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SYMMETRY GEOMETRY BY PAIRINGS

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Abstract

In this paper, we introduce a *symmetry geometry* for all those mathematical structures which can be characterized by means of a generalization (which we call pairing) of a finite rectangular table. In more detail, let Ω be a given set. A *pairing* $\mathfrak{P}$ on Ω is a triple $\mathfrak{P} := (U, F, \Lambda)$, where U and Λ are nonempty sets and $F : U \times \Omega \to \Lambda$ is a map having domain $U \times \Omega$ and codomain Λ. Through this notion, we introduce a local symmetry relation on U and a global symmetry relation on the power set $\mathcal{P}(\Omega)$. Based on these two relations, we establish the basic properties of our symmetry geometry induced by $\mathfrak{P}$. The basic tool of our study is a closure operator $M_{\mathfrak{P}}$, by means of which (in the finite case) we can represent any closure operator. We relate the study of such a closure operator to several types of others set operators and set systems which refine the notion of an abstract simplicial complex.

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1. Introduction

A modern trend in mathematics is to develop a given type of abstract geometries by means of specific set operators and set systems on some fixed set Ω. Classical examples of this trend are represented from the growing development of researches on matroids [24, 27, 29], finite combinatorial geometries [5, 6], closure spaces [4], combinatorial designs [15], association schemes [7, 8], hypergraphs [9, 19] in the finite case, and on abstract convexities in topological spaces [23, 30] in the infinite case.

From another perspective, the notion of symmetry has been developed for several types of geometries concerning graphs [3, 21, 25], digraphs [2], groups [1, 22] and metric spaces [26].

A new type of symmetry geometry has been recently developed for some set operators and set systems induced by graphs [17, 18], digraphs [14, 28], information tables [12], some types of discrete dynamical systems [11, 13, 16] and metric spaces [17]. This geometry is based on a notion of generalized symmetry between subsets of points (for metric spaces) or vertices (for graphs and digraphs), and we refer the reader to [17, 18] for details concerning these notions and corresponding results.

In this paper, we generalize the aforementioned symmetry geometries for any situation where Ω is an arbitrary fixed set, and we have a map $F : U \times \Omega \to \Lambda$, where U and Λ are two given nonempty sets. In this case, we call the triple $\mathfrak{P} := (U, F, \Lambda)$ a *pairing* on Ω and we denote by $\mathrm{PAIR}(\Omega)$ the set of all pairings on Ω.

In a purely combinatorial context, when Ω is finite, we can identify the pairing $\mathfrak{P}$ with the rectangular table in which the elements $u \in U$ and $a \in \Omega$ are respectively the names of the row and of the column determining the value $F(u, a)$ in this table.

Pairings appear everywhere in mathematics. As a first natural example, we can take Ω as the vertex set $V(G)$ of a given graph G and, in such a case, the pairing structure arising is that provided by the adjacency matrix of G, that is $\mathfrak{P}[G] := (V(G), F, \{0, 1\}) \in \mathrm{PAIR}(V(G))$, where $F(u, v) = 1$ if u is adiacent to v and $F(u, v) = 0$ otherwise. A recent study on pairings induced by simple undirected graphs has been undertaken in [18].

An immediate generalization of the pairing induced by a graph is a pairing induced on a metric space (X, d_X), that is $\mathfrak{P}[X, d_X] := (X, F, \{0, 1\}) \in \mathrm{PAIR}(X)$ where $F : X \times X \to \{0, 1\}$ defined by $F(u, u') := 1$ if $d_X(u, u') = 1$ and $F(u, u') := 0$ otherwise.

Furthermore, pairings appear also in the vector space context. As a matter of fact, let V be a vector space over a field $\mathbb{K}$ and $\varphi : V \times V \to \mathbb{K}$ a $\mathbb{K}$-bilinear form on V. Let $\Omega = V$. We associate with V the pairing $\mathfrak{P}[V, \varphi] = (V, \varphi, \mathbb{K}) \in \mathrm{PAIR}(\Omega)$. We call it the $(V, \varphi, \mathbb{K})$-*vector bilinear pairing*, abbreviated $(V, \varphi, \mathbb{K})$-*VBP*.

Again, a group action $\psi : (g, x) \in G \times \Omega \mapsto gx \in \Omega$ is another very natural example of pairing on Ω.

In general, on a pairing $\mathfrak{P} \in \mathrm{PAIR}(\Omega)$ we can consider two generalized notions of symmetry, respectively a *local* and a *global* one. More specifically, for any subset A of Ω, we consider the equivalence relation $\equiv_A$ on U defined by

$$u \equiv_A u' : \text{ if and only if } F(u, a) = F(u', a),$$

for any $a \in A$, for all $u, u' \in U$. Then, we can consider the subset A as a *symmetry block* with respect to which we can compare the elements of U, therefore we call $\equiv_A$ the A-*symmetry relation* on U. However, we can consider $\equiv_A$ as a type of *local* symmetry relation induced by the pairing structure, since it depends on the choice of the subset A (for a detailed study of such a symmetry relation on graphs see [18]). If $u \in U$, we denote by $[u]_A$ the equivalence class of u with respect to $\equiv_A$ and call it A-*symmetry class* of u. Moreover, we denote by $\pi_{\mathfrak{P}}(A) := \{[u]_A : u \in U\}$ the set partition on U induced by $\equiv_A$, which we call the A-*symmetry partition* of U. On the other hand, let $\approx_{\mathfrak{P}}$ be the equivalence relation on $\mathcal{P}(\Omega)$ defined by

$$A \approx_{\mathfrak{P}} A' \text{ if and only if } \pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(A'),$$

for all $A, A' \in \mathcal{P}(\Omega)$. We call $\approx_{\mathfrak{P}}$ the *global symmetry relation* of $\mathfrak{P}$ and we denote by $[A]_{\approx_{\mathfrak{P}}}$ the equivalence class of A with respect to $\approx_{\mathfrak{P}}$.

