STONE LATTICES. II. STRUCTURE THEOREMS

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1. Introduction. Using the triple associated with a Stone algebra L, as introduced in the first part of this paper (**1**), we will investigate certain problems concerning the structure of a Stone lattice.

The following topics will be discussed: prime ideals, topological representation, completeness, relative Stone lattices, and the reduced triple.

It is assumed that the reader is familiar with §§ 2–4 of (**1**). For the sake of convenience, we will write $L = \langle C, D, \phi \rangle$ to indicate that $\langle C, D, \phi \rangle$ is the triple associated with L, and whenever convenient we will write the elements of L as ordered pairs $\langle x, a \rangle$, as it is given in (**1**, § 4, the Construction Theorem).

2. Prime ideals. The first result on Stone lattices was a characterization of Stone lattices as a pseudo-complemented distributive lattice L in which $P \vee Q = L$ for any two distinct minimal prime ideals P and Q; see (**2**). A simple proof of this is given in (**5**), while in (**3**) it is proved that if L is not assumed to be pseudo-complemented, then the conclusion is false.

For $X \subseteq L$, let $(X]$ denote the ideal generated by X, that is $a \in (X]$ if and only if $a \leqq \bigvee(x \mid x \in X_1)$, for some finite $X_1 \subseteq X$. $P(L)$ denotes the set of all prime ideals of the lattice L.

THEOREM 1. *Let* $L = \langle C, D, \phi \rangle$.

(i) *The correspondence*

$$P \to (P], \qquad P \in P(D), \tag{2.1}$$

is a one-to-one mapping from $P(D)$ *onto the set of all prime ideals of* L *not disjoint from* D.

(ii) *The correspondence*

$$P \to (P], \qquad P \in P(C), \tag{2.2}$$

is a one-to-one mapping from $P(C)$ *onto the set of all prime ideals of* L *disjoint from* D*; the latter set is the same as the set of all minimal prime ideals of* D.

Proof. (i) For $P \in P(D)$, $(P]$ is obviously an ideal of L *not disjoint* from D. Since $(P]$ is disjoint from the dual ideal $D - P$, by Stone's theorem (see, e.g., **2**) there exists a prime ideal Q of L containing $(P]$ and disjoint from $D - P$. Thus, $Q \cap D = (P]$. If $x \in Q$, then for any $p \in P$ we obtain $x \vee p \in D$; hence, $x \vee p \in P$ and $x \in (P]$. Thus, $Q = (P]$, proving that $(P]$ is prime and the map in (i) is one-to-one. (It is easily seen that a slightly longer proof can be

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given that does not use Stone's theorem.) If Q is any prime ideal of L with $Q \cap D \neq \emptyset$, then, as above, $Q = (Q \cap D]$ and $Q \cap D \in P(D)$, completing the proof of (i).

(ii) Let $P \in P(C)$. The $(P]$ is an ideal of L. Let $x \wedge y \in (P]$; then $x \wedge y \leqq p$ for some $p \in P$. Thus, $(x \wedge y)^{**} \leqq p^{**} = p$, that is, $x^{**} \wedge y^{**} \in P$. This implies that x^{**} or $y^{**} \in P$, and therefore x or $y \in (P]$. Therefore, $(P]$ is a prime ideal, and $(P] \cap D = \emptyset$. Now, if Q is any prime ideal of L with $Q \cap D = \emptyset$, then for any $a \in Q$, $a^* \neq 0$, $a \wedge a^* \in Q$, hence $a^* \notin Q$. Thus $a^{**} \in Q$, which proves that $Q = (Q \cap C]$.

Let Q be a minimal prime ideal of L. Then $Q \cap C$ is a prime ideal in C, hence $(Q \cap C]$ is a prime ideal of L, contained in Q. Hence, $Q = (Q \cap C]$; thus, Q is disjoint to D. Now let Q be a non-minimal prime ideal of L; let $P \subset Q$, where P is also a prime ideal. Take $x \in Q - P$. Then $0 = x \wedge x^* \in P$, hence $x^* \in P$. Thus, $x \vee x^* \in Q$, proving that $D \cap Q \neq \emptyset$, since $x \vee x^* \in D$ always holds. This completes the proof of Theorem 1.

As an immediate corollary we obtain the result of (**2**) mentioned at the beginning of § 2.

