hand, we have $P_n(X)=\prod_{w\in W_\rho^n}(X-w)$ and $h_n(X)=P_n(X)/P_{n-1}(X)$, and on the other hand, by its very definition (1), the set W_ρ^n depends only on ρ (and not on π). This clearly shows that for any n, the polynomials $P_n(X)$ and $h_n(X)$ and the prime element π_n of H depend only on ρ. □

REMARK 2.3. It is also true that the polynomial (Q_1) does not depend on π. Indeed, if π' is an other prime element of K, then there exists u a unit of $\mathcal{O}_K$ such that $\pi'=u\pi$. Thus we have $\rho_{\pi'}=\rho_u U_1Q_1$. Since ρ_u is a unit in $\mathcal{O}_H\{\{\tau\}\}$, the decomposition of $\rho_{\pi'}$ as a product $U_1'Q_1'$ given by the Weierstrass preparation theorem [8, Theorem 3.2] corresponds to $U_1'=\rho_uU_1$ and $Q_1'=Q_1$.

Let us also remark that

$$P_n^\rho(X+Y)=P_n^\rho(X)+P_n^\rho(Y). \tag{4}$$

PROPOSITION 2.4. *The set $W_\rho^n-W_\rho^{n-1}$ is the set of roots of $h_n(X)$. If $\alpha_0\in W_\rho^n-W_\rho^{n-1}$, then the degree $[H(\alpha_0):H]=(q-1)q^{n-1}$ and $N_{H(\alpha_0)/H}(\alpha_0)=\pi_n$.*

Proof. We have seen that $h_n(X)$ divides $P_n(X)$ and that W_ρ^n is the set of roots of $P_n(X)$. Hence, If $\alpha_0\in\Omega$ is such that $h_n(\alpha_0)=0$, we automatically have $\alpha_0\in W_\rho^n$. To say that $\alpha_0\in W_\rho^{n-1}$ means that α_0 is a root of P_{n-1}. The equality $P_n(X)=h_n(X)P_{n-1}(X)$ given by (3) would imply that α_0 is a multiple root of $P_n(X)$. But we know that P_n is separable. We deduce that $\alpha_0\in W_\rho^n-W_\rho^{n-1}$. Since $W_\rho^n-W_\rho^{n-1}$ has exactly $(q-1)q^{n-1}$ elements, we conclude that this set is the set of all roots of $h_n(X)$. Finally, since the polynomial $h_n(X)$ is Eisenstein and $h_n(0)=\pi_n$, we deduce that the degree of $[H(\alpha_0):H]=(q-1)q^{n-1}$ and the norm $N_{H(\alpha_0)/H}(\alpha_0)=(-1)^{\deg h_n}h_n(0)=\pi_n$ since $(-1)^{\deg h_n}=1$ in all cases. □

Fix an element $\alpha_0\in W_\rho^n$, $\alpha_0\notin W_\rho^{n-1}$. Then the kernel of the map $a\longmapsto\rho_a(\alpha_0)$ is equal to $\mathfrak{p}_K^n$, and therefore, induces an isomorphism

$$\mathcal{O}_K/\mathfrak{p}_K^n\simeq W_\rho^n. \tag{5}$$

This implies that $W_\rho^n=\mathcal{O}_K\cdot_\rho\alpha_0$. Let $End_\rho(W_\rho^n)$ the ring of all endomorphisms of the $\mathcal{O}_K$-module W_ρ^n. To any $a\in\mathcal{O}_K$, we associate an element ε_ρ^a of $End_\rho(W_\rho^n)$ such that $\varepsilon_\rho^a(\beta)=\rho_a(\beta)$. This defines a ring homomorphism $\varepsilon_\rho:\mathcal{O}_K\longrightarrow End_\rho(W_\rho^n)$ which is onto thanks to the equality $W_\rho^n=\mathcal{O}_K\cdot_\rho\alpha_0$. It is easy to see that its kernel is $\mathfrak{p}_K^n$. Hence, we obtain

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a ring isomorphism

$$\mathcal{O}_K/\mathfrak{p}_K^n \simeq End_\rho(W_\rho^n).$$

We also have a group homomorphism $U_K \longrightarrow Aut_\rho(W_\rho^n)$, where U_K is the group of units of $\mathcal{O}_K$ and $Aut_\rho(W_\rho^n)$ is the multiplicative group of all invertible elements of $End_\rho(W_\rho^n)$. We deduce from the above isomorphism the following one:

$$U_K/U_K^{(n)} \simeq Aut_\rho(W_\rho^n), \tag{6}$$

where $U_K^{(n)} = 1 + \mathfrak{p}_K^n$. Let $H_\rho^n = H(W_\rho^n)$ and let $\sigma \in Gal(H_\rho^n/H)$. Since σ is continuous, we have $\sigma(\rho_a(\alpha)) = \rho_a(\sigma(\alpha))$, for all $a \in \mathcal{O}_K$ and $\alpha \in W_\rho^n$. Thus σ induces an element σ' of $Aut_\rho(W_\rho^n)$. This correspondence is an injective group homomorphism $\psi : Gal(H_\rho^n/H) \longrightarrow Aut_\rho(W_\rho^n)$. But Proposition 2.4 and the isomorphism (6) show that $Aut_\rho(W_\rho^n)$ has order at most equal to $[H_\rho^n : H]$. Hence, ψ is actually an isomorphism. Let us summarize the above discussion in the following:

PROPOSITION 2.5. *There exists a surjective group homomorphism*

$$\delta_\rho^n : U_K \longrightarrow Gal(H_\rho^n/H),$$

such that $\delta_\rho^n(u)(\alpha) = \rho_u(\alpha)$, for all $u \in U_K$ and all $\alpha \in W_\rho^n$. The kernel of δ_ρ^n is equal to $U_K^{(n)}$. This gives the equality

$$[H_\rho^n : H] = (q-1)q^{n-1}. \tag{7}$$

Let H_ρ be the union of the fields H_ρ^n, $n \geq 1$. Then the projective limit of the morphisms δ_ρ^n gives a topological isomorphism

$$\delta_\rho : U_K \longrightarrow Gal(H_\rho/H).$$

COROLLARY 2.6. *The extension H_ρ^n/H is totally ramified and $\pi_n^\rho \in N(H_\rho^n/H)$.*