By means of the above two notions, we are able to introduce a *symmetry geometry* by pairings. Such a symmetry geometry can be developed both in a general setting and,

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clearly from an interpretative outlook, in the specific models (graphs, metric spaces, and so on). In [18], it has been widely developed on simple undirect graphs, whereas in [17] on the usual euclidean line.

In order to work in the context of such a symmetry geometry, it is convenient to introduce the notion of a *simplicial operator* on Ω, which is dual to that of a closure operator. The reason for such terminology is that the fixed point set of a simplicial operator is an abstract simplicial complex and, *vice versa*, in the finite case, an abstract simplicial complex induces, in a natural way, a simplicial operator (Theorem 2.2).

In this paper, we first associate with each pairing both a closure operator $M_{\mathfrak{P}}$ and a simplicial operator $C_{\mathfrak{P}}$ and, by means of these two set operators, we establish the basic results for a symmetry geometry induced by an arbitrary pairing (Theorem 3.1).

Next (see (i) and (ii) of Theorem 4.1), we represent respectively any closure operator and any simplicial operator with a nuclearity property (see axiom (OP6)) in terms of $M_{\mathfrak{P}}$ and $C_{\mathfrak{P}}$, for an appropriate pairing $\mathfrak{P} \in \mathrm{PAIR}(\Omega)$. Moreover (see Corollary 4.2), we find a further property (see axiom (OP8)) for a nuclear simplicial operator that is equivalent to establish the matroidality of any closure operator.

The aforementioned results are therefore three representation theorems for the finite case which show the generality of a symmetry geometry induced by pairings.

In the last part of the paper, we represent by pairings two specific set system families naturally associated with any closure operator σ on Ω. For any subset A of Ω, these set systems are defined by

$$\mathrm{ESS}_{\sigma}(A) := \{B \in \mathcal{P}(A) : \sigma(A \setminus B) \subsetneq \sigma(A),\ \sigma(A \setminus B') = \sigma(A)\ \forall B' \subsetneq B\},$$

and

$$\mathrm{BAS}_{\sigma}(A) := \{B \in \mathcal{P}(A) : \sigma(B) = \sigma(A),\ \sigma(B') \subsetneq \sigma(A)\ \forall B' \subsetneq B\}.$$

We call *σ-essential* of A any element $B \in \mathrm{ESS}_{\sigma}(A)$ and *σ-base* of A any element $B \in \mathrm{BAS}_{\sigma}(A)$.

The above set systems can be considered the natural tools to investigate the localization of the closure operator σ relatively to a fixed subset A of Ω (in the graph context the above two set systems have been widely investigated in [18]).

We conclude this introductory section by observing that the effective determination of the set operators and set systems induced by a symmetry geometry on specific types of pairings is (in general) not an easy task. In fact, for any given context from which a specific pairing is considered, one must use the appropriate sectorial techniques of that context in order to provide an effective determination of the aforementioned operators and corresponding structures. For example, in [18] for the Petersen graph a complete classification has been provided of all subgraphs induced by the corresponding essentials and bases described above.

2. Notations, recalls and basic results

Let Ω be a fixed set and $\mathcal{P}(\Omega)$ its power set. If $P = (\Omega, \leq)$ is a partially ordered set and $x, y \in \Omega$, we also write $x < y$ if $x \leq y$ and $x \neq y$. If x, y are two distinct elements of Ω, we say that y *covers* x if $x \leq y$ and there exists no element $z \in \Omega$ such that $x < z < y$.

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A *set operator* on Ω is a map $\sigma : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$. Let $\mathrm{OP}(\Omega)$ be the family of all set operators on Ω. For a generic set operator $\sigma \in \mathrm{OP}(\Omega)$, in this paper, we will consider the following properties.

Let $A, B \in \mathcal{P}(\Omega)$, then the below hold:

(OP1) $A \subseteq \sigma(A)$ (extensivity);
(OP2) $A \supseteq \sigma(A)$ (intensivity);
(OP3) $\sigma(\sigma(A)) = \sigma(A)$ (idempotency);
(OP4) if $A \subseteq B$ then $\sigma(A) \subseteq \sigma(B)$ (direct monotonicity);
(OP5) if $A \subseteq B$ then $\sigma(A) \supseteq A \cap \sigma(B)$ (weak inversion);
(OP6) if $b \in \Omega \setminus (A \cup \sigma(A \cup \{b\}))$ and $c \in \sigma(A \cup \{c\}) \setminus A$, then $c \in \sigma(A \cup \{b, c\})$ (nuclearization);
(OP7) if $\{b, c\} \subseteq \Omega \setminus \sigma(A)$ and $b \in \sigma(A \cup \{c\})$, then $c \in \sigma(A \cup \{b\})$ (Mac Lane–Steinitz exchange property);
(OP8) if $b \in \sigma(A \cup \{b\})$ then $\sigma(A) \cup \{b\} \subseteq \sigma(A \cup \{b\})$ (point attraction).

