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GENERALIZED NEAR-FIELDS

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Introduction

By analogy with the concept of "inverse semi-group" in semi-group theory, in this paper we introduce the concept of "generalized near-field" in near-rings. A near-ring N is called a generalized near-field (GNF) if for each $a \in N$ there exists a unique $b \in N$ such that $a = aba$ and $b = bab$, that is $(N, \cdot)$ is an inverse semi-group. Surprisingly, this concept in rings coincides with that of "strong regularity". But this is not true in the case of near-rings. Every GNF is strongly regular, but in general the converse is not true.

The aim of this paper is to show that for any near-ring N the following are equivalent.

(i) N is a GNF.

(ii) N is regular and each idempotent is central.

(iii) N is fegular and subcommutative.

Also we prove that if N is a near-ring with dcc on ideals, then N is a GNF if and only if it is the direct sum of finitely many near-fields. (ii) is equivalent to $(N, \cdot)$ is a Clifford semi-group. See **[2]** for properties of inverse semi-groups.

Throughout this paper, N stands for a right near-ring. For the basic terminology and notation we refer to **[9]**. Recall that a near-ring N is called regular if for each $a \in N$, $a = aba$ for some $b \in N$.

Lemma 1. *If N is a GNF, then N is zerosymmetric.*

Proof. Since N is a GNF, for each $n \in N$ there is a unique $x \in N$ such that $n0 = n0xn0$, $x = xn0x$. Both 0 and $n0$ satisfy the above equations. So by uniqueness $0 = n0$. Thus N is zerosymmetric.

By **[2**, Theorem 1.2, p. 130] N is a GNF if and only if N is regular and idempotents commute. Recall that N is called strongly regular if for each $a \in N$ there exists $b \in N$ such that $a = ba^2$. For a brief discussion of these near-rings, see **[6]**, **[7]** and **[8]**. In **[7]**, a near-ring N is called subcommutative if $aN = Na$ for all $a \in N$.

Lemma 2. *If N is a GNF, then N has no non-zero nilpotent elements.*

Proof. Let $a \in N$, $a^2=0$, and let a have inverse b. Then $b^2=babbab=bbaabb=0$, since ab, ba are idempotents and hence commute. Also $ba(ba+b)$ is an inverse for a, so $ba(ba+b)=b$ by uniqueness. Thus $0=b^2=ba(ba+b)b=babab=bab=b$. So a must be 0.

We are now ready to prove our main theorems.

Theorem 1. *The following are equivalent:*

(i) *N is a GNF.*

(ii) *N is regular and each idempotent is central.*

(iii) *N is regular and subcommutative.*

Proof. (i)$\Rightarrow$(ii). Let $e=e^2 \in N$ and $a, b \in N$. Since $e^2=e$, $(a-ae)e=0$. By [**9**, Chapter 9a and 9b], since N has no non-zero nilpotent elements by Lemma 2, $(a-ae)be=0$, so $abe=aebe$. But $(eb-ebe)e=0$. For the same reason, $eb(eb-ebe)=0$, $ebe(eb-ebe)=0$ so $(eb-ebe)^2=0$ and $eb=ebe$. Thus $abe=aeb$. Since N is regular, $a=fa$ where f is a suitable idempotent. So $ae=fae=fea=efa=ea$ as idempotents commute. So (ii) holds.

(ii)$\Rightarrow$(iii). Let $a \in N$. Since N is regular, $a=axa$ for some $x \in N$. Since ax and xa are idempotents, by (ii) we have $aN=axaN=aNxa \subseteq Na=Naxa=axNa \subseteq aN$. Thus $aN=Na$ for all $a \in N$.

(iii)$\Rightarrow$(i). Let e, f be idempotents. Then $Ne=eN$. So there exist x, y in N such that $fe=ex$ and $ef=ye$. Hence $efe=fe=ef$. So $ef=fe$ and N is a GNF.

Corollary 1. *Every GNF is a strongly regular near-ring.*

Proof. By (ii) $a=aba=ba^2$ since ba is an idempotent, where b is the inverse of a.

In [**10**], Raphael showed that in a strongly regular ring R, for each $0 \neq a \in R$ there exists a unique $b \in R$ such that $a=aba$ and $b=bab$. Now the converse follows from Corollary 1. Thus in the case of rings the notions "strong regularity" and "GNF" are equivalent. In general the converse of Corollary 1 does not hold in near-rings.

Example 1. Let $(N, +)$ be any group. Define multiplications on N as follows:

$$ab=a \text{ for all } a \text{ and } 0 \neq b \text{ in } N$$

$$a0=0 \text{ for all } a \text{ in } N.$$

Then clearly N is strongly regular but not GNF.

Corollary 2. *Every homomorphic image of a GNF is again a GNF.*

The definition of a GNF shows that the properties are preserved under homomorphisms.

Combining Theorem 1 and a result of Ligh [**5**], we have the following:

Corollary 3. *Every GNF is isomorphic to a subdirect product of near-fields and hence $(N, +)$ is abelian.*

Theorem 2. *N is a GNF and integral if and only if N is a near-field.*

Proof. Suppose N is a GNF and integral. Then clearly each non-zero idempotent is a right identity of N. If e, f are non-zero idempotents then $f = fe = ef = e$. Thus N has a unique non-zero idempotent, say e. Let $0 \neq a \in N$. Then $a = axa$ for some $x \in N$, ax an idempotent. So $ax = e$ and e is the identity of N. Now, by Theorem 3 of [**1**], N becomes a near-field. The converse is immediate.

Combining Theorem 2 and [**9**, Corollary 9.38], we get

Corollary 4. *Suppose N is subdirectly irreducible. Then N is a GNF if and only if N is a near-field.*

In general every GNF is not a near-field.

Example 2. Take a near-field N. Then the direct sum of N with itself is a GNF, but not a near-field.

Corollary 5. *Suppose for each $0 \neq a$ in N there exists a unique $b \in N$ such that $a = aba$. Then N is a near-field.*

Proof. We first show that N has no zero divisors. Let $a, b \in N$ with $ab = 0$ and $b \neq 0$. Then $b = bxb$ for some unique $x \in N$. Now $b(x-a)b = bxb = b$. Hence by the uniqueness of x, we have $a = 0$. Thus N has no zero divisors. Clearly N is a GNF and hence a near-field by Theorem 2.

The following result is an immediate consequence of Corollary 5.

Corollary 6 (Ligh [**4**]). *Let R be a dg near-ring with more than one element. Then R is a division ring if and only if for each $0 \neq a \in R$ there exists a unique $b \in R$ such that $a = aba$.*

In [**7**], a near-ring N is called left simple if for each $0 \neq a \in N$, $Na = N$. Clearly a left simple near-ring contains no zero divisors.

Theorem 3. *Suppose N has dcc on ideals. Then N is a GNF if and only if $N = N_1 \oplus \ldots \oplus N_k$ where each N_i is a near-field.*

Proof. Following the proof of [**3**,Theorem 3.2], we can easily show that the intersection of all maximal ideals is $\{0\}$. Since N has dcc on ideals, there exist maximal ideals $I_1, \ldots, I_n$ such that $\bigcap_{k=1}^{n} I_k = \{0\}$. But from [**9**, Theorem 2.50, p. 57] N is the direct sum of finitely many simple near-rings. Each summand is a GNF by Corollary 2, hence a near-field by Corollary 4. The converse is clear.

Corollary 7. *Suppose N is a GNF and satisfies dcc on ideals. Then*

(i) *N has the identity,*

(ii) $a(-b)=(-a)b=-ab$ *for all* a, b *in* N.

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