Proof. We have $H_\rho^n = H(\alpha_0)$ for any $\alpha_0 \in W_\rho^n - W_\rho^{n-1}$. By Corollary 2.4 the norm group $N(H_\rho/H)$ contains π_n^ρ, which is a prime element of H. This implies that H_ρ^n/H is totally ramified □

3. The fields $\overline{K_u^\infty}(W_\rho^n)$ and $K_u^\infty(W_\rho^n)$. The Galois group $Gal(K_u^\infty/K)$ is topologically generated by the Frobenius automorphism φ. The extension of φ to $\overline{K_u^\infty}$ will be noted by φ too. In this section, we want to study the dependance of the fields $\overline{K_u^\infty}(W_\rho^n)$ on $\rho \in \mathsf{R}$, and also the dependance of $K_u^\infty(W_\rho^n)$, when ρ varies inside R^∞. We use the following two propositions. The first one generalizes [7, Lemma 1] and [6, Proposition 3.12]. The second proposition is [6, Lemma 3.11].

PROPOSITION 3.1. *Let $F \subset K_u^\infty$ be an extension of K, and suppose that $[F : K] < \infty$ or $F = K_u^\infty$. Let a and b be elements of $\mathcal{O}_{\overline{F}}$ such that $v_{\overline{F}}(a) = v_{\overline{F}}(b) = t > 0$. Let f_1 and f_2 be power series in $\mathcal{O}_{\overline{F}}[[X]]$ such that*

$$f_1(X) \equiv aX, \quad f_2(X) \equiv bX \mod \deg 2 \quad \textit{and} \quad f_1(X) \equiv f_2(X) \equiv X^{q^t} \mod \mathfrak{p}_{\overline{F}}.$$

Let m be a positive integer and let $\alpha_1, \ldots, \alpha_m$ be elements of $\mathcal{O}_{\overline{F}}$ such that $\alpha_i^{\varphi^t - 1} = a/b$ for all (i) Then there exists a unique power series $\theta(X_1, \ldots, X_m) \in \mathcal{O}_{\overline{F}}[[X_1, \ldots, X_m]]$ such that

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$$\theta \equiv \alpha_1 X_1 + \cdots + \alpha_m X_m \mod \deg 2 \quad and \quad f_1 \circ \theta = \theta^{\varphi^t} \circ f_2,$$

where $\theta^{\varphi^t} \in \mathcal{O}_{\overline{F}}[[X_1, \ldots, X_m]]$ *is the power series whose coefficients are obtained by applying the automorphism* φ^t *to the coefficients of* θ. *We recall the definition* $(\theta^{\varphi^t} \circ f_2)(X_1, \ldots, X_m) = \theta^{\varphi^t}(f_2(X_1), \ldots, f_2(X_m))$.

Proof. This is the generalization to every $t \geq 1$ of [6, Proposition 3.12] proved for $t = 1$. Since Iwasawa's proof can easily be adapted to the general case, we omit the details here. We point out that [6, Proposition 3.12] generalizes [7, Lemma 1]. □

PROPOSITION 3.2. *Let t be a positive integer. Then the following sequences are exact:*

$$0 \longrightarrow \mathcal{O}_{K_u^t} \longrightarrow \mathcal{O}_{\overline{K_u^\infty}} \xrightarrow{\varphi^t - 1} \mathcal{O}_{\overline{K_u^\infty}} \longrightarrow 0$$

$$1 \longrightarrow U_{K_u^t} \longrightarrow U_{\overline{K_u^\infty}} \xrightarrow{\varphi^t - 1} U_{\overline{K_u^\infty}} \longrightarrow 1.$$

Proof. This is [6, Lemma 3.11]. □

Let $\rho, \rho' \in \mathsf{R}$ be two formal Drinfeld $\mathcal{O}_K$-modules. Since the elements π_i^ρ and $\pi_i^{\rho'}$ are all primes of $\overline{K_u^\infty}$ the above Proposition 3.2 implies that for any integer $n \geq 1$, there exists a unit $\eta_n \in U_{\overline{K_u^\infty}}$ such that

$$\pi_1^{\rho'} \cdots \pi_n^{\rho'} = \eta_n^{\varphi^n - 1} \pi_1^\rho \cdots \pi_n^\rho. \tag{8}$$

Let us apply Proposition 3.1 to $f_1 = P_n^{\rho'}, f_2 = P_n^\rho$ and $m = 1$. We then deduce the existence of a unique power series $\theta_n \in \mathcal{O}_{\overline{K_u^\infty}}[[X]]$ such that

$$\theta_n(X) \equiv \eta_n X \mod \deg 2 \quad and \quad P_n^{\rho'} \circ \theta_n = \theta_n^{\varphi^n} \circ P_n^\rho. \tag{9}$$

The power series θ_n also satisfies the additivity property $\theta_n(X + Y) = \theta_n(X) + \theta_n(Y)$. Indeed, let $M(X, Y) = \theta_n(X + Y)$ and $N(X, Y) = \theta_n(X) + \theta_n(Y)$. Then, by (9), we certainly have $M(X, Y) \equiv N(X, Y) \equiv \eta_n(X + Y) \mod \deg 2$ and $P_n^{\rho'} \circ M = M^{\varphi^n} \circ P_n^\rho$. Moreover, since we have $P_n^{\rho'}(\theta_n(X) + \theta_n(Y)) = P_n^{\rho'}(\theta_n(X)) + P_n^{\rho'}(\theta_n(Y))$, thanks to (4), we deduce by also using (9) the equality $P_n^{\rho'} \circ N = N^{\varphi^n} \circ P_n^\rho$. The unicity assertion in Proposition 3.1 implies that $M = N$. Consequently, we have an isomorphism

$$\theta_n : W_\rho^n \longrightarrow W_{\rho'}^n \tag{10}$$

of $\mathbb{F}_q$-vector spaces.