Let $\sigma \in \mathrm{OP}(\Omega)$. Then, the following points stand:

(CLO) σ is a *closure operator* on Ω if it satisfies the properties (OP1), (OP3) and (OP4). We denote by $\mathrm{CLO}(\Omega)$ the set of all closure operators on Ω.
(MCLO) σ is a *matroidal closure operator* on Ω if $\sigma \in \mathrm{CLO}(\Omega)$ and σ satisfies (OP7). We denote by $\mathrm{MCLO}(\Omega)$ the set of all matroidal closure operators on Ω.
(SO) σ is a *simplicial operator* on Ω if σ satisfies (OP2) and (OP5). We denote by $\mathrm{SO}(\Omega)$ the set of all simplicial operators on Ω.
(NSO) σ is a *nuclear simplicial operator* if $\sigma \in \mathrm{SO}(\Omega)$ and it satisfies the properties (OP6). We denote by $\mathrm{NSO}(\Omega)$ the set of all nuclear simplicial operators on Ω.
(ANSO) σ is an *attractive nuclear simplicial operator* on Ω if $\sigma \in \mathrm{NSO}(\Omega)$ and it satisfies (OP8). We denote by $\mathrm{ANSO}(\Omega)$ the set of all attractive nuclear simplicial operators on Ω.

We denote by $\mathrm{Fix}(\sigma)$ the fixed point set of σ, that is, $\mathrm{Fix}(\sigma) := \{A \in \mathcal{P}(\Omega) : \sigma(A) = A\}$.

We call an element $\mathcal{F} \in SS(\Omega)$ a *set system* on Ω. Let $\mathcal{F} \in SS(\Omega)$. We say that $\mathcal{F}$ is *union-closed* if whenever $\mathcal{F}' \subseteq \mathcal{F}$ then $\cup\mathcal{F}' \in \mathcal{F}$. We denote by $\mathrm{UCL}(\Omega)$ the collection of all union-closed set systems on Ω.

We say that a subset $Y \subseteq \Omega$ is a *transversal* of a set system $\mathcal{F} \in SS(\Omega)$ if $Y \cap A \neq \emptyset$ for each nonempty $A \in \mathcal{F}$. Moreover, a transversal A of $\mathcal{F}$ is *minimal* if no proper subset of A is a transversal of $\mathcal{F}$. We denote by $\mathrm{Tr}(\mathcal{F})$ the family of all minimal transversals of $\mathcal{F}$. We denote by $\max(\mathcal{F})$ the set system of all maximal members of $\mathcal{F}$ and by $\min(\mathcal{F})$ the set system of all minimal members of $\mathcal{F}$. If $A \in \mathcal{P}(\Omega)$, we set $\max(\mathcal{F}|A) := \max\{B \in \mathcal{P}(A) : B \in \mathcal{F}\}$, so that $\max(\mathcal{F}|\Omega) = \max(\mathcal{F})$.

$\mathcal{F}$ is called a *closure system* on Ω if $\Omega \in \mathcal{F}$ and whenever $\mathcal{F}' \subseteq \mathcal{F}$ then $\cap\mathcal{F}' \in \mathcal{F}$. We denote by $\mathrm{CLSY}(\Omega)$ the family of all closure systems on Ω. $\mathcal{F}$ is called an *abstract simplicial complex* on Ω if whenever $A \in \mathcal{F}$ and $B \subseteq A$, then $B \in \mathcal{F}$. We denote by $\mathrm{ASC}(\Omega)$ the family all abstract simplicial complexes on Ω. An abstract

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simplicial complex $\mathcal{F}$ on a finite set Ω is called a *matroid* if for any $U, V \in \mathcal{F}$ such that $|U| = |V| + 1$, then there exists $x \in U \setminus V$ such that $V \cup \{x\} \in \mathcal{F}$ (for further details and results on matroids, we refer to [29]). In this case, we call each member of $\mathcal{F}$ an *independent set*.

We consider the set operators on Ω given by

$$\mathrm{Cl}_{\mathcal{F}}(C) := \bigcap\{A \in \mathcal{F} : C \subseteq A\},$$

and

$$\mathrm{Imx}_{\mathcal{F}}(C) := \bigcap\{A \in \max(\mathcal{F}|C)\},$$

for any $C \in \mathcal{P}(\Omega)$.

Based on the following classical result, it follows that the notions of the closure operator and closure system (on the same set Ω) are equivalent between them.

Theorem 2.1 [10]. *Let* $\mathcal{F} \in \mathrm{CLSY}(\Omega)$ *and* $\sigma \in \mathrm{CLO}(\Omega)$. *Then,* $\mathrm{Cl}_{\mathcal{F}} \in \mathrm{CLO}(\Omega)$, $\mathrm{Fix}(\sigma) \in \mathrm{CLSY}(\Omega)$, $\mathrm{Fix}(\mathrm{Cl}_{\mathcal{F}}) = \mathcal{F}$ *and* $\mathrm{Cl}_{\mathrm{Fix}(\sigma)} = \sigma$.

In the following result we establish the basic links between abstract simplicial complexes and simplicial operators.

