The Stirling Numbers and Polynomials.

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§ 1. If we write the product

$$P_n(x) = x(x+1)(x+2)\dots(x+n-1)$$

in the form

$$C_n^0 x^n + C_n^1 x^{n-1} + \dots + C_n^{n-1} x,$$

the coefficients C_n^r have been called by Professor Nielsen (following Thiele) the *Stirling Numbers of the First Species*, because James Stirling, in his *Methodus Differentialis* (1730), was the first writer to draw attention to their use, and furnished a small table of their initial values.*

Thus C_n^r is simply the sum of the $r-ary$ products of $1, 2, 3, \dots, n-1$. In particular,

$$C_n^0 = 1; \quad C_n^1 = n(n-1)/2; \quad C_n^{n-1} = (n-1)!$$

But the explicit representation of C_n^r in terms of n and r is not very simple.

For identical reasons, when $1/P_n(x)$ is expanded in ascending powers of x in the form $\sum_{s=0}^{\infty}(-1)^s \Gamma_n^s / x^{n+s}$, which is possible if $|x| > n-1$, the positive integers Γ_n^s are termed the *Stirling Numbers of the Second Species.*

Both series of numbers have been studied by various mathematicians of note. Recently Nielsen (*Annali di Matematica*, 1904) has discussed their properties, and shewn their relationship to the Bernoullian Numbers and Polynomials. I propose here to furnish an account of them. By use of a different basis I have been enabled to recast the theory. The relations with the Bernoullian numbers have been brought more into prominence by means of a series of linear transformations that seem peculiar to the Stirling numbers, while a generalisation of both Stirling and Bernoullian numbers is indicated.

* Page 11 and page 8, *Meth. Diff.*

The importance of the rôle of these functions in the theory of the Gamma Functions has been emphasised by Nielsen in his well-known treatise. Since $P_n(x) = \Gamma(x+n) / \Gamma(x)$, the connection with the Gamma Function is obvious enough. Owing to the nature of their formation, the methods of Finite Differences are of peculiar advantage in their discussion.

We may note

$$\Delta P_n(x) = n P_{n-1}(x+1). \quad \text{.....................}(1)$$

$$\Sigma P_n(x) = \frac{1}{n+1} P_{n+1}(x-1). \quad \text{...............}(2)$$

If $n \not< m$, we may write the product $P_m(x) \times P_n(x)$ in the form

$$A_0 P_n(x) + A_1 P_{n+1}(x) + \ldots + A_m P_{m+n}(x) \quad \text{...........}(3)$$

by noting that $P_m(x)$ may be written as

$$P_m(x) = A_0 + A_1(x+n) + A_2(x+n)(x+n+1) + \text{ etc.,}$$

where

$$A_0 = (-1)^m \, {}_nP_m; \quad A_r = (-1)^{m-r} \, {}_nP_{m-r} \times {}_mC_r.$$

In particular,

$$[P_n(x)]^2 = \sum_{r=0}^{n} (-1)^{n-r} (n-r)! \, {}_nC_r^2 \, P_{n+r}(x). \quad \text{.............}(4)$$

Similar conclusions hold for the product $P_l(x) \, P_m(x) \, P_n(x)$, etc.

In particular,

$$[P_n(x)]^s = A_0 P_n(x) + \sum_{k=1}^{n(s-1)} A_k P_{n+k}(x) \quad \text{...................}(5)$$

where $\quad A_0 = [P_n(x)]^{s-1}$ when $x = -n$,

$$A_k = \frac{1}{(n+k)!} \Delta^{n+k} [P_n(x)]^s \quad \text{when } x = -k-n.$$

§ 2. ***Relations connecting the numbers C_n^r, and their successive calculation in terms of n and r.***

If we put $x = 1$ in $P_n(x)$, we find

$$P_n(1) = C_n^0 + C_n^1 + \ldots + C_n^{n-1} = n! \quad \text{..................}(1)$$

Similarly, $P_n(r) = r^n C_n^0 + r^{n-1} C_n^1 + \ldots + r C_n^{n-1} = {}_{n+r-1}C_{r-1} \times n!$ (2)
for integral values of r; and the determinant of order $n+1$ whose rows are the values of

$$r^n, \; r^{n-1}, \; \ldots, \; r, \; {}_{n+r-1}C_{r-1}$$

for any $n+1$ positive integral values of r is equal to zero.

By noting that

$$x(x-1)\ldots(x-n+1) = C_n^0 \, x^n - C_n^1 \, x^{n-1} + C_n^2 \, x^{n-2} - \text{ etc.,}$$

and writing in succession 1, 2, … , $n-1$ for x, we obtain the $n-1$ equations

$$\left.\begin{array}{l} C_n^{n-1} - \ C_n^{n-2} + \ \ C_n^{n-3} + \ldots + (-1)^{n-1} C_n^0 = 0 \\ C_n^{n-1} - 2C_n^{n-2} + 2^2 C_n^{n-3} + \ldots + (-2)^{n-1} C_n^0 = 0 \\ \text{etc.} \end{array}\right\} \ldots\ldots\ldots(3)$$

So that if we take $C_n^0 = 1$, we have $n-1$ linear equations for $C_n^1 \ldots, C_n^{n-1}$, which can then be expressed as the quotients of determinants of a very special type. But the calculation of these quotients is simply the problem of the calculation of C_n^r in another form.

A number of recurrence-formulae for their successive calculation may, however, be readily obtained.

From the identity

$$(x+n)\, P_n(x) = P_{n+1}(x)$$

or

$$\Sigma\,(C_n^r + n\, C_n^{r-1})\, x^{n-r+1} = \Sigma\, C_{n+1}^r \ x^{n-r+1}$$

we deduce

$$C_{n+1}^r = C_n^r + n\, C_n^{r-1}. \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

This relation is of fundamental importance. Moreover, if we take $C_n^n = 0$ and $C_n^0 = 1$ for all positive integral values of n, and restrict the upper index k in C_n^k to be less than the lower index n, then there is only one system of integral numbers thereby determined. It has the defect that to calculate C_{n+1}^k we require to know the values of C_n^r for the lower value n.

We proceed to find a formula not subject to this objection.

From the identity

$$\Delta\, P_n(x) = \frac{n}{x}\, P_n(x)$$

or

$$n\, P_n(x) = x \Delta P_n(x)$$

we find

$$n\,\{C_n^0\ x^n + C_n^1 x^{n-1} + \ldots\}$$
$$= x[\{(x+1)^n - x^n\}\ C_n^0 + \{(x+1)^{n-1} - x^{n-1}\}\ C_n^1 + \text{etc.}].$$

So that, on equating like powers of x on the two sides, we obtain

$$n\, C_n^0 = {}_nC_1\, C_n^0$$

....

$$n\, C_n^r = {}_nC_{r+1}\, C_n^0 + {}_{n-1}C_r\, C_n^1 + {}_{n-2}C_{r-1}\, C_n^2 + \ldots + {}_{n-r}C_1\, C_n^r.$$

Hence

$$r\, C_n^r = {}_nC_{r+1}\, C_n^0 + {}_{n-1}C_r\, C_n^1 + \ldots + {}_{n-r+1}C_2\, C_n^{r-1}\ . \ \ \ldots\ldots\ldots\ldots(5)$$

From this formula, taken with $C_n^0 = 1$, we have a means of obtaining C_n^1, C_n^2, etc., in succession.

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Thus

$$\text{I.}\quad \begin{cases} C_n^0 = 1 \\ C_n^1 = n(n-1)/2 \\ C_n^2 = n(n-1)(n-2)(3n-1)/24 \\ C_n^3 = n^2(n-1)^2(n-2)(n-3)/48 \quad \text{or} \quad {}_nP_4\, n(n-1)/48 \\ C_n^4 = {}_nP_5(15n^3 - 30n^2 + 5n + 2)/5760 \\ C_n^5 = {}_nP_6\, n(n-1)(3n^2 - 7n - 2)/11520 \\ C_n^6 = {}_nP_7(63n^5 - 315n^4 + 315n^3 + 91n^2 - 42n - 16)/252 \times 11520. \end{cases}$$

The calculations begin to get laborious, but from the results given a variety of conclusions are suggested, which are readily established by induction.

The function C_n^r, *as a function of* n, *is an integral function of degree* $2r$.

*It contains the factor** ${}_nP_{r+1} = n(n-1)\ldots(n-r)$.

We observe that C_n^3 and C_n^5 contain the square factor $n^2(n-1)^2$. We proceed to shew *that* C_n^{2k+1} *always contains this factor.*

Dem.—If $\xi = x + n - 1$

$$x(x+1)\ldots(x+n-1) = \xi(\xi-1)\ldots(\xi-n+1)$$

so that

$$\Sigma C_n^r x^{n-r} = \Sigma(-1)^s C_n^s (x+n-1)^{n-s}.$$

Hence

$$C_n^r = (-1)^r C_n^r + (-1)^{r-1} C_n^{r-1}\ {}_{n-r+1}C_1(n-1)$$
$$+ (-1)^{r-2} C_n^{r-2}\ {}_{n-r+2}C_2(n-1)^2 + \ldots,$$

so that, when r is odd and >1,

$$2C_n^r = C_n^{r-1}(n-1)(n-r+1) + (n-1)^2 F(n). \quad \ldots\ldots\ldots(6)$$

But C_n^{r-1} contains the factor $n-1$.