COROLLARY 3.3. *The field* $\overline{K_u^\infty}(W_\rho^n)$ *does not depend on the formal Drinfeld* $\mathcal{O}_K$-*module* $\rho \in \mathsf{R}$.

Proof. Let ρ and ρ' be as above. It suffices to check the inclusion $\overline{K_u^\infty}(W_{\rho'}^n) \subset \overline{K_u^\infty}(W_\rho^n)$. By the isomorphism (10) it suffices to prove that for any $\alpha \in W_\rho^n$, the element $\theta_n(\alpha) \in \overline{K_u^\infty}(W_\rho^n)$. Let us remark that $\theta_n(X)$ has coefficients in $\overline{K_u^\infty}$. In particular, the series $\theta_n(\alpha)$ is a limit of elements in $\overline{K_u^\infty}(W_\rho^n)$. Since $\overline{K_u^\infty}(W_\rho^n)$ is complete, we deduce that $\theta_n(\alpha) \in \overline{K_u^\infty}(W_\rho^n)$. This completes the proof. □

THEOREM 3.4. *The field* $K_u^\infty(W_\rho^n)$ *does not depend on the formal Drinfeld* $\mathcal{O}_K$-*module* $\rho \in \mathsf{R}^\infty$. *It is an abelian extension of K, and* $Gal(K_u^\infty(W_\rho^n)/K)$ *is isomorphic to* $Gal(K_u^\infty/K) \times U_K/U_K^{(n)}$.

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Proof. Let $\rho, \rho' \in \mathsf{R}^\infty$ and let $E = K_u^\infty(W_\rho^n)$ and $E' = K_u^\infty(W_{\rho'}^n)$. By the above Corollary 3.3, we have $\overline{E} = \overline{E'}$. By [6, Lemma 3.1], since the compositum $E(E')$ is a finite separable extension of both E and E', we have

$$E = \overline{E} \cap E(E') = \overline{E'} \cap E(E') = E'.$$

Furthermore, if $\rho' \in \mathsf{R}^1$, the extension $K(W_{\rho'}^n)/K$ is abelian thanks to Proposition 2.5. Now $K_u^\infty(W_\rho^n) = K_u^\infty(W_{\rho'}^n) = K_u^\infty K(W_{\rho'}^n)$ is a compositum of abelian extensions of K. Since $K(W_{\rho'}^n)$ is a totally ramified extension of K, thanks to Corollary 2.6, we have $K(W_{\rho'}^n) \cap K_u^\infty = K$. This gives the isomorphism

$$Gal(K_u^\infty(W_{\rho'}^n)/K) \simeq Gal(K_u^\infty/K) \times Gal(K(W_{\rho'}^n)/K) \simeq Gal(K_u^\infty/K) \times U_K/U_K^{(n)}.$$

The last isomorphism is a direct consequence of Proposition 2.5. □

THEOREM 3.5. *Let $\rho \in \mathsf{R}^\infty$, and let W_ρ be the union of W_ρ^n for all $n \geq 1$. Then $K_u^\infty(W_\rho)$ is the maximal abelian extension K^{ab} of K in Ω.*

Proof. Let $f(X) = X^q + \pi X$, then by Lubin–Tate theory, we know that there exists an injective ring homomorphism $\mathcal{O}_K \longrightarrow End(G_a)$ which associates to $a \in \mathcal{O}_K$ a unique power series $[a]_f$ such that

$$[a]_f(X) \equiv aX \text{ modulo } \deg 2 \quad \text{and} \quad f \circ [a]_f = [a]_f^\varphi \circ f.$$

The reader is invited to consult [6, Chapter IV] for more details. It follows from remark (3.13) at the end of [6, Chapter III] that $[a]_f \in \mathcal{O}_K[[X]]$. Moreover, we easily check that the power series $[a]_f$ has the form $aX + \sum_{i=1}^\infty a_i X^{q^i}$. From this, we deduce the existence of a formal Drinfeld $\mathcal{O}_K$-module $\psi : \mathcal{O}_K \longrightarrow \mathcal{O}_K\{\{\tau\}\}$ such that $\psi_\pi = \tau^d + \pi$ and $\psi_a = a + \sum_{i=1}^\infty a_i \tau^{di}$. In fact, $\psi \in \mathsf{R}^1$. It is obvious that $K(W_\psi^n)$ is the field denoted $L_{\pi,n}$ by Lubin and Tate in [7]. This field is denoted $K_\pi^n = K_\pi^{1,n}$ in [6, page 66 and page 69]. Thus, $K_u^\infty(W_\psi) = K_u^\infty L_\pi = K^{ab}$ by [7, Corollary]. We conclude by using Theorem 3.4. □

REMARK 3.6. The reader can easily check that for any $\rho \in \mathsf{R}^\infty$, the field $K_u^\infty(W_\rho^n)$ is equal to the field denoted L^n in [6, page 66].

4. The logarithm of a formal Drinfeld $\mathcal{O}_K$-module. Let $L \subset \overline{\Omega}$ be an extension of K. Let $L((X))$ be the field of fractions of $L[[X]]$, that is the field of Laurent power series f such that $X^n f \in L[[X]]$, for some nonnegative integer n. Let $L((X))_1$ be the subset of $L((X))$ whose elements are power series convergent on $B' := \mathfrak{p}_{\overline{\Omega}} - \{0\}$, and we let $L[[X]]_1 := L((X))_1 \cap L[[X]]$. Let us endow $L((X))_1$ with the compact-open topology which we denote by $\mathcal{T}$. A sub-basis of $\mathcal{T}$ is given by the sets

$$S_L(\mathcal{C}, U) = \{f \in L((X)) \text{ such that } f(\mathcal{C}) \subset U\},$$

where $\mathcal{C}$ is any compact of B' and U is any open set in $\overline{\Omega}$. It is clear that $\mathcal{T}$ is the topology of uniform convergence on any compact of B'.