Theorem 2.2. *Let* $\sigma \in \mathrm{SO}(\Omega)$ *and* $\mathcal{F} \in \mathrm{ASC}(\Omega)$. *Then, the following conditions hold:*

(i) $\mathrm{Fix}(\sigma) \in \mathrm{ASC}(\Omega)$;

(ii) $\mathrm{Fix}(\mathrm{Imx}_{\mathcal{F}}) = \mathcal{F}$;

(iii) *if* $\mathcal{F} \in \mathrm{ASC}(\Omega) \cap \mathrm{UCL}(\Omega)$, *then* $\mathrm{Imx}_{\mathcal{F}} \in \mathrm{SO}(\Omega)$ *and* $\mathrm{Imx}_{\mathcal{F}}$ *is idempotent.*

Proof. (i) Clearly, $\emptyset \in \mathrm{Fix}(\sigma)$. On the other hand, let $B \in \mathrm{Fix}(\sigma)$ and $A \subseteq B$. By intensivity, it follows that $\sigma(A) \subseteq A$. On the other hand, by (OP5), it follows that $\sigma(A) \supseteq \sigma(B) \cap A = B \cap A = A$, so $A = \sigma(A)$, that is, $A \in \mathrm{Fix}(\sigma)$.

(ii) Just observe that $\mathrm{Imx}_{\mathcal{F}}(A) = A$ if and only if $A \in \mathcal{F}$.

(iii) Clearly, $\mathrm{Imx}_{\mathcal{F}}(A) \subseteq A$ for any $A \in \mathcal{P}(\Omega)$. Assume now that $A \subseteq B$ and let $x \in A \cap \mathrm{Imx}_{\mathcal{F}}(B)$. Then, $x \in A \cap C$ for any $C \in \max(\mathcal{F}|B)$. Let us prove that there exists $D \in \max(\mathcal{F}|A)$ such that $x \in D$. Consider the set system $\mathcal{G} := \{Z \in \mathcal{F} : A \cap C \subseteq D \subseteq A\}$. It is clearly nonempty, since $A \cap C \in \mathcal{G}$. Take a chain $\{Z_i : i \in I\} \subseteq \mathcal{G}$ e and consider $Z := \bigcup_{i \in I} Z_i$. By the fact that $\mathcal{F}$ is union-closed, it follows that $Z \in \mathcal{F}$. Moreover, it is obvious that $A \cap C \subseteq Z \subseteq A$. Hence, we can apply Zorn's lemma and prove that there exists an element $D \in \max \mathcal{G}$. Finally, it is also clear that $x \in D$ and it is straightforward to see that $D \in \max(\mathcal{F}|A)$.

We now show that $x \in E$ for any $E \in \max(\mathcal{F}|A)$. In this regard, assume by contradiction the existence of some $E' \in \max(\mathcal{F}|A)$ for which $x \notin E'$. Let us prove that $E' \subseteq C'$ for some $C' \in \max(\mathcal{F}|B)$. In this regard, set $\mathcal{G}' := \{Y \in \mathcal{F} : E' \subseteq Y \subseteq B\}$. It is clearly nonempty since $E' \in \mathcal{G}'$. So, take a chain $\{Y_i : i \in I\} \subseteq \mathcal{G}'$ and set $Y := \bigcup_{i \in I} Y_i$. It is immediate to see that $E' \subseteq Y \subseteq B$ and $Y \in \mathcal{F}$. By Zorn's lemma, $\mathcal{G}'$ admits a maximal element C' that clearly belongs to $\max(\mathcal{F}|B)$.

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Now, we have that $x \notin C'$, otherwise $C' \cap A \supsetneq E'$ and $C' \cap A \in \max(\mathcal{F}|A)$, contradicting the maximality of E'. But x must belong to each subset of $\max(\mathcal{F}|B)$, so we reach a contradiction. This proves (OP5).

Finally, in order to prove that $\mathrm{Imx}_{\mathcal{F}}$ is idempotent, just observe that $\mathrm{Imx}_{\mathcal{F}}(A) \in \mathcal{F}$ so, by the previous part (ii), we have $\mathrm{Imx}_{\mathcal{F}}(\mathrm{Imx}_{\mathcal{F}}(A)) = \mathrm{Imx}_{\mathcal{F}}(A)$. □

Remark 2.3. It is easy to show that (iii) of Theorem 2.2 holds when Ω is a finite set, without assuming that $\mathcal{F} \in \mathrm{UCL}(\Omega)$. This result justifies the terminology *simplicial operator*, as said in the introductory section.

3. Basic operators and set systems by pairings

In the following result, we establish the fundamental properties of all set systems and set operators of our symmetry geometry induced by pairings.

Theorem 3.1. *The following conditions hold:*

(i) *for any $A \in \mathcal{P}(\Omega)$, the set system $[A]_{\approx_{\mathfrak{P}}}$ is union-closed;*

(ii) *the map $M_{\mathfrak{P}} : A \in \mathcal{P}(\Omega) \mapsto \bigcup\{A' : A' \in [A]_{\approx_{\mathfrak{P}}}\} \in [A]_{\approx_{\mathfrak{P}}}$ is a closure operator on Ω such that*

$$M_{\mathfrak{P}}(A) = \{b \in \Omega : A \cup \{b\} \approx_{\mathfrak{P}} A\}; \tag{3.1}$$

(iii) *the subset family*

$$\mathrm{MAXP}(\mathfrak{P}) := \{C \in \mathcal{P}(\Omega) : M_{\mathfrak{P}}(C) = C\} = \{M_{\mathfrak{P}}(A) : A \in \mathcal{P}(\Omega)\}$$

is a closure system on Ω;

(iv) *the map $C_{\mathfrak{P}} : A \in \mathcal{P}(\Omega) \mapsto \{a \in A : A \setminus \{a\} \not\approx_{\mathfrak{P}} A\} \in \mathcal{P}(\Omega)$ is a nuclear simplicial operator on Ω satisfying (OP3);*