COROLLARY. *Let L be a pseudo-complemented, distributive lattice. Then L is a Stone lattice if and only if $P \vee Q = L$ for any two distinct, minimal prime ideals P and Q.*

Proof. Let L be a Stone lattice, P and Q distinct minimal prime ideals, that is (by Theorem 1) prime ideals disjoint to D. Then by Theorem 1 (ii), $P \cap C$ and $Q \cap C$ are distinct prime ideals (i.e., maximal ideals) of the Boolean algebra C. Therefore, $x \in P \cap C$, $y \in Q \cap C$ exist with $x \vee y = 1$ in C, hence in L, proving that $P \vee Q = L$.

Conversely, if L is not Stone, then $a^* \vee a^{**} < 1$, for some $a \in L$. Let P be a prime ideal containing $a^* \vee a^{**}$. It is easily seen that $a^*, a^{**} \notin (L - P) \vee D$. (Indeed, if for instance $a^* \in (L - P) \vee D$, then $a^* = x \wedge d$, for some $x \in L - P, d \in D$. Then $a^* = a^{***} = (x \wedge d)^{**} = x^{**} \wedge d^{**} = x^{**} \wedge 0^* = x^{**}$, and hence $x \in P$, a contradiction.) Thus, there exist prime ideals Q, R of L containing a^* and a^{**}, respectively, and disjoint to $(L - P) \vee D$. Since $Q \neq R$ and $Q \vee R \subseteq P \neq L$, we have completed the proof.

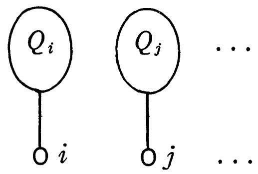
Theorem 1 enables us to characterize the partially ordered set $P(L)$ of all prime ideals of a Stone lattice L modulo the characterization problem of $P(D)$, where D is a distributive lattice with 1. The latter problem is a special case of the former one.

Let us call a partially ordered set Q *representable* if there exists a distributive lattice K with 1, such that $Q \cong P(K)$ (as partially ordered sets).

A partially ordered set Q is in *standard form* if the following conditions are satisfied:

(1) $Q = \bigcup(Q_i | i \in I) \cup I$;

(2) the Q_i are pairwise disjoint, and all the Q_i are disjoint to I;
(3) $i < x$ for $x \in Q_i$ and $i \nleq x$ for $x \in Q_j$, $i \neq j$.
It follows from (3) that
(4) $x \nleq y$ for any $x \in Q_i$, $y \in Q_j$, $i \neq j$.
The following diagram illustrates Q:



THEOREM 2. *Let L be a Stone lattice. Then $P(L)$ can be written in standard form* (1)–(3) *above such that*
(*) *all Q_i are representable.*
Conversely, if Q is a partially ordered set in the standard form (1)–(3), *and* (*) *is satisfied, then there exists a Stone lattice L such that Q and $P(L)$ are isomorphic.*

Proof. Let L be a Stone lattice, $C = C(L)$, $D = D(L)$, and

$$I = \{(P] \mid P \in P(C)\}.$$

Set

$$P \in Q_i, \quad \text{if } i \subseteq P, P \cap D \neq \emptyset, P \in P(L). \tag{2.3}$$

Then $P(L) = \bigcup(Q_i \mid i \in I) \cup I$ by Theorem 1, and (2) is obvious. By the Corollary to Theorem 1, a prime ideal contains exactly one minimal prime ideal, verifying (3).

To prove (*) we have to construct a distributive lattice L_i with $Q_i \cong P(L_i)$.

For any ideal I of L there exists a *smallest* congruence relation Θ such that $x \equiv y(\Theta)$ for all $x, y \in I$; the notation L/I stands for L/Θ. The following lemma can be easily verified.

LEMMA 1. *Let P be a prime ideal of a distributive lattice L. Then there is an order isomorphism between the prime ideals of L containing P and the prime ideals of L/P.*

This implies that in L/i, the prime ideals not $\{0\}$ form a partially ordered set isomorphic with Q_i. However, the 0 of L/i is meet irreducible (since $\{0\}$ is prime), hence $L_i = (L/i) - \{0\}$ will satisfy $P(L_i) \cong Q_i$. This completes the proof of the first part of Theorem 2.