In this section, we show that the logarithm of ρ where $\rho \in \mathsf{R}^\infty$ is the limit in $H[[X]]_1$ of the sequence $\frac{\rho_{\pi^n}(X)}{\pi^n}$. We recall that the logarithm of ρ is introduced in [8, Proposition 2.1] for any ρ. The construction we give here is inspired by [10, Lemma 3] and also by [4, Lemma 21 (i)]. If $a \in \overline{\Omega}$, we set $|a| = q^{-v_{\overline{\Omega}}(a)}$.

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Proposition 4.1. *Let $\rho \in \mathsf{R}^m$ for some positive integer m and let $H = K_u^m$. Then the sequence of power series $\frac{\rho_{\pi^n}(X)}{\pi^n}$ converges in $H[[X]]_1$. The limit λ_ρ belongs to $H\{\{\tau\}\}$ and satisfies $D(\lambda_\rho) = 1$.*

Proof. Let us first observe that $H[[X]]_1$ is complete for the compact-open topology. Indeed, if $(f_n(X))_{n\in\mathbb{N}} \in H[[X]]_1$ is a Cauchy sequence, then $(f_n(X))_{n\in\mathbb{N}}$ is a Cauchy sequence for the topology of uniform convergence on any closed ball $D \subset B'$. By [1, Théorème 4.1.6 and Lemme 4.1.8] the set

$$A(D) = \{f \in \overline{\Omega}[[X]] \text{ such that } f \text{ converges on } D\}$$

is a Banach space for the norm of uniform convergence on D. Hence, there exists a formal power series $f_D \in A(D)$ such that $(f_n(X))_{n\in\mathbb{N}}$ converges uniformly to f_D on the closed ball D. Since the individual coefficients of $(f_n(X))_{n\in\mathbb{N}}$ converge to those of f_D and since H is complete, we see that $f_D \in H[[X]]$ and does not depend on D. This limit $f = f_D$ converges on D for any D. This means that $f \in H[[X]]_1$ and proves that $H[[X]]_1$ is complete. Therefore, to prove the proposition, we only have to check that $\frac{\rho_{\pi^n}(X)}{\pi^n}$ is a Cauchy sequence. For this, we adapt the proofs of [4, Lemma 20 (ii) and Lemma 21 (i)]. First we observe that for any $b \in B'$, we have

$$|\rho_\pi(b)| \leq \max(|\pi b|, |b^q|).$$

Indeed, we have $\rho_\pi(X) = \pi X + \sum_{i=1}^{\infty} \alpha_i X^{q_0^i}$, where the coefficients α_i are in $\mathcal{O}_H$, $|\alpha_d| = 1$ and $|\alpha_i| \leq |\pi|$ for any $i \in \{1, \ldots, d-1\}$. This implies

$$|\pi b + \alpha_1 b^{q_0} + \cdots + \alpha_{d-1} b^{q_0^{d-1}}| = |\pi b| \quad \text{and} \quad |\sum_{i=d}^{\infty} \alpha_i b^{q_0^i}| = |b^{q_0^d}| = |b^q|.$$

By arguing as in [4, page 107], we deduce that for any positive real number $R < 1$ there exists a constant C_R such that

$$|\rho_{\pi^n}(b)| < |\pi^n| C_R, \tag{11}$$

for all $n \geq 1$ and all b such that $|b| \leq R$. Now, for positive integers $m > n$, we have

$$\frac{\rho_{\pi^m}(X)}{\pi^m} - \frac{\rho_{\pi^n}(X)}{\pi^n} = \pi^{-n} f_{m,n} \circ \rho_{\pi^n}(X),$$

where $f_{m,n}(X) = \frac{\rho_{\pi^{m-n}}(X)}{\pi^{m-n}} - X$. It is clear that $f_{m,n}(X) = X^2 h_{m,n}(X)$, with $h_{m,n}(X) \in H[[X]]$. We deduce that if $|b| \leq R$ then

$$|\pi^{-n} f_{m,n}\big(\rho_{\pi^n}(b)\big)| = |\pi|^{-n} |\rho_{\pi^n}(b)|^2 |h_{m,n}(\rho_{\pi^n}(b))| \leq |\pi|^n C_R^2 |h_{m,n}(\rho_{\pi^n}(b))|,$$

thanks to (11). On the other hand, since $|\rho_{\pi^n}(b)| \leq |b|$, we have for $|b| \leq R$

$$|h_{m,n}(\rho_{\pi^n}(b))| \leq \sup_{|x|\leq R} \frac{|f_{m,n}(x)|}{|x^2|} = \sup_{|x|=R} \frac{|f_{m,n}(x)|}{|x^2|} \leq \frac{C_R + R}{R^2},$$

thanks to (11). The equality in the middle is the maximum principle satisfied by the elements of $H((X))_1$ for any H. We deduce from above that our sequence $\frac{\rho_{\pi^n}(X)}{\pi^n}$ is a Cauchy sequence and hence is convergent to, say λ_ρ. Since the coefficients of the power series $\frac{\rho_{\pi^n}(X)}{\pi^n}$ converge in H to the coefficients of λ_ρ, we deduce that $\lambda_\rho \in H\{\{\tau\}\}$. □

The following properties of λ_ρ are already proved in [8, proposition 2.2], but one may also deduce them from the above proposition.

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(1) $\lambda_\rho \circ \rho_a = a\lambda_\rho$, for all $a \in \mathcal{O}_K$.
(2) The power series λ_ρ belongs to $H[[X]]_1$.
(3) The kernel of λ_ρ in B' is equal to $W_\rho \setminus \{0\}$.
We recall that λ_ρ is called the logarithm of ρ because of property (1).