(v) *we have that*

$$M_{\mathfrak{P}}(A) = \{b \in \Omega : b \notin C_{\mathfrak{P}}(A \cup \{b\})\} \tag{3.2}$$

and

$$C_{\mathfrak{P}}(A) = \{a \in A : a \notin M_{\mathfrak{P}}(A \setminus \{a\})\}; \tag{3.3}$$

(vi) *the subset family*

$$\mathrm{MINP}(\mathfrak{P}) := \bigcup\{\min([A]_{\approx_{\mathfrak{P}}}) : A \in \mathrm{MAXP}(\mathfrak{P})\}$$

is an abstract simplicial complex on Ω such that

$$\begin{aligned}\mathrm{MINP}(\mathfrak{P}) &= \{A \in \mathcal{P}(\Omega) : \ a \notin M_{\mathfrak{P}}(A \setminus \{a\})\ \forall a \in A\} \\ &= \{B \in \mathcal{P}(\Omega) : C_{\mathfrak{P}}(B) = B\}.\end{aligned} \tag{3.4}$$

Proof. (i) Let $u, u' \in U$ such that $u \equiv_A u'$ and let $\{A_j : j \in J\} \subseteq [A]_{\approx_{\mathfrak{P}}}$. Then, $u \equiv_{A_j} u'$ for all $j \in J$ in view of the definition of the relation $\approx_{\mathfrak{P}}$. If $z \in \bigcup_{j\in J} A_j$ there exists some index $j \in J$ such that $z \in A_j$, so that $F(u, z) = F(u', z)$ because $u \equiv_{A_j} u'$. Hence, $u \equiv_{\bigcup_{j\in J} A_j} u'$. This implies that $\pi_{\mathfrak{P}}(A) \leq \pi_{\mathfrak{P}}(\bigcup_{j\in J} A_j)$. On the other hand, it is straightforward

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to see that $\pi_{\mathfrak{P}}(\bigcup_{j\in J} A_j) \preceq \pi_{\mathfrak{P}}(A_j) = \pi_{\mathfrak{P}}(A)$ because $A_j \approx_{\mathfrak{P}} A$. Moreover, we have that $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(\bigcup_{j\in J} A_j)$, which is equivalent to the condition $\bigcup_{j\in J} A_j \in [A]_{\approx_{\mathfrak{P}}}$.

(ii) Let $M_{\mathfrak{P}}(A) := \bigcup\{B : B \in [A]_{\approx_{\mathfrak{P}}}\}$. By part (i), it follows that $M_{\mathfrak{P}}(A) \in [A]_{\approx_{\mathfrak{P}}}$, moreover we also have that $B \subseteq M_{\mathfrak{P}}(A)$ for all $B \in [A]_{\approx_{\mathfrak{P}}}$. Uniqueness is obvious.

Now, let us show that (3.1) holds. In this regard, let $b \in M_{\mathfrak{P}}(A)$. Then, there exists $A' \in [A]_{\approx_{\mathfrak{P}}}$ such that $b \in A'$. Let now $u, u' \in U$ such that $u \equiv_{A\cup\{b\}} u'$, then $u \equiv_A u'$ because A is a subset of $A \cup \{b\}$. On the other hand, assume that $u \equiv_A u'$. Since $A' \in [A]_{\approx_{\mathfrak{P}}}$, we have $u \equiv_{A'} u'$, and therefore $F(u, b) = F(u', b)$ because $b \in A'$. Hence, $u \equiv_{A\cup\{b\}} u'$, so we deduce that $A \approx_{\mathfrak{P}} A \cup \{b\}$.

Let now $b \in \Omega$ such that $A \approx_{\mathfrak{P}} A \cup \{b\}$. Then, $b \in A \cup \{b\} \in [A]_{\approx_{\mathfrak{P}}}$ and we deduce that $b \in M_{\mathfrak{P}}(A)$.

Let us prove that

$$M_{\mathfrak{P}}(A) = \{a \in \Omega : (u, u' \in \Omega \wedge u \equiv_A u') \Longrightarrow F(u, a) = F(u', a)\}. \tag{3.5}$$

Let $B := \{b \in \Omega : (u, u' \in U \wedge u \equiv_A u') \Longrightarrow F(u, b) = F(u', b)\}$. We show that $B \approx_{\mathfrak{P}} A$. In fact, let $u, u' \in U$ such that $u \equiv_A u'$ and let $b \in B$. In view of the definition of B we have that $F(u, b) = F(u', b)$, so that $u \equiv_B u'$. Hence, $\pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(B)$. Let now $C \in [A]_{\approx_{\mathfrak{P}}}$ and $c \in C$. Then, for all $u, u' \in U$ such that $u \equiv_A u'$, by definition of $\approx_{\mathfrak{P}}$ we have that $u \equiv_C u'$, so that $F(u, c) = F(u', c)$. It follows that $C \subseteq B$ for all $C \in [A]_{\approx_{\mathfrak{P}}}$, in particular $A \subseteq M_{\mathfrak{P}}(A) \subseteq B$. It is also immediate to see that $\pi_{\mathfrak{P}}(B) \preceq \pi_{\mathfrak{P}}(A)$. Therefore $\pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(A)$, that is, $B \in [A]_{\approx_{\mathfrak{P}}}$, so we deduce that $B \subseteq M_{\mathfrak{P}}(A)$. Hence, $B = M_{\mathfrak{P}}(A)$.