Now, let us assume that Q is in a standard form (1)–(3), and (*) holds. Since the Q_i are representable, there exist distributive lattices D_i with 1 such that $P(D_i) \cong Q_i$ for all $i \in I$. Let E be the (complete) direct product of the D_i, and

$$D = \{f \mid f \in E, |\{i \mid f(i) \neq 1\}| < \aleph_0\}.$$

Then D is a distributive lattice with 1; we can identify $a \in D_i$ with the function f defined by $f(i) = a$, $f(j) = 1$ for $j \neq i$. Thus, D_i becomes a sublattice of D.

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A prime ideal of D_i generates in D an ideal, and it is a prime ideal of D. These and only these are the prime ideals of D, hence with a slight change of notation we can write $P(D) = \bigcup(Q_i | i \in I)$.

There exists a Boolean algebra C with $|P(C)| = |I|$, thus we can assume that $P(C) = I$. We define a map ϕ of C into $\mathscr{D}(D)$ as follows: let $a \in C$, set

$$a\phi = \{f | f \in D \text{ and } f(i) = 1 \text{ for } a \in i\}. \tag{2.4}$$

Then $0\phi = \{1\}$ and $1\phi = D$ are obvious. Furthermore,

$$\begin{aligned}(a \wedge b)\phi &= \{f | f(i) = 1 \text{ for } a \wedge b \in i\} \\ &= \{f | f(i) = 1 \text{ for } a \in i\} \cap \{f | f(i) = 1 \text{ for } b \in i\} = a\phi \wedge b\phi.\end{aligned}$$

This also implies that $(a \vee b)\phi \supseteq a\phi \vee b\phi$. Now let $f \in (a \vee b)\phi$, that is $f(i) = 1$ if $a \vee b \in i$. Define $g, h \in D$ by $g(i) = f(i)$ for $a \in i$, $g(i) = 1$, otherwise; $h(i) = f(i)$ for $b \in i$, $g(i) = 1$, otherwise. Then $g \in b\phi$, $h \in a\phi$, and $g \wedge h = f$, thus $f \in a\phi \vee b\phi$, completing the proof of $(a \vee b)\phi = a\phi \vee b\phi$.

Thus, we have proved that $\langle C, D, \phi\rangle$ is a triple. By the Construction Theorem there exists a Stone lattice L such that $\langle C, D, \phi\rangle$ is the associated triple. We claim that $P(L) \cong Q$.

We have already noted that $P(D) = \bigcup(Q_i | i \in I)$ and $P(C) = I$. Thus, it is enough to prove that $(i] \subseteq (P]$ for $P \in Q_j$ if and only if $i = j$ (compare this with Theorem 1). Let $a \in i$. Then $f \in a\phi$ implies that $f(i) = 1$, thus $f \notin P$ for any $P \in Q_i$. Therefore, $a\phi$ is disjoint from P, hence $a' \notin (P]$, and therefore $a \in (P]$, proving that $(i] \subseteq (P]$ for $P \in Q_i$. Since L is a Stone lattice, $(P]$ contains exactly one minimal prime ideal, thus $(j] \subseteq (P]$ implies that $i = j$. This completes the proof of Theorem 2.

Theorem 2 describes the structure of $P(L)$ using conditions (1)–(4) and (*) which, it is hoped, make it easy to visualize $P(L)$. A briefer version of (1)–(4) and (*) is given in the following corollary.

COROLLARY. *Let Q be a partially ordered set. There exists a Stone lattice L such that $Q \cong P(L)$ if and only if every element of Q contains exactly one minimal element and for every minimal element m, the partially ordered set*

$$\{x | x > m, x \in Q\}$$

is representable.

Note that $\{x | x > m, x \in Q\}$ could be replaced by $\{x | x \geqq m, x \in Q\}$.

3. Topological representation. In (**4**), Stone gave a topological representation theorem for arbitrary distributive lattices. Let L be a distributive lattice, $P(L)$ the set of all prime ideals of L. For $x \in L$ we set

$$R_L(x) = \{P | P \in P(L), x \notin P\}.$$

The *Stone space* $S(L)$ of L is $P(L)$ with $\{R_L(x) | x \in L\}$ as a base for open sets. $S(L)$ determines L; in fact, L is isomorphic to the lattice of all compact open sets of $S(L)$.