5. The trace and norm operators of Coleman. In this section, we fix a formal Drinfeld $\mathcal{O}_K$-module $\rho \in \mathsf{R}^m$, where m is a positive integer. In our situation also, there is a trace operator $\mathcal{S}_{\rho,\pi}$ and a norm operator $\mathcal{N}_{\rho,\pi}$ analogous to the trace and norm defined by Coleman in [4]. Since our construction is strongly inspired by Coleman's approach, we will only give an outline of the construction. Let $G_{\infty,\rho}$ be the Galois group of $H(W_\rho)/H$, where $H = K_u^m$. Let

$$\kappa : G_{\infty,\rho} \longrightarrow U_K,$$

be the inverse of the isomorphism δ_ρ defined in Proposition 2.5. let $\Lambda_H = \mathcal{O}_H[[G_{\infty,\rho}]]$ be the Iwasawa algebra of $G_{\infty,\rho}$ over $\mathcal{O}_H$, that is,

$$\Lambda_H = \varprojlim_n \mathcal{O}_H[G_{n,\rho}],$$

where $G_{n,\rho} = Gal(H(W_\rho^n)/H)$. The proof of [4, Theorem 1] is still valid to check that $H((X))_1$ has a unique structure of Λ_H-module such that

$$\sigma.f = f \circ \rho_{\kappa(\sigma)},$$

for any $\sigma \in G_{\infty,\rho}$ and any $f \in H((X))_1$. This action is continuous. Furthermore, exactly as in [4, Lemma 3], if $f \in \mathcal{O}_H[[X]]$ is such that $f(X+w) = f(X)$ for any $w \in W_\rho^1$, then there exists a unique $g \in \mathcal{O}_H[[X]]$ such that

$$f = g \circ \rho_\pi.$$

5.1. The trace operator.

PROPOSITION 5.1. *There exist a unique map $\mathcal{S}_{\rho,\pi} : H((X))_1 \longrightarrow H((X))_1$ such that*

$$\mathcal{S}_{\rho,\pi}(f) \circ \rho_\pi(X) = \sum_{u \in W_\rho^1} f(X+u)$$

The map $\mathcal{S}_{\rho,\pi}$ is a continuous Λ_H-endomorphism of $H((X))_1$.

Proof. See [4, Theorem 4] or [3, Theorem 7]. □

Let us remark that for any $m \geq n$ the map $\alpha \longmapsto \rho_{\pi^{m-n}}(\alpha)$ induces a surjective homomorphism of $\mathcal{O}_K$-modules $W_\rho^m \longrightarrow W_\rho^n$. The inverse limit $\varprojlim_n W_\rho^n$ with respect to these maps is easily seen to be isomorphic to $\mathcal{O}_K$, thanks to (5). We fix a generator $(v_n)_n$ of $\varprojlim_n W_\rho^n$ as an $\mathcal{O}_K$-module. In particular, we have $\rho_\pi(v_{n+1}) = v_n$ and $W_\rho^n = \{\rho_a(v_n),\ a \in \mathcal{O}_K\} = \mathcal{O}_K \cdot_\rho v_n$. Moreover, v_n is a prime of H_ρ^n. Thus the maximal ideal $\mathfrak{p}_{H_\rho^n} = v_n\mathcal{O}_{H_\rho^n}$.

REMARK 5.2. The following properties of $\mathcal{S}_{\rho,\pi}$ are easy to check. The reader may also consult [4, Corollary 5 (i)] and [4, Lemma 6].

(1) $\mathcal{S}_{\rho,\pi}^n(f) \circ \rho_{\pi^n}(X) = \sum_{u \in W_\rho^n} f(X+u)$.

(2) $\mathcal{S}_{\rho,\pi}(f)(v_n) = T_{n+1,n}(f(v_{n+1}))$, where $T_{n+1,n}$ is the trace map from H_ρ^{n+1} to H_ρ^n.

(3) If $f \in \mathcal{O}_H((X))$ then $\mathcal{S}_{\rho,\pi}^n(f) \equiv 0$ modulo $\pi^n\mathcal{O}_H((X))$.

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Let $\mathcal{F}_\rho$ be the set of $G_{\infty,\rho}$-equivariant maps $f : W'_\rho \longrightarrow H(W_\rho)$, where W_ρ is the union of W_ρ^n for all $n \geq 1$ and $W'_\rho = W \setminus \{0\}$. Then $\mathcal{F}_\rho$ is naturally a Λ_H-module, where the action is given by

$$(\lambda.f)(w) = \lambda(f(w)), \text{ for any } f \in \mathcal{F}_\rho,\, w \in W'_\rho \text{ and } \lambda \in \Lambda_H.$$

Moreover, any power series $f \in H((X))_1$ defines an element $\Phi(f) \in \mathcal{F}_\rho$. This gives a Λ_H-homomorphism

$$\Phi : H((X))_1 \longrightarrow \mathcal{F}_\rho.$$

We immediately deduce from [4, Lemma 2a] that $\ker(\Phi) \cap \mathcal{O}_H((X)) = \{0\}$. To compute the image of $\mathcal{O}_H((X))$ by Φ, we need some preliminary remarks and results. Let $h \in \mathcal{F}_\rho$. Since h is $G_{\infty,\rho}$-equivariant, we have

$$\sum_{\substack{w \in W_\rho^n \\ w \neq 0}} h(w) = \sum_{i=1}^{n} T_i(h(v_i)),$$

where T_i is the trace map from H_ρ^i to H. Let us explain this equality. We have

$$\sum_{\substack{w \in W_\rho^n \\ w \neq 0}} h(w) = \sum_{i=1}^{n} \sum_{w \in W_\rho^i \setminus W_\rho^{i-1}} h(w).$$

But the elements of $W_\rho^i \setminus W_\rho^{i-1}$ are the roots of the irreducible polynomial $h_i(X)$ defined in (2). Hence the elements of $W_\rho^i \setminus W_\rho^{i-1}$ are the conjugates over H of v_i. This implies

$$\sum_{w \in W_\rho^i \setminus W_\rho^{i-1}} h(w) = \sum_{\sigma \in Gal(H_\rho^i/H)} h(v_i^\sigma) = \sum_{\sigma \in Gal(H_\rho^i/H)} h(v_i)^\sigma = T_i(h(v_i)).$$