$\therefore$ C_n^r contains the factor $(n-1)^2$ when r is odd.

Now $C_n^r = C_{n+1}^r - n\,C_n^{r-1}$ and C_{n+1}^r contains the factor n^2 when r is odd. Hence C_n^r contains the factors n^2, and therefore the factor $n^2(n-1)^2$ when r is odd.

The same conclusion may be obtained as follows.

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* These theorems are only true if r is independent of n. For example, since $n! = 1 + C_n^1 + C_n^2 + \ldots + C_n^{n-1}$, *all* the numbers $C_n^1 \ldots C_n^{n-1}$ cannot be divisible by $n(n-1)$. A similar restriction applies to the presence of a square factor in the Bernoullian Polynomials of even degree.

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By Newton's Interpolation Formula

$$P_n(x)/n! = x + {}_{n-1}C_1\, {}_xC_2 + {}_{n-1}C_2\, {}_xC_3 + \ldots + {}_xC_n\,. \quad \ldots\ldots\ldots (7)$$

Pick out the terms in x^{n-r}.

$$\therefore \quad \frac{C_n^r}{n!} = (-1)^r \frac{C_n^r}{n!} + (-1)^{r-1}\, {}_{n-1}C_1 \frac{C_{n-1}^{r-1}}{(n-1)!}$$

$$+ (-1)^{r-2}\, {}_{n-1}C_2 \frac{C_{n-2}^{r-2}}{(n-2)!} + \ldots + {}_{n-1}C_r \frac{C_{n-r}^0}{(n-r)!}\,. \quad \ldots\ldots (8)$$

When r is odd, the factor $(n-1)^2$ appears in each term of the equivalent of $2C_n^r$, and we may reason as before to complete the demonstration.

By introducing the Bernoullian numbers and polynomials another recurrence formula may be obtained.

In $P_n(x) = \Sigma C_n^r x^{n-r}$ write in succession 1, 2, ... , $x-1$ for x and add.

$$\therefore \quad (x-1)\, P_n(x)/(n+1) = \Sigma\, C_n^r S_{n-r} \quad \ldots\ldots\ldots\ldots (9)$$

where

$$S_m = 1^m + 2^m + \ldots + (x-1)^m$$

$$= \frac{x^{m+1}}{m+1} - \frac{x^m}{2} + \sum_{k=1}^{\leqq \frac{m}{2}} (-1)^{k-1} \frac{{}_mC_{2k}}{m+1-2k} B_k\, x^{m+1-2k},$$

in which B_1, B_2, etc., are the Bernoullian numbers.

Substitute the corresponding values for S_n, etc., in (9) and equate the coefficients of x^{n-r+1} on the two sides, when we deduce

$$\frac{C_n^r - C_n^{r-1}}{n+1} = \frac{C_n^r}{n-r+1} - \frac{C_n^{r-1}}{2} + \sum_{s=1}^{\leqq \frac{r}{2}} (-1)^{s-1} \frac{{}_{n-r+2s}C_{2s}}{n-r+1} B_s\, C_n^{r-2s}$$

or

$$\frac{C_{n+1}^r}{n+1} = \frac{C_n^r}{n-r+1} + \frac{C_n^{r-1}}{2} + \frac{1}{n-r+1} \sum_{s=1}^{\leqq \frac{r}{2}} (-1)^{s-1}\, {}_{n-r+2s}C_{2s} B_s\, C_n^{r-2s} \quad (10)$$

Cor.—If r is odd $= 2k+1$, then

$$C_n^{2k} + \frac{(-1)^{k-1}}{n-2k}\, {}_{n-1}C_{2k}\, n(n-1)\, B_k$$

is divisible by n^2, since all the terms with an odd upper index > 1 in (10) are divisible by n^2.

From the identity

$$P_{m+n}(x) = x(x+1)\ldots(x+m-1) \times (x+m)\ldots(x+m+n-1)$$

or

$$\Sigma C^{t}_{m+n}\, x^{m+n-t} = \Sigma C^{r}_{m}\, x^{m-r} \times \Sigma C^{s}_{n}\, (x+m)^{n-s}$$

we find

$$C^{\,t}_{m+n} = \sum_{k=0}^{t} C^{k}_{m} \left[C^{t-k}_{n} + C^{t-k-1}_{n}\ {}_{n-t+k+1}C_1 \times m + \ldots + C^{0}_{n}\ {}_{n}C_{t-k}\, m^{t-k}\right].$$

For example,

$$C^{t}_{m+1} = C^{\,t}_{m}\ C_1^{\,0} + C^{t-1}_{m}\left[C_1^{\,0} \times m\right]$$
$$= C^{\,t}_{m} + m\, C^{t-1}_{m}.$$

More complicated expressions follow from such identities as

$$P_{l+m+n}(x) = P_l(x) \times P_m(x+l) \times P_n(x+l+m).$$

§ 3. *The Stirling Numbers of the Second Species.*

These are defined by the relation

$$1/P_n(x) = \sum_{s=0}^{\infty} (-1)^s\ \Gamma^{s}_{n}\ /\ x^{n+s}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

In particular,

$$\Gamma^{0}_{n} = 1\ ; \quad \Gamma^{k}_{1} = 0\ ; \quad \Gamma^{k}_{2} = 1\ ; \quad \Gamma^{k}_{3} = 2^{k+1} - 1\ ; \quad \text{etc.}$$

Since $(x+n)/P_{n+1}(x) = 1/P_n(x)$,

$$\therefore \quad \Gamma^{r}_{n+1} - n\, \Gamma^{r-1}_{n+1} = \Gamma^{r}_{n}\ . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

This is the fundamental relation corresponding to (4) § 2 for $C_n{}^r$.

Moreover, there is only one system of positive integers satisfying (2), provided $\Gamma^{0}_{n} = 1$ for all positive integral values of n and $\Gamma^{1+k}_{1} = 0$.

A number of other relations follow from the definition.

Since, for $|x| > n$,

$$1/(x+n) = 1/x - n/x^2 + \text{ etc.}$$

$$\therefore \quad \Gamma^{r}_{n+1} = \Gamma^{r}_{n} + n\, \Gamma^{r-1}_{n} + n^2\, \Gamma^{r-2}_{n} + \ldots + n^r\, \Gamma^{0}_{n}\ , \quad \ldots\ldots\ldots\ldots (3)$$

$$e.g. \quad \Gamma^{r}_{4} = \Gamma^{r}_{3} + 3\ \Gamma^{r-1}_{3} + \ldots + 3^r\ \Gamma^{0}_{3} = \tfrac{1}{2}\,(3^{r+2} - 2^{r+3} + 1).$$

From the representation of $n!\,/\,P_{n+1}(x)$ as a sum of partial fractions in the form

$$\sum_{s=0}^{n} (-1)^s\ {}_nC_s\,/\,(x+s) = \Sigma\,(-1)^{n-s}\ {}_nC_s\,/\,(x+n-s)$$

it follows that

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$$\Gamma_{n+1}^{r} = \frac{1}{n\,!} \sum_{s=0}^{n} (-1)^{s}\, {}_{n}C_{s}\, (n-s)^{n+r}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Moreover, since for $p > n$,

$$\Delta^{n}\,(x^{p}) = \sum_{s=0}^{s=n} (-1)^{s}\, {}_{n}C_{s}\, (x+n-s)^{p}$$

$$= \Sigma\,(-1)^{s}\, {}_{n}C_{s} \left[x^{p} + {}_{p}C_{1}\, x^{p-1}\,(n-s) + \ldots + (n-s)^{p}\right]$$

and since

$$\Sigma\,(-1)^{s}\,{}_{n}C_{s}\,(n-s)^{a} = \Sigma\,(-1)^{s}\,{}_{n}C_{s}\, s^{a}$$

$$= \Delta^{n}\,(0^{a}) = 0 \quad \text{for} \quad a < n$$

$$\therefore\ \Delta^{n}\, x^{p} = n\,!\ \sum_{r=n}^{p} {}_{p}C_{r}\, \Gamma_{n+1}^{r-n}\, x^{p-r}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

In particular,

$$\Delta^{n}\,(0^{n+r}) = n\,!\ \Gamma_{n+1}^{r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

From Newton's Interpolation Formula

$$x^{n} = x\,\Delta\,(0^{n}) + \sum_{r=2}^{n} {}_{x}C_{r}\, \Delta^{r}\,(0^{n}),$$

and therefore, from (6)

$$x^{n} = \Gamma_{2}^{n-1}\, x + \Gamma_{3}^{n-2}\, x\,(x-1) + \ldots + \Gamma_{n+1}^{0}\, x\,(x-1)\ldots(x-n+1) \quad (7)$$

Recurrence Formula for the successive calculation of Γ_{n}^{1}, Γ_{n}^{2}, *etc.*

Clearly, for suitable values of x,

$$(x+n-1) \sum_{r=0}^{\infty} (-1)^{r}\, \Gamma_{n}^{r} / x^{n+r}$$

$$= 1/P_{n-1}\,(x)$$

$$= (x-1)\, \Sigma\,(-1)^{s}\, \Gamma_{n}^{s}\ /\ (x-1)^{n+s}$$

$$= \Sigma\,(-1)^{s}\,\Gamma_{n}^{s}\, \left[1/x^{n+s-1} + \sum_{t=1}^{\infty} {}_{n+s-1}H_{t}\, / \,x^{n+s-1+t}\right]. \quad \ldots\ldots\ldots(8)$$