A characterization of $S(L)$ is given in (**4**). This immediately yields a characterization of $S(L)$, where L is a Stone lattice. To the conditions in (**4**) we have to add that: (i) $S(L)$ is compact (since $1 \in L$); (ii) if X is compact open, then the interior of $S(L) - X$ is closed (or, equivalently, the closure of any compact open set is open).

In this section we will be concerned with the relationship between $S(C)$, $S(D)$, and $S(L)$, as it was set up in Theorem 1. By Theorem 1, the map $\phi: P \rightarrow (P]$ is one-to-one and onto between $S(C) \cup S(D)$ and $S(L)$. Thus, $|S(C)| + |S(D)| = |S(L)|$.

THEOREM 3. *$S(C)$ is homeomorphic to $S(C)\phi$, as a subspace of $S(L)$; in fact, ϕ is a homeomorphism. $S(C)\phi$ is everywhere dense in $S(L)$. In particular, $S(C)\phi = S(L)$ if and only if $C = L$. $S(C)\phi$ is open in $S(L)$ if and only if D has a smallest element.*

Proof. The homeomorphism follows from the following formula, which is immediate from Theorem 1:

$$R_L(x) \cap S(C)\phi = R_C(x^{**})\phi.$$

To prove that $S(C)\phi$ is everywhere dense in $S(L)$, take a $P \in S(L)$ and $R_L(x)$ containing P; we have to show that $R_L(x) \cap S(C)\phi \neq \emptyset$. Indeed, since $x \notin P$, we obtain $x \notin Q$, where Q is the minimal prime ideal contained in P, and $Q \in R_L(x) \cap S(C)\phi$.

If D has a least element, say d, then $S(C)\phi = R_L(d)$, thus $S(C)\phi$ is open.

Now, assume that $S(C)\phi$ is open. Then we can form $I = \{x| \ x \leqq d$ for every $d \in D\}$. Obviously, I is an ideal of L. If I and D have an element, say d, in common, then $D = [d)$ as claimed. The assumption that I and D are disjoint leads to a contradiction. Indeed, then there exists a prime ideal P, with $P \supseteq I, P \cap D = \emptyset$. Then $P \in S(C)\phi$. Since $S(C)\phi$ is open, there exists an $x \in L$ with $P \in R_L(x) \subseteq S(C)\phi$. This implies that $x \notin P$, hence $x \notin I$, and therefore $x \nleqq d$ for some $d \in D$. Thus, there exist a prime ideal Q with $x \notin Q, d \in Q$. We then have $Q \in R_L(x), Q \notin S(C)\phi$, a contradiction. This completes the proof of Theorem 3.

To investigate $S(D)\phi$, we need a notation. Let $x \leqq D$ mean that $x \leqq d$ for all $d \in D$; $x \nleqq D$ is the negation of $x \leqq D$.

THEOREM 4. *The map ϕ is a homeomorphism of $S(D)$ with the subspace $S(D)\phi$ of $S(L)$ if and only if for $x \nleqq D$, the set $\{d| \ x \leqq d, d \in D\}$ has a least element d_x.*

Proof. Let us assume that ϕ is a homeomorphism, and let $x \in L$ such that $x \nleqq D$ and $\{d| \ x \leqq d, d \in D\}$ has no smallest element. Then for all $d \in D$ with $x \leqq d$ there exists a $d_1 \in D$ with $x \leqq d_1 < d$, and therefore there exists a prime ideal P of L with $d \notin P$ and $d_1 \in P$. Hence $P \cap D \in R_D(d)$, but $P \cap D \notin \{Q \cap D| \ Q \in R_L(x)\}$. Thus, $R_D(d) \nsubseteq \{Q \cap D| \ Q \in R_L(x)\}$ for all $d \in D$, and therefore the latter is not open in $S(D)\phi$; hence, ϕ is not a homeomorphism.

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Conversely, if the condition holds, then the formula

$$\{P \cap D \mid P \in R_L(x)\} = R_D(d_x), \tag{3.1}$$

shows that ϕ is a homeomorphism. To establish (3.1), note that the left side is always contained in the right side. Now let $P \in R_D(d_x)$. Note that $x \notin (P]$, since $x \in (P]$ implies that $x \leqq p$ for some $p \in P$, and thus $d_x \leqq p$, contradicting $d_x \notin P$. Therefore, $x \notin (P]$; that is, $(P] \in R_L(x)$, and hence by Theorem 1, $(P] \cap D = P$, proving the reverse inclusion in (3.1). This concludes the proof of Theorem 4.