For any positive integer n, we let $L_n(W_\rho)$ be the Λ_H-submodule of $\mathcal{F}_\rho$ whose elements are those $h \in \mathcal{F}_\rho$ for which we have

$$\sum_{i=1}^{n} T_i(g(v_i)h(v_i)) \equiv 0 \text{ modulo } \pi^n \mathcal{O}_H, \tag{12}$$

for all $g \in X\mathcal{O}_H[[X]]$. By definition, we set $L_\infty(W_\rho) = \bigcap_{n \geq 1} L_n(W_\rho)$. Let us remark that for any n

$$\Phi(\mathcal{O}_H[[X]]) \subset L_n(W_\rho). \tag{13}$$

Indeed, if $f \in \mathcal{O}_H[[X]]$ and $g \in X\mathcal{O}_H[[X]]$ then

$$\sum_{\substack{w \in W_\rho^n \\ w \neq 0}} g(w)f(w) = \sum_{w \in W_\rho^n} g(w)f(w) = \sum_{w \in W_\rho^n} (gf)(w) = \mathcal{S}_{\rho,\pi}^n(gf)(0) \equiv 0 \text{ modulo } \pi^n \mathcal{O}_H.$$

The third equality follows from property (1) in Remark 5.2 in which we take $X = 0$. The final congruence is property (3) of the same remark.

LEMMA 5.3. *Let n be a positive integer and let $f \in \mathcal{O}_H[[X]]$ be such that $X^{-1}f(X) \in L_n(W_\rho)$, for some positive integer n. Then there exists $g \in \mathcal{O}_H[[X]]$ such that $g(w) = w^{-1}f(w)$, for all $w \in W_\rho^n$.*

Proof. Let us first prove that $f(0) \in \pi^n \mathcal{O}_H$. By taking $g = X$ in (12), we obtain that

$$\sum_{\substack{w \in W_\rho^n \\ w \neq 0}} f(w) \equiv 0 \text{ modulo } \pi^n \mathcal{O}_H.$$

Moreover, we have

$$f(0) = \sum_{w \in W_\rho^n} f(w) - \sum_{\substack{w \in W_\rho^n \\ w \neq 0}} f(w) = \mathcal{S}_{\rho,\pi}^n(f)(0) - \sum_{\substack{w \in W_\rho^n \\ w \neq 0}} f(w).$$

Since $\mathcal{S}_{\rho,\pi}^n(f)(0) \in \pi^n \mathcal{O}_H$ by the third property of Remark 5.2, we deduce from above that $f(0) \in \pi^n \mathcal{O}_H$. Therefore, the following power series

$$g(X) = X^{-1} f(X) - \frac{f(0)}{\pi^n} X^{-2} \rho_{\pi^n}(X)$$

belongs to $\mathcal{O}_H[[X]]$ and satisfies the desired property. □

LEMMA 5.4. *Let $n > 0$ be a positive integer. Let $\alpha_1, \ldots, \alpha_n$ be such that $\alpha_i \in \pi^{n-i} v_1 \mathcal{O}_{H_\rho^i}$. Then there exists $f \in \mathcal{O}_H[[X]]$ such that $f(v_i) = \alpha_i$, for all $i \in \{1, \ldots, n\}$.*

Proof. Since $\rho_{\pi^n}(X) = \rho_{\pi^{n-k}}(\rho_{\pi^k}(X))$, the power series

$$g_{n,k}(X) = \frac{\rho_{\pi^n}(X)}{\rho_{\pi^k}(X)} \cdot \rho_{\pi^{k-1}}(X)$$

belongs to $\mathcal{O}_H[[X]]$ and satisfies

$$g_{n,k}(v_i) = \begin{cases} 0 & \text{if } i \neq k \\ \pi^{n-k} v_1 & \text{if } i = k. \end{cases}$$

Now to obtain f, we use the equality $\mathcal{O}_{H_\rho^i} = \mathcal{O}_H[v_i]$ which is a consequence of the fact that the extension H_ρ^i/H is totally ramified. □

THEOREM 5.5. *Let $S^{(k)}$ be the set of power series $f \in \mathcal{O}_H((X))$ such that $X^k f \in \mathcal{O}_H[[X]]$. Then for any $k \in \mathbb{Z}$, $h \in \mathcal{F}_\rho$ and $1 \leq n \leq \infty$*

$$X^k h \in L_n(W_\rho) \iff \exists f \in S^{(k)} \text{ such that } f(w) = h(w) \text{ for all } w \in W_\rho^n \setminus \{0\}.$$

Proof. We just repeat Coleman's proof of [4, Theorem 8], page 101 at the end of section III. Moreover, it is sufficient to give a proof when $k = 0$, because if $h \in \mathcal{F}_\rho$, then $X^k h \in \mathcal{F}_\rho$ for any rational integer k. Let $n > 0$ be a positive integer. By (13), we see that if $f \in \mathcal{O}_H[[X]]$ and $f(w) = h(w)$ for all $w \in W_\rho^n \setminus \{0\}$, then $h \in L_n(W_\rho)$. Conversely, let $h \in L_n(W_\rho)$ and fix a positive integer r such that

$$v_i^r h(v_i) \in \pi^{n-i} v_1 \mathcal{O}_{H_\rho^i}, \qquad \text{for all } 1 \leq i \leq n.$$

By Lemma 5.4, there exists $f \in \mathcal{O}_H[[X]]$ such that $f(v_i) = v_i^r h(v_i)$ for all $1 \leq i \leq n$. Since $r \geq 1$, we easily check that $X^{-1} f(X) \in L_n(W_\rho)$. Therefore, using Lemma 5.3 iteratively, we deduce the existence of $f \in \mathcal{O}_H[[X]]$ satisfying $f(w) = h(w)$ for all $w \in W_\rho^n \setminus \{0\}$. This proves the theorem for $n < \infty$. Now suppose that $h \in L_\infty(W_\rho)$. By we have just proved, for any positive integer n, there exists $f_n \in \mathcal{O}_H[[X]]$ such that $f_n(w) = h(w)$ for all $w \in W_\rho^n \setminus \{0\}$. By [4, lemma 2a], the sequence (f_n) is convergent to $f \in \mathcal{O}_H[[X]]$ for the compact-open topology, and we necessarily have $f(w) = h(w)$ for all $w \in W_\rho \setminus \{0\}$. □

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5.2. The norm operator.