Pick out the terms in $1/x^{n+r-1}$ on both sides of (8) ;

$$\therefore\quad \Gamma_{n}^{r} - (n-1)\, \Gamma_{n}^{r-1}$$

$$= \Gamma_{n}^{r} - \Gamma_{n}^{r-1} \cdot {}_{n+r-2}H_{1} + \Gamma_{n}^{r-2} \cdot {}_{n+r-3}H_{2} - \ldots + (-1)^{r}\, \Gamma_{n}^{0} \cdot {}_{n-1}H_{r}$$

$$\therefore\quad (r-1)\, \Gamma_{n}^{r-1} = \Gamma_{n}^{r-2} \cdot {}_{n+r-3}H_{2} - \ldots$$

or

$$r\, \Gamma_{n}^{r} = {}_{n+r-2}H_{2}\, \Gamma_{n}^{r-1} - {}_{n+r-3}H_{3}\, \Gamma_{n}^{r-2} + \ldots + (-1)^{r+1}\, {}_{n-1}H_{r+1}\, \Gamma_{n}^{0}. \quad (9)$$

From (9) we can readily make the following conclusions :—

Γ_{n}^{r} , *as a function of n, is, in general, of degree* $2r$.

It is the product of $(n-1)\,n\,(n+1)\ldots(n+r-1)$ *by an integral function of* n *of degree* $r-1$.

In particular,

$$\text{II.}\quad\left\{\begin{array}{l} \Gamma^1_n = (n-1)\,n/2, \quad \text{and} \;\therefore\; = C^1_n\,. \\ \Gamma^2_n = (n-1)\,n\,(n+1)\,(3n-2)\,/\,24. \\ \Gamma^3_n = (n-1)^2\,n^2\,(n+1)\,(n+2)\,/\,48. \\ \qquad\quad \text{etc.,} \quad \text{etc.} \end{array}\right.$$

Like C^r_n, Γ^r_n *is divisible by* $n^2(n-1)^2$ *when* r *is odd and* >1.

It is sufficient to prove that Γ^r_n then contains $(n-1)^2$ as a factor, for in such a case Γ^r_{n+1} contains n^2, and $\Gamma^r_n = \Gamma^r_{n+1} - n\,\Gamma^{r-1}_{n+1}$ also contains n^2.

Dem.

If $$x+n-1 = -\xi, \; P_n(x) = (-1)^n P_n(\xi)$$
so that
$$1/P_n(x) = \Sigma\,(-1)^r\;\Gamma^r_n\,/\,x^{n+r} = \Sigma\,\Gamma^s_n\;/\,(x+n-1)^{s+n}$$
$$= \Sigma\,\Gamma^s_n\;\left\{\frac{1}{x^{n+s}} + \frac{\Sigma\,(-1)^t\;{}_{n+s}H_t\,(n-1)^t}{x^{n+s+t}}\right\}.$$

Hence
$$(-1)^r\,\Gamma^r_n = \Gamma^r_n\;-(n-1)\;{}_{n+r-1}H_1\;\Gamma^{r-1}_n\;+(n-1)^2 k, \quad \ldots\ldots(10)$$
so that, when r is odd,
$$2\,\Gamma^r_n = (n-1)\,(n+r-1)\;\Gamma^{r-1}_n - (n-1)^2\,k.$$
But Γ^{r-1}_n, for $r>1$, contains $n-1$ as a factor. $\therefore$ etc.

§ 4 The analogy between the properties of C^r_n and Γ^r_n suggested by the results so far given is carried farther by the following analysis.

Consider the functional equation
$$F(x+1) = F(x) + x f_m(x)$$
where $f_m(x)$ is an integral function of degree m. If $F(x)$ is to be an integral function of x and if $F(0)=0$, it is easy to shew that it is a unique function of degree $m+2$, when $f_m(x)$ is given.

Hence if we take $f_0(x)=1$, we obtain a definite function $f_2(x)$. By using the same equation, but for $m=2$, we get a function $f_4(x)$; similarly $f_6(x)$, etc.; and the system of functions may be considered as solutions of
$$f_{2r}(x+1) = f_{2r}(x) + x f_{2r-2}(x).$$

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Now $C^r_{n+1} = C^r_n + n\,C^{r-1}_n$ for *all* integral values of n, and since $f_{2r}(x)$ is of finite degree $2r$, the functions found from the *algebraic* equation in x

$$f_{2r}(x+1) = f_{2r}(x) + x f_{2r-2}(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

are simply the Stirling Numbers C^r_n when x is an integer $= n$.

Similarly Γ^r_n satisfies the equation

$$\psi_{2r}(x) = \psi_{2r}(x+1) - x\,\psi_{2r-2}(x+1) \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

In (2) write $-y$ for x, when it becomes

$$\psi_{2r}(-y) = \psi_{2r}(1-y) + y\,\psi_{2r-2}(1-y). \quad \ldots\ldots\ldots\ldots\ldots(3)$$

Write $\psi_{2r}(1-y) = f_{2r}(y)$ for all values of y, when (3) becomes

$$f_{2r}(y+1) = f_{2r}(y) + y f_{2r-2}(y),$$

i.e. an equation identical with (1) but with y written for x. We have therefore the important theorem.

If C^r_n, *as a function of* n, $= f_{2r}(n)$, *then* $\Gamma^r_n = f_{2r}(1-n)$: *or if* $\Gamma^r_n = f_{2r}(n)$, $C^r_n = f_{2r}(1-n)$.

From this relation we may clearly deduce all the properties of the one number from those of the other.

When r is $= n$ or $> n$, Γ^r_n has a definite value, and C^r_n may then simply be assumed to be zero, having the vanishing factor $n - n$.

The following are examples of the use of Stirling Numbers not here discussed :—

(1) If $Y = \phi(y)$, and $y = e^x$,

$$d^n Y / dx^n = \sum_{s=0}^{n-1} \Gamma^s_{n-s+1}\, y^{n-s}\, \frac{d^{n-s} Y}{d y^{n-s}}.$$

(2) If $Y = \phi(y)$, and $y = \log x$,

$$\frac{d^n Y}{d x^n} = \frac{1}{x^n} \sum_{s=0}^{n-1} (-1)\; C^s_n \frac{d^{n-s} Y}{d y^{n-s}}.$$ (Schlömilch).

(3) $$\frac{1}{x^p} = \sum_{s=p-1}^{\infty} C^{s-p+1}_s \,/\, x(x+1) \ldots (x+s)$$ if $R(x) > 0$. (Stirling).

An error in Stirling's second table (105056 instead of 118124 in the last row—Stirling has omitted to add 13068 from the row above—see page 11, *Meth. Diff.*) has been noted by Binet (*Jour. de l'Ecole Poly.*, 1838). Stirling's two tables, in corrected form, were, however, reproduced by Emerson in *Method of Increments*, 1763—a work based on the Treatises of Taylor and Stirling.

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§ 5. *Identities connecting the two species of numbers.*

The Stirling numbers are connected by a great variety of identities, some of which we proceed to give with a view to further application.

Since $$P_n(x) = \Sigma\, C_n^r\, x^{n-r}$$

$$1/P_n(x) = \sum_{s=0}^{\infty} (-1)^s\, \Gamma_n^s \,/\, x^{n+s}$$

and $$P_n(x) \times 1/P_n(x) = 1,$$

we find immediately the system

$$\left.\begin{array}{l} C_n^0\, \Gamma_n^0 = 1 \\ C_n^0\, \Gamma_n^1 - C_n^1 \Gamma_n^0 = 0 \\ \dots\dots\dots\dots\dots\dots \\ C_n^0 \Gamma_n^r - C_n^1\, \Gamma_n^{r-1} + \dots + (-1)^r C_n^r\, \Gamma_n^0 = 0,\ r < n \\ C_n^0\, \Gamma_n^{n+s} - C_n^1 \Gamma_n^{n+s-1} + \dots + (-1)^{n-1} C_n^{n-1}\, \Gamma_n^{s+1} = 0. \end{array}\right\} \quad (1)$$

Cor. $\Gamma_n^r + (-1)^r C_n^r$ is divisible by $n^2(n-1)^2$ if $r < n$.

From these we may express, say, the quantities Γ_n^r in terms of C_n^0, C_n^1, etc., and deduce their properties.

Moreover, if from these we find

$$\Gamma_n^r = F(C_n^0, C_n^1, \dots) \,/\, (C_n^0)^{r+1}$$

then $$C_n^r = F(\Gamma_n^0, \Gamma_n^1, \dots) \,/\, (\Gamma_n^0)^{r+1}$$

where F is an integral function in which each term is of weight r in the upper indices.