Finally, we show that the set operator $M_{\mathfrak{P}}$ is a closure operator. It is obvious that $A \subseteq M_{\mathfrak{P}}(A)$. Let now $A' \in \mathcal{P}(\Omega)$ such that $A \subseteq A'$. Let $b \in M_{\mathfrak{P}}(A)$. Then, $A \approx_{\mathfrak{P}} A \cup \{b\}$. Let $v, v' \in \Omega$ such that $v \equiv_{A'} v'$. Since A is a subset of A', we have that $v \equiv_A v'$, and this implies $F(v, b) = F(v', b)$ because $A \approx_{\mathfrak{P}} A \cup \{b\}$. This shows that $v \equiv_{A'\cup\{b\}} v'$, that is $b \in M_{\mathfrak{P}}(A')$. Hence, $M_{\mathfrak{P}}(A) \subseteq M_{\mathfrak{P}}(A')$. Finally, we must prove that $M_{\mathfrak{P}}(M_{\mathfrak{P}}(A)) = M_{\mathfrak{P}}(A)$, for this it is sufficient to show that $M_{\mathfrak{P}}(M_{\mathfrak{P}}(A)) \subseteq M_{\mathfrak{P}}(A)$. Let then $b \in M_{\mathfrak{P}}(M_{\mathfrak{P}}(A))$ and $v, v' \in \Omega$ such that $v \equiv_A v'$. Since $A \approx_{\mathfrak{P}} M_{\mathfrak{P}}(A)$, it follows that $v \equiv_{M_{\mathfrak{P}}(A)} v'$, therefore $F(v, b) = F(v', b)$ in view of (3.5), because b belongs to $M_{\mathfrak{P}}(M_{\mathfrak{P}}(A))$. Hence, $b \in M_{\mathfrak{P}}(A)$ again by (3.5).

(iii) MAXP($\mathfrak{P}$) is the family of all closed sets for the closure operators $M_{\mathfrak{P}}$. Therefore the result follows by Theorem 2.1.

(iv) Clearly, $C_{\mathfrak{P}}(A) \subseteq A$, that is, (OP2) holds. Let $a \in C_{\mathfrak{P}}(B) \cap A$, then $a \in A$ and $\pi_{\mathfrak{P}}(B \setminus \{a\}) \neq \pi_{\mathfrak{P}}(A)$. Suppose that $a \in M_{\mathfrak{P}}(A \setminus \{a\})$, then $a \in M_{\mathfrak{P}}(B \setminus \{a\})$, contradicting the fact that $a \in C_{\mathfrak{P}}(B)$. Thus, $a \in C_{\mathfrak{P}}(A)$. This proves (OP5) holds.

We now prove that (OP6) holds. In this regard, let $A \in \mathcal{P}(\Omega)$ and $b, c \in \Omega \setminus A$. Assume that $b \in \Omega \setminus (A \cup \sigma(A \cup \{b\}))$. This means that $b \in M_{\mathfrak{P}}(A \cup \{b\} \setminus \{b\})$, that is, $b \in M_{\mathfrak{P}}(A) \setminus A$. Moreover, assume that $c \in \sigma(A \cup \{c\})$. This implies that $c \in \Omega \setminus M_{\mathfrak{P}}(A \cup \{c\} \setminus \{c\}) = M_{\mathfrak{P}}(A)$. In particular, it results that $M_{\mathfrak{P}}(A \cup \{b, c\} \setminus \{c\}) = M_{\mathfrak{P}}(A \cup \{b\}) = M_{\mathfrak{P}}(A)$, so $c \in \Omega \setminus M_{\mathfrak{P}}(A)$ or, equivalently, $c \in M_{\mathfrak{P}}(A \cup \{b, c\})$. Finally, let us show that $C_{\mathfrak{P}}$ is an idempotent operator. Clearly, we have that $C_{\mathfrak{P}}(C_{\mathfrak{P}}(A)) \subseteq C_{\mathfrak{P}}(A)$. On the other hand, let $a \in C_{\mathfrak{P}}(A)$, then $\pi_{\mathfrak{P}}(A \setminus \{a\}) \neq \pi_{\mathfrak{P}}(A)$, that is, $M_{\mathfrak{P}}(A \setminus \{a\}) \subsetneq M_{\mathfrak{P}}(A)$. Now, since $C_{\mathfrak{P}}(A) \subseteq A$, we deduce that $a \in M_{\mathfrak{P}}(C_{\mathfrak{P}}(A) \setminus \{a\}) \subseteq M_{\mathfrak{P}}(A \setminus \{a\})$

but this would imply that $A \subseteq M_{\mathfrak{P}}(A \setminus \{a\})$ or, equivalently, $M_{\mathfrak{P}}(A \setminus \{a\}) = M_{\mathfrak{P}}(A)$, contradicting the fact that $a \in C_{\mathfrak{P}}(A)$.

(v) We simply observe that $M_{\mathfrak{P}}(A) = M_{\mathfrak{P}}(A \cup \{b\} \setminus \{b\})$ so $b \in M_{\mathfrak{P}}(A)$ if and only if $b \notin C_{\mathfrak{P}}(A \cup \{b\})$. In this way, we have shown (3.2).

On the other hand, to prove (3.3), let us observe that $a \notin M_{\mathfrak{P}}(A \setminus \{a\})$ if and only if $M_{\mathfrak{P}}(A \setminus \{a\}) \neq M_{\mathfrak{P}}(A)$, that is, if and only if $\pi_{\mathfrak{P}}(A) \neq \pi_{\mathfrak{P}}(A \setminus \{a\})$.