PROPOSITION 5.6. *There exist a unique map* $\mathcal{N}_{\rho,\pi} : \mathcal{O}_H((X)) \longrightarrow \mathcal{O}_H((X))$ *such that*

$$\mathcal{N}_{\rho,\pi}(f) \circ \rho_\pi(X) = \prod_{u \in W_\rho^1} f(X+u).$$

The map $\mathcal{N}_{\rho,\pi}$ *is continuous.*

Proof. See [4, Theorem 11] or [3, Theorem 7]. □

REMARK 5.7. The following properties of $\mathcal{N}_{\rho,\pi}$ are immediate. They are also the analogous of the properties proved in [4, section IV].

(1) $\mathcal{N}_{\rho,\pi}^n(f) \circ \rho_{\pi^n}(X) = \prod_{u \in W_\rho^n} f(X+u)$.
(2) $\mathcal{N}_{\rho,\pi}(f)(v_n) = N_{n+1,n}(f(v_{n+1}))$, where $N_{n+1,n}$ is the norm map from H_ρ^{n+1} to H_ρ^n.
(3) $v_X(\mathcal{N}_{\rho,\pi}(f)(X)) = v_X(f)$ where $v_X(f)$ is the order of f with respect to X.
(4) If $f \equiv 1$ modulo $\pi^i \mathcal{O}_H[[X]]$, then $\mathcal{N}_{\rho,\pi}(f) \equiv 1$ modulo $\pi^{i+1}\mathcal{O}_H[[X]]$.

For a positive integer t, we define $\rho^{\varphi^t} \in \mathsf{R}^m$ to be the formal Drinfeld $\mathcal{O}_K$-module such that $(\rho^{\varphi^t})_a = (\rho_a)^{\varphi^t}$, for any $a \in \mathcal{O}_K$. We recall that $(\rho_a)^{\varphi^t}$ is the element of $\mathcal{O}_H\{\{\tau\}\}$ whose coefficients are obtained by applying the automorphism φ^t to the coefficients of ρ_a. Then we have

$$\mathcal{N}_{\rho^{\varphi^t},\pi}(f^{\varphi^t}) = \mathcal{N}_{\rho,\pi}(f)^{\varphi^t}, \tag{14}$$

for any $f \in \mathcal{O}_H((X))$.

5.3. A class of formal Drinfeld modules. In this section, we make the following two assumptions and notations:

- $\mathcal{A}_1$: ρ is a given formal Drinfeld $\mathcal{O}_K$-module that belongs to R^m, for some fixed positive integer m.
- $\mathcal{A}_2$: There exists a positive integer $m_0 \mid m$ and $\eta \in \mathcal{O}_K$ such that $v_K(\eta) = m_0$ and $\rho_\eta \equiv \tau^{dm_0}$ modulo $\pi\mathcal{O}_{K_u^m}\{\{\tau\}\}$. This is equivalent to say that $\rho_\eta(X) \equiv X^{q^{m_0}}$ modulo $\pi\mathcal{O}_{K_u^m}[[X]]$.

We draw the attention of the reader that the formal Drinfeld modules coming from Drinfeld modules of rank one and appearing for instance in [2, 3] all satisfy the above three conditions, with $m = m_0$. The case $m_0 = m = 1$ is Lubin–Tate theory, and the results described in the sequel are all already proved by Coleman in his famous article [4].

By its very definition, there exists u a unit of K such that $u\eta = \pi^{m_0}$. We consider the operator $\widetilde{\mathcal{N}}$ defined by

$$\widetilde{\mathcal{N}}(f) = \mathcal{N}_{\rho,\pi}^{m_0}(f) \circ \rho_u(X),$$

so that $\widetilde{\mathcal{N}}(f) \circ \rho_\eta(X) = \prod_{u \in W_\rho^{m_0}} f(X+u)$. We also define $\widetilde{\varphi} = \varphi^{m_0}$ and we denote by H the unramified extension K_u^m of K. Then, exactly as in [4, Section IV], one may prove that for any $f \in \mathcal{O}_H[[X]]$, we have

$$\widetilde{\mathcal{N}}(f) \equiv f^{\widetilde{\varphi}}(X) \text{ modulo } \pi\mathcal{O}_H[[X]]. \tag{15}$$

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Let $\mathcal{O}_H((X))^*$ be the group of invertible elements of $\mathcal{O}_H((X))$. We deduce from (15)

$$\frac{\widetilde{\mathcal{N}}^i(f)}{\widetilde{\mathcal{N}}^{i-1}(f^{\widetilde{\varphi}})} \equiv 1 \text{ modulo } \pi^i\mathcal{O}_H[[X]] \text{ for all } i \geq 1 \text{ and all } f \in \mathcal{O}_H((X))^*.$$

Therefore, the limit

$$\widetilde{\mathcal{N}}^\infty(f) = \lim_{i\to\infty} \widetilde{\mathcal{N}}^i(f^{\widetilde{\varphi}^{-i}})$$

exists in $\mathcal{O}_H((X))^*$ and satisfies $\widetilde{\mathcal{N}}(\widetilde{\mathcal{N}}^\infty(f)) = \widetilde{\mathcal{N}}^\infty(f^{\widetilde{\varphi}})$.