From the identity

$$P_n(x)/P_{n-r}(x) = (x+n-r)(x+n-r+1)\dots(x+n-1)$$

or $$\Sigma\, C_n^s\, x^{n-s} \times \Sigma(-1)^s \Gamma_{n-r}^s \,/\, x^{n-r+s} = x^r + \dots,\ \text{etc.},$$

it follows that, for $t \nless 1$,

$$C_n^0\, \Gamma_{n-r}^{r+t} - C_n^1\, \Gamma_{n-r}^{r+t-1} + C_n^2\, \Gamma_{n-r}^{r+t-2} - \dots = 0\,; \quad \dots\dots\dots\dots (2)$$

for the coefficient of $1/x^t$ on the left side must vanish.

Also

$$C_n^0\, \Gamma_{n-r}^r - C_n^1\, \Gamma_{n-r}^{r-1} + \dots + (-1)^r C_n^r\, \Gamma_{n-r}^0 = (-1)^r\, {}_{n-1}P_r. \qquad (3)$$

The identities found from

$$P_n(x) \,/\, P_{n+r}(x) = 1/P_r(x+n)$$

are more complicated in expression.

From the particular case

$$\frac{1}{x+n} = P_n(x) \,/\, P_{n+1}(x)$$
$$= \Sigma\, C_n^r\, x^{n-r} \times \Sigma(-1)_s\, \Gamma_{n+1}^s \,/\, x^{n+1+s}$$

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we find

$$n^r = C_n^0\ \Gamma_{n+1}^r - C_n^1\ \Gamma_{n+1}^{r-1} + \ldots + (-1)^r C_n^r\ \Gamma_{n+1}^0, \quad \ldots\ldots\ldots(4)$$

a result which holds for $r > n$ if we assume $C_n^{n+k} = 0$.

From the identity

$$x^n = \Gamma_2^{n-1} x + \Gamma_3^{n-2} x(x-1) + \ldots + \Gamma_{n+1}^0 x(x-1)\ldots(x-n+1)$$

the coefficient of x^{n-r} on the right side must vanish.

$$\therefore\quad \Gamma_{n+1}^0 C_n^r - \Gamma_n^1\ C_{n-1}^{r-1} + \Gamma_{n-1}^2 C_{n-2}^{r-2} - \ldots (-1)^r\ \Gamma_{n-r+1}^r C_{n-r}^0 = 0. \quad (5)$$

§ 6. *Some equivalent systems of linear equations.*

The following two systems of equations are equivalent :—

$$\text{(A)}\begin{cases} a_1 = C_1^0 b_1 = b_1 \\ a_2 = C_2^0\ b_2 \pm C_2^1\ b_1 \\ a_3 = C_3^0\ b_3 \pm C_3^1\ b_2 + C_3^2 b_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_n = C_n^0\ b_n \pm C_n^1\ b_{n-1} + \ldots + (\pm 1)^{n-1} C_n^{n-1}\ b_1 \\ \text{and} \\ b_1 = \Gamma_2^0 a_1 = a_1 \\ b_2 = \Gamma_3^0\ a_2 \mp \Gamma_2^1 a_1 \\ b_3 = \Gamma_4^0\ a_3 \mp \Gamma_3^1 a_2 + \Gamma_2^2 a_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ b_n = \Gamma_{n+1}^0 a_n \mp \Gamma_n^1 a_{n-1} + \ldots + (\mp 1)^{n-1}\ \Gamma_2^{n-1} a_1. \end{cases}$$

For, substitute the values of $b_1, b_2, \ldots b_n$, in terms of

$$a_1, a_2, \ldots a_n \text{ in } a_n = C_n^0 b_n \pm \text{etc.},$$

when the coefficient of a_{n-r} vanishes by (2) § 5, save for $r = 0$, when it is unity.

Or substitute from the first system in

$$b_n = \Gamma_{n+1}^0\ a_n \mp \text{etc.},$$

when the result follows from (5) § 5.

We can thence deduce an interesting conclusion regarding the interdependence of (2) § 5 and (5) § 5.

Ex. Since

$$x = C_1^0 x$$

$$x(x+1) = C_2^0 x^2 + C_2^1 x$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \ldots\ldots$$

$$x(x+1)\ldots(x+n-1) = C_n^0\ x^n + \ldots + C_n^{n-1} x$$

$$\therefore\quad x^n = \Gamma_{n+1}^0 P_n(x) - \Gamma_n^1\ P_{n-1}(x) + \ldots + (-1)^{n-1}\Gamma_2^{n-1}\ x. \quad (1)$$

Similarly, since

$$x(x-1)\ldots(x-n+1) = C_n^0\ x^n - C_n^1\ x^{n-1} + \text{etc.},$$

it follows that

$$x^n = \Gamma_2^{n-1}\, x + \Gamma_3^{n-2} x(x-1) + \ldots + \Gamma_{n+1}^0\, x(x-1)\ldots(x-n+1). \quad (2)$$

Cor. The sum

$$1^n + 2^n + \ldots + (x-1)^n$$

$$= \frac{1}{n+1}\,\Gamma_{n+1}^0\, P_{n+1}(x-1) - \frac{1}{n}\,\Gamma_n^1 P_n(x-1) + \ldots\ldots$$
$$+ (-1)^{n-1} \tfrac{1}{2}\,\Gamma_2^{n-1} P_2(x-1) \quad (3)$$

or

$$= \frac{1}{n+1}\,\Gamma_{n+1}^0 P_{n+1}(x-n) + \frac{1}{n}\,\Gamma_n^1\, P_n(x-n+1) + \ldots\ldots$$
$$+ \tfrac{1}{2}\,\Gamma_2^{n-1}\, P_2(x-1). \quad (4)$$

The equivalence of the two linear systems in (A), which is to be found in Schlömilch's *Compendium der H. Analysis*, is only one of several linear equivalences.

A generalisation of it is given by the following:—

If

$$a_r = C_n^0 b_r + C_n^1\, b_{r-1} + C_n^2\, b_{r-2} + \ldots + C_n^{r-k}\, b_k$$
$$a_{r-1} = C_{n-1}^0 b_{r-1} + C_{n-1}^1 b_{r-2} + \ldots + C_{n-1}^{r-k-1} b_k$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

(B) $a_k = C_{n-r+k}^0\, b_k$

then

$$b_r = \Gamma_{n+1}^0 a_r - \Gamma_n^1 a_{r-1} + \Gamma_{n-1}^2 a_{r-2} - \ldots + (-1)^{r-k}\,\Gamma_{n-r+k+1}^{r-k}\, a_k$$
$$b_{r-1} = \Gamma_n^0 a_{r-1} - \Gamma_{n-1}^1 a_{r-2} + \text{ etc.}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$b_k = a_k.$$

For

$$\Gamma_{n+1}^0 C_n^{r-k} - \Gamma_n^1\, C_{n-1}^{r-k-1} + \ldots + (-1)^{r-k}\,\Gamma_{n-r+k+1}^{r-k}\, C_{n-r+k}^0 = 0$$

for $0 < k < r$, by (5) § 5, and $\Gamma_{n+1}^0 C_n^0 = 1$:

$$\left(\text{or because } \sum_{m=0}^{r-k} (-1)^m C_n^m\, \Gamma_{n-r+k+1}^{r-k-m} = 0\right).$$

Again, if

$$a_r = C_n^0\, b_r + C_n^1\, b_{r-1} + \ldots\ldots + C_n^{r-k}\, b_k$$
$$a_{r-1} = C_n^0\, b_{r-1} + C_n^1\, b_{r-2} + \ldots + C_n^{r-k-1} b_k$$
$$\ldots\ldots\ldots\ldots\ldots\ \ \ldots\ldots\ldots\ldots\ldots\ldots$$

(C) $a_k = C_n^0\, b_k$,

then, by (1) § 5,

$$b_r = \Gamma_n^0\, a_r - \Gamma_n^1\, a_{r-1} + \Gamma_n^2\, a_{r-2} + \ldots (-1)^{r-k}\,\Gamma_n^{r-k}\, a_k$$
$$b_{r-1} = \Gamma_n^0\, a_{r-1}, \ - \text{ etc.}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$b_k = \Gamma_n^0\, a_k.$$

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Also, if the signs of the b's in the first system are alternately positive and negative, the signs in the second system are all positive.

Ex. 1. From

$$\Gamma^r_{n+1} = \Gamma^0_n\ n^r + \Gamma^1_n\ n^{r-1} + \ldots + \Gamma^r_n \cdot 1$$
$$\Gamma^{r-1}_{n+1} = \Gamma^0_n\ n^{r-1} + \ldots\ldots\ldots\ldots + \Gamma^{r-1}_n \cdot 1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\Gamma^0_{n+1} = \Gamma^0_n \cdot 1$$

we deduce

$$n^r = C^0_n\ \Gamma^r_{n+1} - C^1_n\ \Gamma^{r-1}_{n+1} + C^2_n\ \Gamma^{r-2}_{n+1} - \ldots (-1)^r\, C^r_n\ \Gamma^0_{n+1}, \quad \ldots (5)$$

a result which holds even when $r \not< n$, provided we then assume $C^{n+k}_n = 0$.