COROLLARY. *If the conditions of Theorem 4 are satisfied, then (and only then) ϕ is a homeomorphism of $S(C) \cup S(D)$ and $S(L)$, where $S(C) \cup S(D)$ is the disjoint union of $S(C)$ and $S(D)$ with sets of the form $\{R_C(a) \cup R_D(d_x)\}$ with $a \in C, x \in F_a$ as a base for the topology. (We set $d_x = 1$ for $x \leqq D$.)*

Proof. Using the formula

$$d_x \wedge d_y = d_{x \wedge y}, \tag{3.2}$$

it is easy to check that $(R_C(a) \cup R_D(d_x))\phi = R_L(x)$, proving the Corollary. The proof of (3.2) is straightforward, and will be left to the reader.

4. Completeness. Suppose that the Stone lattice L is given by the triple $\langle C, D, \phi \rangle$ (we assume that the elements of L are of the form $\langle x, a \rangle$, $x \in a\phi$, as in the Construction Theorem). How can we tell whether L is complete?

THEOREM 5. *The Stone lattice L is complete if and only if the following conditions are satisfied:*

(1) C is complete;

(2) D is conditionally complete;

(3) for each $E \subseteq D$, the set $C_E = \{a \mid a \in C$ and $\bigwedge (d\rho_a \mid d \in E)$ exists$\}$ has a greatest element in C.

Proof. Let L be complete. (1) easily follows from $x \leqq x^{**}$, since this means that any lower bound of some $H \subseteq C$ can be majorized by an element of C; thus C is a complete sublattice of L. (2) is trivial. To verify (3), let $E \subseteq D$, and

$$\bigwedge (\langle d, 1 \rangle \mid d \in E) = \langle e, a \rangle.$$

We claim that a is the largest element of C_E. We obtain $a \in C_E$ easily, since $d\rho_a = d \vee a' \geqq e \vee a' = e$ (recall that $e \in a\phi = \{x \mid x \in D, x \geqq a'\}$); hence, $E\rho_a$ is bounded from below by e, and thus $\bigwedge E\rho_a$ exists. Now assume that $f = \bigwedge E\rho_b$ exists. Since $x\rho_b \geqq b'$ for all $x \in D$, we have $f \in b\phi$, and hence $f\rho_b = f$. We claim that $\langle f, b \rangle$ is a lower bound for E; indeed, $E\rho_b = \{x \vee b' \mid x \in E\}$, hence $f \leqq x \vee b'$ for all $x \in E$, and therefore $f \wedge b \leqq (x \vee b') \wedge b = x \wedge b \leqq x$, proving $\langle f, b \rangle = f \wedge b \leqq \langle x, 1 \rangle$, for all $x \in E$. Since $\bigwedge E = \langle e, a \rangle$, we conclude that $\langle f, b \rangle \leqq \langle e, a \rangle$, in particular, $b \leqq a$, which was to be proved.

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To prove the converse, let (1)–(3) hold. As a first step we verify the following formula:

(4.1) $\quad (\bigwedge (x| \, x \in D_1))\rho_a = \bigwedge (x\rho_a| \, x \in D_1) \quad \text{for } a \in C, D_1 \subseteq D,$

provided D_1 is bounded from below.

Indeed, for $d \in D_1$, $\bigwedge D_1 \leqq d$, hence $(\bigwedge D_1)\rho_a \leqq d\rho_a$, and therefore $(\bigwedge D_1)\rho_a \leqq \bigwedge (d\rho_a| \, d \in D_1)$, proving that the left side is contained in the right side in (4.1).