(vi) We firstly show that given $A \in \mathcal{P}(\Omega)$ and $C \subseteq B \subseteq A$ with $B \in \mathrm{MINP}(\mathfrak{P})$, then

$$\pi_{\mathfrak{P}}(C \setminus \{x\}) \neq \pi_{\mathfrak{P}}(C) \tag{3.6}$$

for any $x \in C$.

Let us suppose by contradiction that there exists $x \in C$ such that $\pi_{\mathfrak{P}}(C \setminus \{x\}) = \pi_{\mathfrak{P}}(C)$. Then, since $C \setminus \{x\} \subseteq B \setminus \{x\}$, we have

$$\pi_{\mathfrak{P}}(B \setminus \{x\}) \preceq \pi_{\mathfrak{P}}(C \setminus \{x\}) = \pi_{\mathfrak{P}}(C). \tag{3.7}$$

Therefore, if $v, v' \in \Omega$, in view of (3.7), it follows that

$$v \equiv_{B\setminus\{x\}} v' \implies v \equiv_C v' \implies F(v, x) = F(v', x) \implies v \equiv_B v',$$

therefore

$$\pi_{\mathfrak{P}}(B \setminus \{x\}) \preceq \pi_{\mathfrak{P}}(B). \tag{3.8}$$

On the other hand, since we also have $\pi_{\mathfrak{P}}(B) \preceq \pi_{\mathfrak{P}}(B \setminus \{x\})$, in view of (3.8) we deduce that $\pi_{\mathfrak{P}}(B \setminus \{x\}) = \pi_{\mathfrak{P}}(B)$, which is in contrast with the hypothesis that $B \in \mathrm{BAS}_{\mathfrak{P}}(A)$. This concludes the proof of the claim.

Let now $C \in \mathrm{MINP}(\mathfrak{P})$. Then, there exists $B \in \mathrm{MAXP}(\mathfrak{P})$ such that $C \in \min([B]_{\approx_{\mathfrak{P}}})$. Let $K \subsetneq C$, then there exists $B' \in \mathrm{MAXP}(\mathfrak{P})$ such that $K \in [B']_{\approx_{\mathfrak{P}}}$. Suppose by contradiction that $K \notin \min([B']_{\approx_{\mathfrak{P}}})$, then there exists $K' \subsetneq K$ such that $K' \in \min([B']_{\approx_{\mathfrak{P}}})$. Hence, there is $x \in K \setminus K'$. We deduce that

$$K' \subseteq K \setminus \{x\} \subseteq K,$$

that is,

$$\pi_{\mathfrak{P}}(K) \preceq \pi_{\mathfrak{P}}(K \setminus \{x\}) \preceq \pi_{\mathfrak{P}}(K').$$

But since $\pi_{\mathfrak{P}}(K) = \pi_{\mathfrak{P}}(K') = \pi_{\mathfrak{P}}(B)$, we conclude that $\pi_{\mathfrak{P}}(K) = \pi_{\mathfrak{P}}(K \setminus \{x\})$, contradicting (3.6). Hence, $K \in \mathrm{MINP}(\mathfrak{P})$.

Furthermore, we have that $\mathrm{MINP}(\mathfrak{P}) = \{A \in \mathcal{P}(\Omega) : a \notin M_{\mathfrak{P}}(A \setminus \{a\})\ \forall a \in A\}$. In fact, let $A \in \mathrm{MINP}(\mathfrak{P})$ and $a \in A$. Then, $A \setminus \{a\} \notin [A]_{\approx_{\mathfrak{P}}}$, so $M_{\mathfrak{P}}(A \setminus \{a\}) \notin [A]_{\approx_{\mathfrak{P}}}$. Since $A \setminus \{a\} \subsetneq A$, we have $\pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(A \setminus \{a\})$. Moreover, suppose by contradiction that $a \in M_{\mathfrak{P}}(A \setminus \{a\})$. Then, $A \subseteq M_{\mathfrak{P}}(A \setminus \{a\})$ and $M_{\mathfrak{P}}(A) \subseteq M_{\mathfrak{P}}(A \setminus \{a\})$. We clearly have $\pi_{\mathfrak{P}}(A \setminus \{a\}) \preceq \pi_{\mathfrak{P}}(A)$, so $\pi_{\mathfrak{P}}(A \setminus \{a\}) = \pi_{\mathfrak{P}}(A)$ and $A \setminus \{a\} \in [A]_{\approx_{\mathfrak{P}}}$, that is a contradiction. So $a \notin M_{\mathfrak{P}}(A \setminus \{a\})$ and $\mathrm{MINP}(\mathfrak{P}) \subseteq \{A \in \mathcal{P}(\Omega) : a \notin M_{\mathfrak{P}}(A \setminus \{a\})\ \forall a \in A\}$. On the other hand, let $A \in \mathcal{P}(\Omega)$ such that $a \notin M_{\mathfrak{P}}(A \setminus \{a\})$ for any $a \in A$. Suppose by contradiction that $A \in \mathrm{MINP}^c(\mathfrak{P})$, that is, there exists $B \subsetneq A$ such that $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(B)$. Then, there exists $a \in A$ such that $B \subseteq A \setminus \{a\} \subseteq A$. This implies that $\pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(A \setminus \{a\}) \preceq \pi_{\mathfrak{P}}(B)$, that is, $\pi_{\mathfrak{P}}(A \setminus \{a\}) = \pi_{\mathfrak{P}}(A)$. In other terms, we have $M_{\mathfrak{P}}(A \setminus \{a\}) = M_{\mathfrak{P}}(A) \supseteq A$, which is a contradiction.