Ex. 2. Since, by (5) § 2,

$$r \,.\, C^r_n = {}_nC_{r+1}\, C^0_n + {}_{n-1}C_r\, C^1_n + \ldots + {}_{n-r+1}C_2\, C^{r-1}_n$$
$$(r-1)\, C^{r-1}_{n-1} = {}_{n-1}C_r\, C^0_{n-1} + {}_{n-2}C_{r-1}\, C^1_{n-1} + \text{ etc.}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$1 \,.\, C^1_{n-r+1} = {}_{n-r+1}C_2\, C^0_{n-r+1},$$

it follows that

$${}_nC_{r+1} = r\, C_n{}^r\, \Gamma^0_{n+1} - (r-1)\, C^{r-1}_{n-1}\, \Gamma^1_n + \text{ etc.}$$

and

$$\therefore\quad {}_nC_r = (r-1)\, C^{r-1}_n\, \Gamma^0_{n+1} - (r-2)\, C^{r-2}_{n-1}\, \Gamma^1_n + \ldots + (-1)^{r-2}\, C^1_{n-r+2}\, \Gamma^{r-2}_{n-r+3}, \quad \ldots\ldots (6)$$

Ex. 3. Similarly, from

$$r\,\Gamma^r_n = (-1)^{r-1}\left[{}_{n-1}H_{r+1}\, \Gamma^0_n - {}_nH_r\, \Gamma^1_n + \text{ etc.}\right]$$

we obtain

$$(-1)^r\, {}_nH_r = (r-1)\, \Gamma^{r-1}_{n+1}\, C^0_n - (r-2)\, \Gamma^{r-2}_{n+2}\, C^1_{n+1} + (r-3)\, \Gamma^{r-3}_{n+3}\, C^2_{n+2} - \ldots + (-1)^{r-2}\, \Gamma^1_{n+r-1}\, C^{r-2}_{n+r-2}. \quad (7)$$

The expression for ${}_nP_r$ is to be found from (3) § 5.

§ 7. *Relations connecting the Stirling Numbers with the Bernoullian Numbers and Polynomials.*

Since the equation

$$C^0_n\ x^{n-1} - C^1_n\ x^{n-2} + C^2_n\ x^{n-3} + \ldots (-1)^{n-1}\, C^{n-1}_n = 0$$

has for roots 1, 2, 3, ... $n-1$, write

$$S_p = 1^p + 2^p + \ldots + (n-1)^p,$$

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when we have from the theory of symmetric functions the system of equations

$$S_1 - C_n^1 = 0$$
$$S_2 - C_n^1 S_1 + 2 C_n^2 = 0$$
$$\dots\dots\dots\dots\dots\dots$$
$$S_p - C_n^1 S_{p-1} + \dots + (-1)^p p C_n^p = 0 \quad \dots\dots\dots\dots(1)$$

up to $p = n - 1$.

$$\dots\dots\dots$$
$$S_{n+k} - C_n^1 S_{n+k-1} + \dots + (-1)^{n-1} S_{k+1} C_n^{n-1} = 0.$$

Hence, consider the linear system

$$C_n^0 S_p - C_n^1 S_{p-1} + C_n^2 S_{p-2} - \dots + (-1)^p S_0 C_n^p = 0$$
$$C_n^0 S_{p-1} - \dots \dots + (-1)^{p-1} S_0 C_n^{p-1} = (-1)^{p-1} C_n^{p-1}$$
$$C_n^0 S_{p-2} - \dots\dots + (-1)^{p-2} S_0 C_n^{p-2} = (-1)^{p-2} . 2 . C_n^{p-2}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$C_n^0 S_1 - S_0 C_n^1 = -(p-1) C_n^1$$
$$C_n^0 S_0 = p C_n^0$$

(which hold for $p > n$, provided we assume $C_n^n = C_n^{n+1} = \dots = 0$).

Multiply respectively by Γ_n^0, Γ_n^1, ..., Γ_n^p, and add.

$$\therefore \; S_p = (-1)^{p-1} [\Gamma_n^1 C_n^{p-1} - 2 \Gamma_n^2 C_n^{p-2} + \dots + (-1)^{p-1} p \Gamma_n^p C_n^0]. \; (2)$$

Similarly,

$$S_{p-1} = (-1)^{p-2} [\text{ etc. }]$$
$$\dots\dots\dots\dots\dots\dots$$
$$S_2 = -[\Gamma_n^1 C_n^1 - 2 \Gamma_n^2 C_n^0]$$
$$S_1 = \Gamma_n^1 C_n^0 .$$

Hence, multiply in the new linear system by Γ_n^0, Γ_n^1, ..., Γ_n^{p-1}, and add.

$$\therefore \quad \Gamma_n^0 S_p + \Gamma_n^1 S_{p-1} + \Gamma_n^2 S_{p-2} + \dots + \Gamma_n^{p-1} S_1 = p \Gamma_n^p . \dots(3)$$

Form a new linear system by writing $p-1$, $p-2$, etc., for p in (3).

Multiply respectively by C_n^0, $-C_n^1$, $+C_n^2$, etc., and add.

$$\therefore \quad S_p = C_n^1 \Gamma_n^{p-1} - 2 C_n^2 \Gamma_n^{p-2} + \dots + (-1)^{p-1} p C_n^p \Gamma_n^0 . \quad (4)$$

If we form a new linear system from (4), as before, and multiply by C_n^0, $-C_n^1$, etc., we find, on adding,

$$C_n^0 S_p - C_n^1 S_{p-1} + \dots = (-1)^{p-1} p . C_n^p ,$$

which is the identity from which we started.

It is worthy of note that the relations obtained, holding for all integral values of n while the degree in p is limited, furnish algebraical identities in n true for all values of n.

From (1) and (3) it also follows that, *in general, the following functions are divisible by* $n^2(n-1)^2$:—

$$S_p + (-1)^p p\, C_n^p\,; \quad S_p - p\,\Gamma_n^p\,; \quad \Gamma_n^p + (-1)^p C_n^p\,.$$

If we substitute for S_p from (2) in $C_n^0\, S_p - C_n^1\, S_{p-1} +$ etc. $=0$, there results

$$\begin{aligned} &\Gamma_n^1\,[C_n^0\, C_n^{p-1} + C_n^1\, C_n^{p-2} + \dots + C_n^{p-1}\, C_n^0] \\ &-2\,\Gamma_n^2\,[C_n^0\, C_n^{p-2} + C_n^1\, C_n^{p-3} + \dots + C_n^{p-2}\, C_n^0] \\ &+3\,\Gamma_n^3\,[C_n^0\, C_n^{p-3} + \dots] \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &+(-1)^{p-1} p\,\Gamma_n^p\, C_n^0\, C_n^0 \qquad = p\, C_n^p. \quad \dots\dots\dots\dots\dots\dots (5) \end{aligned}$$

Also from

$$\Gamma_n^0\, S_p + \dots + \Gamma_n^{p-1}\, S_1 = p\,\Gamma_n^p$$

it follows that

$$\begin{aligned} &\Gamma_n^1\,[\Gamma_n^0\, C_n^{p-1} - \Gamma_n^1\, C_n^{p-2} + \dots + (-)^{p-1}\,\Gamma_n^{p-1} C_n^0] \\ &-2\,\Gamma_n^2\,[\Gamma_n^0\, C_n^{p-2} - \dots] \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &+(-1)^{p-1} p\,\Gamma_n^p = (-1)^{p-1}\, p\,\Gamma_n^p\,, \end{aligned}$$

which is obvious from (1) § 5.

From $\qquad S_p = C_n^1\,\Gamma_n^{p-1} - 2\,C_n^2\,\Gamma_n^{p-2} +$ etc.

and $\qquad \Gamma_n^0\, S_p + \Gamma_n^1\, S_{p-1} + \dots = p\,\Gamma_n^p$,

it follows that

$$\begin{aligned} &C_n^1\,[\Gamma_n^0\,\Gamma_n^{p-1} + \Gamma_n^1\,\Gamma_n^{p-2} + \dots + \Gamma_n^{p-1}\,\Gamma_n^0] \quad \dots\dots\dots\dots (6) \\ &-2\,C_n^2\,[\Gamma_n^0\,\Gamma_n^{p-2} + \text{etc.}] + \dots + (-1)^{p-1} p\, C_n^p\,\Gamma_n^0\,\Gamma_n^0 = p\,\Gamma_n^p\,. \end{aligned}$$

§ 8. *Expression of Bernoullian Numbers in terms of Stirling Numbers.*

Write $x+1$ for x in (3) § 6.

$$\therefore\ \Gamma_{p+1}^0\, P_{p+1}(n)\,/\,(p+1) - \Gamma_p^1\, P_p(n)\,/\,p + \dots + (-1)^{p-1}\,\Gamma_2^{p-1}\, P_2(n)\,/\,2$$

$$= S_p + n^p$$

$$= \frac{n^{p+1}}{p+1} + \frac{n^p}{2} + \Sigma\, \frac{(-1)^{k-1}\,{}_pC_{2k}}{(p+1)-2k} B_k\, n^{p+1-2k}.$$

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Equate like powers of n on the two sides (the relation being an identity in n).

$$\therefore \frac{(-1)^{k-1}\,{}_pC_{2k}}{p+1-2k} B_k = \Gamma^{2k}_{p+1-2k}\, C^0_{p+1-2k} / (p+1-2k),$$

$$-\Gamma^{2k-1}_{p+2-2k}\, C^1_{p+2-2k} / (p+2-2k) + \ldots + (-1)^{2k}\, \Gamma^0_{p+1}\, C^{2k}_{p+1} / (p+1) \quad (1)$$

and

$$0 = \Gamma^{2k+1}_{p-2k}\, C^0_{p-2k} / (p-2k) - \Gamma^{2k}_{p+1-2k}\, C^1_{p+1-2k} / (p+1-2k)$$
$$+ \ldots + (-1)^{2k-1}\, \Gamma^0_{p+1}\, C^{2k+1}_{p+1} / (p+1). \quad \ldots\ldots\ldots\ldots(2)$$