On the other hand, for $d \in D_1$, $d_1 = \bigwedge (d\rho_a| \, d \in D_1) \leqq d\rho_a$, hence

$$d_1 \wedge d\rho_{a'} \leqq d\rho_a \wedge d\rho_{a'} = d\rho_{a \vee a'} = d$$

and thus

$$d_1 \wedge \bigwedge (d\rho_{a'}| \, d \in D_1) \leqq \bigwedge D_1.$$

This implies that

$$(d_1 \wedge (\bigwedge D_1)\rho_{a'})\rho_a \leqq (\text{using} \leqq \text{in (4.1) for } a')[d_1 \wedge (\bigwedge (d\rho_{a'}| \, d \in D_1)]\rho_a$$
$$\leqq (\bigwedge D_1)\rho_a;$$

thus,

$$d_1 \leqq d_1\rho_a = d_1\rho_a \wedge (\bigwedge D_1)\rho_{a' \wedge a} \leqq (\bigwedge D_1)\rho_a,$$

proving the reverse inclusion in (4.1). Thus, (4.1) is true.

Now let $M \subseteq L$, $C_1 = \{x| \, \langle d, x \rangle \in M \text{ for some } d \in D\}$, $a = \bigwedge C_1$, and let b be the greatest element of C_E, where $E = \{d| \, \langle d, x \rangle \in M, \text{ for some } x \in C\}$. Finally, put $d = \bigwedge (y\rho_b| \, y \in E)$. We claim that $\bigwedge M$ exists and equals $\langle d, a \wedge b \rangle$. Let $g \in C_1$, and $h \in E$. Then $g \geqq a \wedge b$, $h\rho_{a \wedge b} = h\rho_a\rho_b \geqq d$, hence

$$\langle d, a \wedge b \rangle \leqq \langle g, h \rangle,$$

and therefore $\langle d, a \wedge b \rangle$ is a lower bound for M. Suppose that $\langle e, c \rangle$ is a lower bound for M. Then $c \leqq b$ for $b \in C_1$, and $e \leqq d\rho_c$. But then $e \leqq d\rho_c \leqq d\rho_a\rho_c$, and thus $\bigwedge ((d\rho_a)\rho_c| \, d \in E)$ exists, which implies, by the definition of b, that $c \leqq b$. Now, we have $a \wedge b = (\bigwedge C_1) \wedge b \geqq c$, and

$$d\rho_c = (\bigwedge (d\rho_a\rho_b| \, d \in E))\rho_c = (\text{by (4.1)}) \bigwedge (d\rho_a\rho_b\rho_c| \, d \in E)$$
$$= \bigwedge (d\rho_a\rho_c| \, d \in E) \geqq e,$$

establishing $\langle e, c \rangle \leqq \langle d, a \wedge b \rangle$. Thus, $\langle d, a \wedge b \rangle = \bigwedge M$. Therefore, arbitrary meets exist in L, that is, L is complete.

Remark. It is easily seen that if Theorem 5 (2) holds, then the set C_E defined in Theorem 5 (3) is always an ideal. Hence, Theorem 5 (3) requires that C_E be principal.

A similar proof yields the analogue of Theorem 5 for $\mathfrak{m}$-complete lattices.

COROLLARY 1. *Let L be given by the triple $\langle C, D, \phi \rangle$. If C and D are complete, then so is L.*

In this case, $C_E = C$, which has 1 as a greatest element.

COROLLARY 2. *Let L be given by* $\langle C, D, \phi\rangle$. *If C is finite and D is conditionally complete, then L is complete.*

In this case, C_E is an ideal of a finite Boolean algebra, hence C_E is principal.

For complete Stone lattices, C and D are not independent. The following result describes those Boolean algebras C for which C and D are always independent.

THEOREM 6. *Let C be a complete Boolean algebra. For any conditionally complete distributive lattice D with* 1 *there exists a complete Stone lattice with* $C = C(L)$, $D = D(L)$, *if and only if C has an atom.*

Proof. Suppose that C has an atom, p, and D is a conditionally complete distributive lattice with 1. Define ϕ: $C \to \mathscr{D}(D)$ by

$$a\phi = \begin{cases} \{1\}, & \text{for } a \leqq p', \\ D, & \text{for } a \nleqq p'. \end{cases}$$

Then Theorem 5 (3) is obvious for $\langle C, D, \phi\rangle$ since the largest element of C_E will always be p' (if E is not bounded from below) or 1.