5.3.1. The case $\rho \in \mathsf{R}^{m_0}$

When $\rho \in \mathsf{R}^{m_0}$, the equality (14) implies the equation $\widetilde{\mathcal{N}}(f^{\widetilde{\varphi}}) = \widetilde{\mathcal{N}}(f)^{\widetilde{\varphi}}$. This formula is used below to prove a theorem that may be considered as a generalization of [**4**, Theorem A] to the case $m_0 > 1$. It also generalizes [**3**, Theorem 11]. For any $n > 0$, we denote by E_n the local field $H_\rho^{nm_0}$ and we set $\widetilde{v}_n = \rho_{u^n}(v_{nm_0})$. In particular, we have $\rho_\eta(\widetilde{v}_{n+1}) = \widetilde{v}_n$.

THEOREM 5.8. *Suppose we have $\rho \in \mathsf{R}^{m_0}$. Let $X_\infty = \varprojlim_n E_n^*$ be the projective limit of the multiplicative groups E_n^* with respect to the norm maps. Let $\mathcal{M}_\infty = \{f \in \mathcal{O}_H((X))^*,\ \widetilde{\mathcal{N}}(f) = f^{\widetilde{\varphi}}\}$. There exists a topological isomorphism*

$$ev_{\widetilde{\mathcal{N}}} : \mathcal{M}_\infty \longrightarrow X_\infty,$$

defined by $ev_{\widetilde{\mathcal{N}}}(f) = \big(f^{\widetilde{\varphi}^{-n}}(\widetilde{v}_n)\big)_n$.

Proof. Since we obviously have $N^{m_0}_{\rho,\pi}(g \circ \rho_u(X)) = N^{m_0}_{\rho,\pi}(g) \circ \rho_u(X)$, the property (2) in Remark 5.7 gives us

$$\widetilde{\mathcal{N}}(g)(\widetilde{v}_n) = N_{E_{n+1}/E_n}\big(g(\widetilde{v}_{(n+1)})\big),$$

for any $g \in \mathcal{O}_H((X))$. Moreover, if $f \in \mathcal{M}_\infty$, then we have

$$\widetilde{\mathcal{N}}(f^{\widetilde{\varphi}^{-(n+1)}}) = f^{\widetilde{\varphi}^{-n}}.$$

Therefore, the sequence $\big(f^{\widetilde{\varphi}^{-n}}(\widetilde{v}_n)\big)_n$ belongs to X_∞. Hence, the map $ev_{\widetilde{\mathcal{N}}_\rho}$ is well defined. It is also an injection thanks to [**4**, Lemma 2a]. Let us prove that $ev_{\widetilde{\mathcal{N}}_\rho}$ is onto. In this, we take our inspiration from the proof of [**3**, Theorem 11] and the proof of [**9**, Theorem 13.38]. Let $(u_n)_n$ be an element of X_∞. Suppose first that $(u_n)_n \in \varprojlim_n \mathcal{O}^*_{E_n}$. For any positive integer $k \in \mathbb{N}$, choose $g \in \mathcal{O}_H[[X]]^*$ such that $g^{\widetilde{\varphi}^{-2k}}(\widetilde{v}_{2k}) = u_{2k}$. Since

$$\widetilde{\mathcal{N}}^k(g^{\widetilde{\varphi}^{-k}}) \equiv \widetilde{\mathcal{N}}^{2k-i}(g^{\widetilde{\varphi}^{-(2k-i)}}) \text{ modulo } \pi^k\mathcal{O}_H[[X]],$$

for all $1 \leq i \leq k$, we deduce that the power series $f_k = \widetilde{\mathcal{N}}^k(g^{\widetilde{\varphi}^{-k}})$ is such that

$$f_k^{\widetilde{\varphi}^{-i}}(\widetilde{v}_i) \equiv \widetilde{\mathcal{N}}^{2k-i}(g^{\widetilde{\varphi}^{-2k}})(\widetilde{v}_i) \text{ modulo } \pi^k\mathcal{O}_{E_i}.$$

But we have

$$\widetilde{\mathcal{N}}^{2k-i}(g^{\widetilde{\varphi}^{-2k}})(\widetilde{v}_i) = N_{E_{2k}/E_i}\big(g^{\widetilde{\varphi}^{-2k}}(\widetilde{v}_{2k})\big) = N_{E_{2k}/E_i}\big(u_{2k}\big) = u_i.$$

Finally, we obtain $f_k^{\widetilde{\varphi}^{-i}}(\widetilde{v}_i) \equiv u_i$ modulo $\pi^k\mathcal{O}_{E_i}$, for $1 \leq i \leq k$. We deduce that the sequence $(f_k)_k$ is a Cauchy sequence. Let $f \in \mathcal{O}_H[[X]]^*$ be its limit. Then we necessarily have

$$f^{\widetilde{\varphi}^{-i}}(\widetilde{v}_i) = u_i.$$

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It is also immediate that $f \in \mathcal{M}_\infty$ and $ev_{\widetilde{\mathcal{N}}}(f) = (u_n)_n$. Now, if $(u_n)_n$ is a general element of X_∞, then there exists an integer e such that $u_n \in \widetilde{v}_n^e \mathcal{O}_{E_n}^*$, for all $n \geq 1$ because the fields E_n are totally ramified over H. As Coleman already proceeded to complete his proof of [4, Theorem 15], we consider the power series $G(X) = \widetilde{\mathcal{N}}^\infty(X)$. We have $G(X) \in \mathcal{O}_H[[X]]$ and $G(X) \equiv X$ modulo $\pi \mathcal{O}_H[[X]]$. Thus $G(\widetilde{v}_n)$ is a prime of E_n. Moreover, $\left(u_n G(\widetilde{v}_n)^{-e}\right)_n \in \varprojlim_n \mathcal{O}_{E_n}^*$. Let $f \in \mathcal{M}_\infty$ be such that $ev_{\widetilde{\mathcal{N}}}(f) = \left(u_n G(\widetilde{v}_n)^{-e}\right)_n$. Then we have $ev_{\widetilde{\mathcal{N}}}(fG^e) = (u_n)_n$.

The continuity of $ev_{\widetilde{\mathcal{N}}}$ is immediate. The continuity of its inverse is a consequence of [4, Lemma 2a]. □

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