We may, by suitably selecting p and k, obtain therefrom the following equations :—

$$0 = \Gamma^{2k+1}_{p+1-2k}\, C^0_{p+1-2k} / (p+1-2k) - \ldots$$
$$\ldots - \Gamma^0_{p+2}\, C^{2k+1}_{p+2} / (p+2)$$

$$\frac{(-1)^{k-1}\, p\, C_{2k}\, B_k}{p+1-2k} = \Gamma^{2k}_{p+1-2k}\, C^0_{p+1-2k} / (p+1-2k) - \ldots$$
$$\ldots + \Gamma^0_{p+1} C^{2k}_{p+1} / (p+1),$$

$$0 = \Gamma^{2k-1}_{p+1-2k}\, C^0_{p+1-2k} / (p+1-2k) - \text{etc.}$$

$$\frac{(-1)^{k-2}\,{}_{p-2}C_{2k-2} B_{k-1}}{p+1-2k} = \Gamma^{2k-2}_{p+1-2k}\, C^0_{p+1-2k} / (p+1-2k) - \text{etc.}$$

....

$$\frac{{}_{p-2k+2}C_2\, B_1}{p+1-2k} = \text{etc.}$$

$$-\tfrac{1}{2} = \frac{\Gamma^1_{p+1-2k}\, C^0_{p+1-2k}}{p+1-2k} - \frac{\Gamma^0_{p+2-2k}\, C^1_{p+2-2k}}{p+2-2k}$$

$$\frac{1}{p+1-2k} = \Gamma^0_{p+1-2k}\, C^0_{p+1-2k} / (p+1-2k).$$

Multiply these respectively by

$$C^0_{p+1},\ -C^1_{p+1},\ \ldots,\ (-1)^{2k+1}\, C^{2k+1}_{p+1},\ \text{and add.}$$

$$\therefore\ (-1)^k \{C^1_{p+1}\,{}_pC_{2k} B_k - C^3_{p+1}\,{}_{p-2}C_{2k-2} B_{k-1} + \ldots$$
$$+ (-1)^{k-1} C^{2k-1}_{p+1}\,{}_{p-2k+2}C_2 B_1\} / (p+1-2k)$$
$$= -C^{2k+1}_{p+2} / (p+2) + \tfrac{1}{2} C^{2k}_{p+1} + C^{2k+1}_{p+1} / (p+1-2k)$$

or $$(-1)^k \{\ldots\} = \frac{2k+1}{p+2} C^{2k+1}_{p+1} - \frac{p(p+1-2k)}{2(p+2)} C^{2k}_{p+1}. \quad \ldots\ldots\ldots\ldots(3)$$

Similarly, if we omit the first equation of the linear system, multiply the rest in succession by C_p^0, $-C_p^1$, etc., and add, we find, after a few reductions,

$$(-1)^k [C^0_p\,{}_pC_{2k} B_k - C^2_p\,{}_{p-2}C_{2k-2} B_{k-1} + \ldots + (-1)^{k-1} C^{2k-2}_p\,{}_{p-2k+2}C_2 B_1]$$
$$= C^{2k}_p \frac{2k}{p+1} - C^{2k-1}_p \frac{(p-1)(p+1-2k)}{2(p+1)}. \quad \ldots\ldots\ldots(4)$$

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Cor.—From these very general results we may, by putting $p=2k$, deduce in particular the following :—

$$(-1)^k B_k = \frac{1}{2}\Gamma_2^{2k-1} - \frac{2\,!}{3}\Gamma_3^{2k-2} + \frac{3\,!}{4}\Gamma_4^{2k-3} - \dots - \frac{(2k)\,!}{2k+1}\Gamma_{2k+1}^0 . \quad \dots(5)$$

$$0 = \frac{1}{2}\Gamma_2^{2k} - \frac{2\,!}{3}\Gamma_3^{2k-1} + \dots + \frac{(2k+1)\,!}{2k+2}\Gamma_{2k+2}^0 . \quad \dots\dots\dots\dots(6)$$

Also

$$(-1)^{k-1}\left[C_{2k+1}^1 B_k - C_{2k+1}^3 B_{k-1} + \dots + (-1)^{k-1} C_{2k+1}^{2k-1} B_1\right]$$
$$= k\,(2k)\,!\,/\,(2k+2), \quad \dots\dots\dots\dots\dots\dots\dots\dots(7)$$

and

$$C_{2k}^0 B_k - C_{2k}^2 B_{k-1} + \dots = (-1)^{k-1}\frac{(2k-1)}{2\,(2k+1)}(2k-1)\,!\,. \quad \dots\dots(8)$$

§ 9. *Some Recurrence Formulae for* B_1, B_2, etc

The three recurrence formulae for calculating B_1, B_2, etc., given in *Pascal's Repertorium* may be easily found as follows :—

In $F_{p+1}(x) = (p+1)\,S_p$

$$= x^{p+1} - \frac{(p+1)\,x^p}{2} + \Sigma\,(-1)^{k-1}\,{}_{p+1}C_{2k}\,B_k x^{p+1-2k}$$

express that $x-1$ is a factor

(i) when $p=2m$, when we obtain Demoivre's formula (1730), viz.,

$${}_{2m+1}C_1\,B_m - {}_{2m+1}C_3\,B_{m-1} + \dots (-1)^{m-1}\,{}_{2m+1}C_{2m-1}\,B_1$$
$$+ (-1)^m\,(m-\tfrac{1}{2}) = 0. \quad (1)$$

If we differentiate $S_{2m+1}(x)$ and express that $S'_{2m+1}(1)$ is $=0$, we simply find Demoivre's formula over again ;

(ii) when $p=2m+1$, when we obtain Jacobi's formula (1834),

$${}_{2m+2}C_2\,B_m - {}_{2m+2}C_4\,B_{m-1} + \dots (-1)^{m-1}\,{}_{2m+2}C_{2m}\,B_1$$
$$+ (-1)^m\,m = 0. \quad (2)$$

(iii) Subtract (1) from (2) and note that ${}_{2m+2}C_r = {}_{2m+1}C_r + {}_{2m+1}C_{r-1}$, when we obtain the formula of Stern (*Crelle*, 84),

$${}_{2m+1}C_2\,B_m - {}_{2m+1}C_4\,B_{m-1} + \dots + (-1)^{m-1}\,{}_{2m+1}C_{2m}\,B_1$$
$$+ (-1)^m\,\tfrac{1}{2} = 0. \quad (3)$$

Further, $F_{2m+1}(\tfrac{1}{2}) = 0$.

$$\therefore\ 2m - {}_{2m+1}C_2\,.\,2^2 B_1 + {}_{2m+1}C_4\,2^4 B_2 - \dots + (-1)^m\,{}_{2m+1}C_{2m}\,2^{2m} B_m = 0. \quad (4)$$

Since $F_{2m+1}(x)$ contains the factors $x\,(x-1)\,(2x-1)$, therefore $F_{2m+1}(x)\,/\,(1-x)\,(1-2x)$ is an integral function of x of degree $2m-1$.

But, for a suitable continuum for x, $1/(1-x)(1-2x) = \Sigma\,\Gamma_3^s x^s$, and for the same continuum $F_{2m+1}(x) \times \Sigma\,\Gamma_3^s x^s$ is equivalent to an integral function of x of degree $2m-1$.

Hence the coefficients, in the product, of x^{2m}, x^{2m+1}, etc., must vanish, so that

$$\Gamma_3^0 \frac{2m+1}{2} - \Gamma_3^1 \; {}_{2m+1}C_2 B_1 + \Gamma_3^3 \; {}_{2m+1}C_4 B_2$$
$$\ldots + (-1)^m \Gamma_3^{2m-1} \; {}_{2m+1}C_{2m} B_m = 0, \quad (5)$$

and

$$\Gamma_3^{a} - \frac{2m+1}{2} \Gamma_3^{a+1} + \sum_{k=1}^{m} (-1)^{k-1} \Gamma_3^{a+2k} \; {}_{2m+1}C_{2k} B_k = 0. \quad \ldots\ldots(5)^1$$

Since $\Gamma_3^a = 2^{a+1} - 1$, the former of these may be written as

$$(2^{2m}-1)\; {}_{2m+1}C_1 B_m - (2^{2m-2}-1)\; {}_{2m+1}C_3 B_{m-1}$$
$$+ \ldots + (-1)^{m-1}(2^2-1)\; {}_{2m+1}C_{2m-1} B_1 + (-1)^m (m+\tfrac{1}{2}) = 0. \quad (6)$$

Added to Demoivre's formula, it gives

$$2^{2m}\; {}_{2m+1}C_1 B_m - \ldots + (-1)^{m-1} 2^2\; {}_{2m+1}C_{2m-1} B_1 + (-1)^m 2m = 0, \quad (7)$$

as in (4).