Conversely, suppose that for any conditionally complete distributive lattice D with 1 there exists a ϕ: $C \to \mathscr{D}(D)$ satisfying Theorem 5 (3). Choose a D such that D is not complete and the centre of $\mathscr{D}(D)$ contains two elements only (e.g., let $\mathscr{D}$ be the chain of negative integers). Choose a ϕ: $C \to \mathscr{D}(D)$ satisfying Theorem 5 (3). Then C_D has a greatest element, say a. It is easy to see that a' must be an atom, since the kernel of the homomorphism ϕ is a prime ideal P of C, and $P = (a]$. This completes the proof of Theorem 6.

5. Closing remarks.

A lattice L is called a *relative Stone lattice* if every closed interval of L is a Stone lattice; see (**2**).

THEOREM 7. $L = \langle C, D, \phi\rangle$ *is a relative Stone lattice if and only if D is a relative Stone lattice.*

Proof. Since D is a convex sublattice of L, "only if" is obvious. Now assume that D is a relative Stone lattice. For every $x \in L$, $[0, x]$ is always a Stone lattice; hence, to show that L is relatively Stone, it suffices to prove that $[y, 1]$ is Stone for all $y \in L$. Put $y = \langle d, b\rangle$, let $\langle x, a\rangle \in [\langle d, b\rangle, \langle 1, 1\rangle]$. Let x_0 be the pseudo-complement of $x\rho_b$ in $[d, 1]$ ($\subseteq D$, hence Stone). A routine computation shows that the pseudo-complement of $\langle x, a\rangle$ in $[\langle d, b\rangle, \langle 1, 1\rangle]$ is $\langle x_0, a' \vee b\rangle$. Thus, if x_1 is the pseudo-complement of x_0 in $[d, 1]$, then $\langle x_1, a\rangle$ is the pseudo-complement of $\langle x_0, a' \vee b\rangle$ in $[\langle d, b\rangle, \langle 1, 1\rangle]$. Compute:

$$\begin{aligned} \langle x_0, a' \vee b\rangle \vee \langle x_1, a\rangle &= \langle (x_0\rho_{a'} \wedge x_1) \vee (x_1\rho_{a \wedge b'} \wedge x_0), 1\rangle \\ &= \langle (x_0\rho_b\rho_{a'} \wedge x_1) \vee (x_1\rho_a\rho_{b'}\rho_b \wedge x_0), 1\rangle = \langle x_0 \vee x_1, 1\rangle = \langle 1, 1\rangle, \end{aligned}$$

completing the proof.

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Let $L = \langle C, D, \phi \rangle$. Let Θ be the congruence relation of C induced by ϕ, that is, $x \equiv y(\Theta)$ if and only if $x\phi = y\phi$. Set $C_1 = C/\Theta$. We call $\langle C_1, D, \phi_1 \rangle$ the *reduced triple* associated with L, where ϕ_1 is defined by $([x]\Theta)\phi_1 = x\phi$.

Note that every reduced triple is a triple, hence it defines a Stone lattice L_1, which is a homomorphic image of L by the homomorphism $\langle \chi, 1 \rangle$, where χ is the natural homomorphism of C onto C/Θ; for the notation see (**1**, § 5). Furthermore, C_1 is isomorphic to a subalgebra of the centre of $\mathscr{D}(D_1)$.

There are several unsolved problems in connection with the triple approach to Stone lattices and the results given in (**1**) and in this paper. We list a few of these problems.

Problem 1. Determine the triple $\langle C(n), D(n), \phi(n) \rangle$ associated with the free Stone lattice FSL(n) on n generators. ($C(n)$ is easily seen to be the free Boolean algebra on n generators.)

Problem 2. Determine the reduced triple associated with FSL(n).

Problem 3. Determine the injective and projective Stone lattices.

Problem 4. Characterize the partially ordered set $P(D)$ of prime ideals of a distributive lattice D.

Problem 5. Characterize $P(D)$ if D is assumed to have 1. Is the only difference between the conditions of Problems 4 and 5 that in the latter every element of $P(D)$ is contained in a maximal one?

Problem 6. Solve the "fill in" problems for complete Stone lattices.

Note added in proof. FSL(n) has been determined by R. Balbes and A. Horn (*Stone lattices*, Duke Math. J., to appear). Injective Stone lattices and finite projective Stone lattices have been described by R. Balbes and G. Grätzer (*Injective and projective Stone algebras*, Notices Amer. Math. Soc. *16* (1969), 407).

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