On subtracting Demoivre's formula, we find

$$2(2^{2m-1}-1)\; {}_{2m+1}C_1 B_m - \ldots$$
$$+ (-1)^{m-1} 2(2-1)\; {}_{2m+1}C_{2m-1} B_1 + (-1)^m = 0. \quad (8)$$

These last two formulae are found by trigonometrical series in *Saalschutz, Die Bernoullischen Zahlen.* The other, $(5)^1$, is only apparently more general, for after subtraction of $(-1)^m$ of (1), the factor 2^a may be removed, when it reduces to (6).

§10. *The Stirling Polynomial.*

It has been seen that

$$C_n^r = n(n-1)\ldots(n-r)\,\phi_{r-1}(n)$$

where $\phi_{r-1}(n)$ is an integral function of n of degree $r-1$; and at the same time Γ_n^r admits of expression in the form

$$\Gamma_n^r = (-1)^{r+1}(n-1)\,n\,(n+1)\ldots(n+r-1)\,\phi_{r-1}(1-n).$$

If, therefore, a full discussion of the function ϕ were possible, a complete knowledge of the Stirling Numbers would ensue. Nielsen has therefore proposed to call this function the *Stirling Polynomial*, though the analogy with the Bernoullian Numbers and Polynomials is not perfect.

From the relation

$$C_{n+1}^{r+1} = C_n^{r+1} + n\,C_n^{r}$$

follows that

$$(n+1)\,\phi_r(n+1) = (n-r-1)\,\phi_r(n) + n\,\phi_{r-1}(n). \quad \ldots\ldots\ldots(1)$$

This functional equation may be used to deduce a number of properties of the Stirling Polynomial.

In fact, if we assume $\phi_0(n) = 1/2$ and restrict the solutions of (1) to be integral functions, it is easy to shew that only one series of integral functions is found : $\phi_0(n)$, $\phi_1(n)$, $\phi_2(n)$, etc. ; and they are therefore the Stirling Polynomials.

Some particular values of ϕ for special values of n and r may be noted.

Since C_n^{2m+1} contains the factor $n^2(n-1)^2$,

$$\therefore \quad \phi_{2m}(0) = 0\,; \quad \phi_{2m}(1) = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Also $S_p + (-1)^p\,p\,C_n^p$ is divisible by $n^2(n-1)^2$, and

$$S_{2m} = n^{2m+1}/(2m+1) + \ldots + (-1)^{m-1}B_m\,n,$$

$$\therefore \quad \phi_{2m-1}(0) = (-1)^m B_m / 2m \times (2m)\,!\,. \quad \ldots\ldots\ldots\ldots(3)$$

Moreover, from (1)

$$\phi_r(1) = -(r+1)\,\phi_r(0),$$

so that

$$\phi_{2m+1}(1) = (-1)^m B_{m+1}/(2m+2)\,!\,. \quad \ldots\ldots\ldots\ldots\ldots(4)$$

Since $\quad (n-1)\,! = C_n^{n-1} = n\,!\,\phi_{n-2}(n),$

$$\therefore \quad \phi_n(n+2) = 1/(n+2). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Also $\quad (n-1)\,!\left(1 + \frac{1}{2} + \ldots + \frac{1}{n-1}\right) = C_n^{n-2} = n\,!\,\phi_{n-3}(n),$

$$\therefore \quad \phi_n(n+3) = \frac{1}{n+3}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n+2}\right). \quad \ldots\ldots(6)$$

Similarly

$$\phi_n(n+4) = \frac{1}{n+4}\left[\left(1 + \frac{1}{2} + \ldots + \frac{1}{n+3}\right)^2 - \left(1^2 + \frac{1}{2^2} + \ldots + \left(\frac{1}{n+3}\right)^2\right)\right]$$

Since $\quad \Gamma_n^r = (-1)^{r-1}(n-1)\,n\ldots(n+r-1)\,\phi_{r-1}(1-n). \quad \ldots\ldots(7)$

$$\therefore \quad \Gamma_2^r = (-1)^{r-1}(r+1)\,!\,\phi_{r-1}(-1)$$

$$\text{or} \quad \phi_r(-1) = (-1)^r/(r+2)\,!\,.$$

Similarly $\quad \phi_r(-2) = (-1)^r(2^{r+2}-1)/(r+3)\,! \quad \ldots\ldots\ldots\ldots\ldots(8)$

§ 11. *Nature of the coefficients of the Stirling Polynomial* $\phi^r(x)$.

The determination of the coefficients of $\phi_r(x)$ is not simple, but recurrence formulae for their calculation are easily furnished.

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I note only two of these:

$$(x+1)\,\phi_r(x+1) - x\,\phi_r(x) + (r+1)\,\phi_r(x) = x\,\phi_{r-1}(x), \qquad (1)$$

and from (5), § 2,

$$(r+1)\,\phi_r(x) = \frac{x-r}{2\,!}\,\phi_{r-1}(x) + \frac{x-r+1}{3\,!}\,\phi_{r-2}(x)$$

$$+ \frac{x-r+2}{4\,!}\,\phi_{r-3}(x) + \ldots + \frac{x-1}{(r+1)\,!}\,\phi_0(x) + \frac{1}{(r+2)\,!}. \qquad (2)$$

Assume $\phi_r(x) = S_r^0 x^r + S_r^1 x^{r-1} + \ldots + S_r^r$.

Substitution in (1) leads to the equations

$$2(r+1)\,S_r^0 = S_{r-1}^0,$$

$${}_{r+1}C_2\,S_r^0 + {}_rC_1\,S_r^1 + (r+1)\,S_r^1 = S_{r-1}^1,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$${}_{r+1}C_k\,S_r^0 + {}_rC_{k-1}\,S_r^1 + \ldots + {}_{r-k+2}C_1\,S_r^{k-1} + (r+1)\,S_r^{k-1} = S_{r-1}^{k-1},$$

etc., etc.

Substitution in (2) leads to

$$(r+1)\,S_r^0 = \tfrac{1}{2}\,S_{r-1}^0,$$

$$(r+1)\,S_r^1 = \tfrac{1}{2}\,S_{r-1}^1 - \frac{r}{2}\,S_{r-1}^0 + \tfrac{1}{6}\,S_{r-2}^0,$$

$$(r+1)\,S_r^2 = \tfrac{1}{2}\,S_{r-1}^2 - \frac{r}{2}\,S_{r-1}^1 + \tfrac{1}{6}\,S_{r-2}^1 - \frac{r-1}{6}\,S_{r-2}^0 + \tfrac{1}{24}\,S_{r-3}^0,$$

etc., etc.

Hence $\qquad S_r^0 = 1/2^{r+1}(r+1)\,!\,.$ ……………………(3)

Combine the second equations of the two systems to eliminate S_{r-1}^1, and there results

$$S_r^1 = -1/12\,.\,2^r(r-1)\,!\,. \qquad \ldots\ldots(4)$$

The elimination of S_{r-1}^2 from the third equations furnishes

$$S_r^2 = 1/144\,.\,2^r(r-3)\,!\,. \qquad \ldots\ldots(5)$$

Similarly

$$S_r^3 = -(r^2 - 7r + 24/5)\,/\,3^4\,.\,2^5\,.\,2^r(r-3)\,!\,. \qquad \ldots\ldots(6)$$

From the two equations for S_r^4 the elimination of S_{r-1}^4 leaves S_r^4 with a numerical coefficient independent of r, and it should have in the denominator $(r-3)\,!$, with a numerator of degree higher by two than that of S_r^3. The nature of this equation shews that the denominator should reduce to $(r-4)\,!$. But since $S_4^4 = 0$, the simplified numerator also contains the factor $r-4$, so that the final form of S_r^4 is $f_2(r)/2^{r+1}(r-5)\,!$. By continuing this reasoning we obtain the result

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$$2^{r+1}\phi_r(x) = \frac{x^r}{(r+1)!} - \frac{x^{r-1}}{6(r-1)!} + \frac{x^{r-2}}{72(r-3)!}$$
$$- \frac{r^2 - 7r + 24/5}{1296(r-3)!} x^{r-3} + \frac{x^{r-5}}{(r-5)!}\{x f_2(r) + F_4(r)\}$$
$$+ \frac{x^{r-7}}{(r-7)!}\{x f_4(r) + F_6(r)\} + \text{etc.}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

in which $f_2(r)$, etc., are integral functions of r.

We easily find the following particular results :

$$S_{2r}^{2r} = \phi_{2r}(0) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

$$S_{2r+1}^{2r+1} = \phi_{2r+1}(0) = (-1)^{r+1} B_{r+1} / (2r+2) \times (2r+2)! \quad \ldots\ldots\ldots (9)$$

$$\phi_{2m}(1) = 0, \quad \therefore \quad \sum_{k=0}^{2m} S_{2m}^{k} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Similarly $\displaystyle\sum_{k=0}^{2m+1} S_{2m+1}^{k} = \phi_{2m+1}(1) = (-1)^m B_{m+1} / (2m+2)! \ldots\ldots (11)$

and $S_r^0 - S_r^1 + S_r^2 - \ldots = 1/(r+2)!$. $\ldots\ldots\ldots\ldots\ldots\ldots (12)$

§12. *Generalisation of Stirling Numbers and Bernoullian Polynomials.*

The relation $C_{n+1}^r = C_n^r + n\, C_n^{r-1}$

leads to the functional equation

$$F_{2r}(x+1) = F_{2r}(x) + x\, F_{2r-2}(x), \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

which furnishes a unique series of integral functions $F_2(x)$, $F_4(x)$, etc., when we assume $F_0(x) = 1$; $F_2(0) = F_4(0) =$ etc. $= 0$.

The presence of the factorial $(x-1)(x-2)\ldots(x-r)$ in $F_{2r}(x)$ is easily established.

For $F_{2r}(0) = 0$, by hypothesis.

$$F_{2r}(1) = F_{2r}(0) + 0 = 0$$
$$F_{2r}(2) = F_{2r}(1) + F_{2r-2}(1)$$
$$= 0, \quad \text{provided } r > 1,$$
$$F_{2r}(3) = F_{2r}(2) + 2F_{2r-2}(2)$$
$$= 0, \quad \text{provided } r > 2,$$
etc., etc.

Cor. 1.—Equivalent initial conditions for the functions $F(x)$ are

I. $F_0(x) = 1$; $F_{2k}(0) = 0$.

II. $F_0(x) = 1$; $F_{2k}(1) = 0$.

III. $F_0(x) = 1$; $F_2(1) = F_4(2) = \ldots = F_{2r}(r) = 0$.

Cor. 2.—The integral functions furnished by

$$F_{2r}(x+1) = F_{2r}(x) + Cx\,F_{2r-2}(x)$$

with the same initial conditions are those already found multiplied by the constant C.

Cor 3.—Similar reasoning applies to the system of integral functions furnished by the functional equation

$$\phi_{nr}(x+1) = \phi_{nr}(x) + x^{n-1}\phi_{n(r-1)}(x) \quad \ldots\ldots\ldots\ldots\ldots (2)$$

with the initial conditions

$$\phi_0(x) = 1\ ; \quad \phi_{nk}(0) = 0.$$

The first function obtained is simply the Bernoullian Polynomial

$$\phi_n(x) = 1^{n-1} + 2^{n-1} + \ldots + (x-1)^{n-1}.$$

The other functions are the sums of such Bernoullians of different degrees.

The function $\phi_{nr}(x)$ *contains the factor* $x(x-1)\ldots(x-r)$.

Also, if $\phi_{nr}(x)$ *contains the factor* x^a, *it contains the factor* $(x-1)^a$ *as well, provided* $\alpha \not> n$.

For

$$\left.\begin{aligned} \phi_{nr}(1) &= \phi_{nr}(0) \\ \phi'_{nr}(1) &= \phi'_{nr}(0) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \phi_{nr}^{(n-1)}(1) &= \phi_{nr}^{(n-1)}(0) + (n-1)!\ \phi_{nr-n}(0) \end{aligned}\right\}, \quad \ldots\ldots\ldots (3)$$

so that when the right sides of these equations vanish, so do the left sides.

Note.—It will be proved presently that when $n-1$ is odd, the alternate functions are all divisible by $x^2(x-1)^2$.

With respect to Bernoullian Polynomials, we may note that $\Sigma(\Sigma x^{2n})$ is divisible by $x(x-1)^2$.

For $\Sigma x^{2n} = \dfrac{x^{2n+1}}{2n+1} - \dfrac{x^{2n}}{2}$ + odd powers of $x + (-1)^{n-1}B_n x$.

On summing again we need only examine for $-\dfrac{x^{2n}}{2} + (-1)^{n-1}B_n x$, since Σx^{2n+1} always contains $x^2(x-1)^2$.

Now $\psi(x) = \Sigma\left(-\dfrac{x^{2n}}{2}\right) + (-1)^{n-1}B_n\dfrac{x(x-1)}{2}$ possesses a term in x, viz., $(-1)^n B_n x$, and is therefore not divisible by x^2.

It is known that $\dfrac{d}{dx}\left[\Sigma - \dfrac{x^{2n}}{2}\right]$ reduces to $-(-1)^{n-1}B_n/2$ when $x=1$.

Hence $\psi'(1)=0$. But $\psi(1)=0$. $\therefore$ $\psi(x)$ contains the factor $(x-1)^2$.

If $\phi_{nr}(x) = \sum_{0}^{nr-1} a_k x^{nr-k} / (nr-k)!$, we find from (3)

$$\left.\begin{aligned} &a_0/(nr)! + \ldots\ldots\ldots + a_{nr-1} = 0 \\ &a_0/(nr-1)! \quad + \ldots + a_{nr-2} = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &a_0/(nr-n+1)! + \ldots + a_{nr-n} = (n-1)!\ \phi_{n(r-1)}(0) \end{aligned}\right\} . \quad \ldots\ldots(4)$$

For example, for

$$\phi_n(x+1) - \phi_n(x) = x^{n-1}/(n-1)!$$

we obtain the equations

$$\left.\begin{aligned} &a_0 = 1 \\ &\frac{a_0}{2!} + \frac{a_1}{1!} = 0 \\ &\ldots\ldots\ldots\ldots\ldots \\ &\frac{a_0}{n!} + \frac{a_1}{(n-1)!} + \ldots + a_{n-1} = 0 \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

which may be used to determine the Bernoullian numbers.

Cor.—The functional equation (2) simply corresponds to a recurrence formula for the successive calculation of the coefficients of

$$x(x+1^{m-1})(x+2^{m-1})\ldots(x+\overline{n-1}^{m-1}) = \sum_{r=0}^{n-1} s_n^r\ x^{n-r} \quad \ldots\ldots\ldots(6)$$

and of

$$1/x(x+1^{m-1})\ldots(x+\overline{n-1}^{m-1}) = \sum_{s=0}^{\infty} (-1)^s \sigma_n^s / x^{n+s}, \quad \ldots\ldots\ldots(7)$$

in which s_n^r and σ_n^r may be called the *Generalised Stirling Numbers.*

Thus $\qquad s_{n+1}^r = s_n^r + n^{m-1} s_n^{r-1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$

corresponding to $\phi_{mr}(x+1) = \phi_{mr}(x) + x^{m-1}\phi_{m(r-1)}(x)$.

For the functions σ we find

$$\sigma_{n+1}^r - n^{m-1}\sigma_{n+1}^{r-1} = \sigma_n^r, \quad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

corresponding, say, to

$$\psi_{mr}(x) = \psi_{mr}(1+x) - x^{m-1}\psi_{mr-m}(1+x) \quad \ldots\ldots\ldots(10)$$

There are two cases to distinguish, according as m is even or odd.

If $m = 2\mu$, write $-\xi$ for x in (10).

$$\therefore \quad \psi_{mr}(-\xi) = \psi_{mr}(1-\xi) + \xi^{m-1}\psi_{mr-m}(1-\xi)$$

Put $\psi_{mr}(1-\xi) = \phi_{mr}(\xi)$ for all values of r and ξ, and therefore $\psi_{mr}(-\xi) = \phi_{mr}(1+\xi)$.

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Hence $\phi_{mr}(1+\xi)=\phi_{mr}(\xi)+\xi^{m-1}\phi_{mr-m}(\xi)$. ....(11)

If $m=2\mu+1$,

$$\psi_{mr}(-\xi)=\psi_{mr}(1-\xi)-\xi^{m-1}\psi_{mr-m}(1-\xi).$$

Write $\psi_{mr}(1-\xi)=(-1)^{r-1}\phi_{mr}(\xi)$

$\therefore$ $\phi_{mr}(1+\xi)=\phi_{mr}(\xi)+\xi^{m-1}\phi_{mr-m}(\xi)$. ....(12)

Hence, if m is even and $s_n^r=\phi_{mr}(n)$, then $\sigma_n^r=\phi_{mr}(1-n)$; but if m is odd, $\sigma_n^r=(-1)^{r-1}\phi_{mr}(1-n)$.

A variety of identities may then be established connecting the generalised numbers, admitting, in particular, of the expression of the one system of numbers in terms of the other. In particular, (1) § 5, with Corollary, holds unchanged.

The numbers s_n^r are expressible as homogeneous integral functions of C_n^0, C_n^1 ..., C_n^{a-1} of degree $m-1$.

Let $Q_n^{m-1}(x)=x(x-1^{m-1})\ldots(x-\overline{n-1}^{m-1})$

$$Q_n(x)=x(x-1)\ldots\ldots(x-n+1),$$

and ω a primitive $(m-1)^{\text{th}}$ root of unity.

Then $Q_n^{m-1}(\xi^{m-1})=\pm Q_n(\omega\xi)\times Q_n(\omega^2\xi)\ldots Q_n(\omega^{m-1}\xi)$, ...(13)

from which the statement at once follows.

Also when $m-1$ is an odd number and r is odd, the generalised Stirling numbers s_n^r and σ_n^r are divisible by $n^2(n-1)^2$.

For example, when $m-1=3$, s_n^1 consists of ternary products such as $C_n^0\,C_n^0\,C_n^3$ and terms involving at least two numbers distinct from C_n^0, each of which is divisible by $n(n-1)$, while C_n^3 contains the factor $n^2(n-1)^2$. Similarly, s_n^3 involves terms in $C_n^0\,C_n^0\,C_n^9$ and terms involving at least two numbers distinct from C_n^0. $\therefore$ etc.

The proof for σ_n^r follows from (1) § 5, or from (11) and (12).

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