We prove now that $\mathrm{MINP}(\mathfrak{P}) = \{B \in \mathcal{P}(\Omega) : C_{\mathfrak{P}}(B) = B\}$. Let $B \in \mathrm{MINP}(\mathfrak{P})$. Clearly, $C_{\mathfrak{P}}(B) \subseteq B$. *Vice versa*, let $b \in B$ and assume by contradiction that $\pi_{\mathfrak{P}}(B \setminus \{b\}) = \pi_{\mathfrak{P}}(B)$. This means that B is not a minimal partitioner of $\mathfrak{P}$, contradicting our assumption on B.

In contrast, let $B \in \mathcal{P}(\Omega)$ such that $C_{\mathfrak{P}}(B) = B$ and suppose by contradiction that $B \notin \mathrm{MINP}(\mathfrak{P})$. Then, there exists $B' \subsetneq B$ such that $B' \approx_{\mathfrak{P}} B$ and $B' \in \mathrm{MINP}(\mathfrak{P})$. In particular, $B' \subseteq B \setminus \{b\}$ for some $b \in B$. This entails that $\pi_{\mathfrak{P}}(B \setminus \{b\}) = \pi_{\mathfrak{P}}(B)$, that is, $b \notin C_{\mathfrak{P}}(B)$ or, equivalently, $B \neq C_{\mathfrak{P}}(B)$, contradicting the choice of B. This proves that $\mathrm{MINP}(\mathfrak{P}) = \{B \in \mathcal{P}(\Omega) : C_{\mathfrak{P}}(B) = B\}$. □

At this point, it is useful to introduce a relativization of the notion of maximum partitioner. Therefore, for any $A \in \mathcal{P}(\Omega)$ we set

$$\mathrm{MAXP}_{\mathfrak{P}}(A) := \{B \cap A : \; B \in \mathrm{MAXP}(\mathfrak{P})\}, \quad \mathbb{M}_{\mathfrak{P}}(A) := (\mathrm{MAXP}_{\mathfrak{P}}(A), \subseteq^*),$$

therefore we obtain a map $\mathrm{MAXP}_{\mathfrak{P}} : \mathcal{P}(\Omega) \to \mathcal{P}(\mathcal{P}(\Omega))$ such that

$$\mathrm{MAXP}_{\mathfrak{P}}(\Omega) = \mathrm{MAXP}(\mathfrak{P})$$

and

$$\mathrm{MAXP}_{\mathfrak{P}}(A) \in \mathrm{CLSY}(A),$$

for any $A \in \mathcal{P}(\Omega)$.

Remark 3.2. Let us note that if $A \in \mathrm{MAXP}(\mathfrak{P})$, then $\mathrm{MAXP}_{\mathfrak{P}}(A) = \{B \in \mathrm{MAXP}(\mathfrak{P}) : B \subseteq A\}$.

We now express the relation between the set operator $C_{\mathfrak{P}}$ and $\mathrm{MAXP}_{\mathfrak{P}}$.

Theorem 3.3. *Let $A \in \mathcal{P}(\Omega)$. Then, the below hold.*

(i) $C_{\mathfrak{P}}(A) = \{a \in A : A \setminus \{a\} \in \mathrm{MAXP}_{\mathfrak{P}}(A)\}$;

(ii) $C_{\mathfrak{P}}(M_{\mathfrak{P}}(A)) \subseteq A$.

Proof. (i) Let $a \in C_{\mathfrak{P}}(A)$. We must show that $A \setminus \{a\} \in \mathrm{MAXP}_{\mathfrak{P}}(A)$, that is, there exists $B \in \mathrm{MAXP}(\mathfrak{P})$ such that $A \setminus \{a\} = A \cap B$. Let us assume by contradiction that $A \setminus \{a\} \neq A \cap B$ for any $B \in \mathrm{MAXP}(\mathfrak{P})$. Since $\pi_{\mathfrak{P}}(A) \neq \pi_{\mathfrak{P}}(A \setminus \{a\})$, then $A \setminus \{a\} \not\approx_{\mathfrak{P}} A$. Let $B = M_{\mathfrak{P}}(A \setminus \{a\})$. Clearly, $A \setminus \{a\} \subseteq B$ and we have $A \setminus \{a\} \subseteq A \cap B$ therefore, by our assumption, it follows that $A \setminus \{a\} \subsetneq A \cap B$. Hence, $A \cap B = A$, that is, $A \subseteq B$. It follows that $\pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(A \setminus \{a\}) \preceq \pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(A \setminus \{a\}) = \pi_{\mathfrak{P}}(B)$, that is, $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(B)$, which is equivalent to saying that $A \approx_{\mathfrak{P}} B \approx_{\mathfrak{P}} A \setminus \{a\}$, which is a contradiction. Thus, $A \setminus \{a\} \in \mathrm{MAXP}_{\mathfrak{P}}(A)$.

Conversely, let $a \in A$ such that $A \setminus \{a\} \in \mathrm{MAXP}_{\mathfrak{P}}(A)$. Hence, there exists $B \in \mathrm{MAXP}(\mathfrak{P})$ such that $A \setminus \{a\} = B \cap A$. If $B = A \setminus \{a\}$, then $A \setminus \{a\} \in \mathrm{MAXP}(\mathfrak{P})$. In this case, in view of the maximality of $A \setminus \{a\}$, then $A \notin [A \setminus \{a\}]_{\approx_{\mathfrak{P}}}$, so $\pi_{\mathfrak{P}}(A) \neq \pi_{\mathfrak{P}}(A \setminus \{a\})$, therefore $a \in C_{\mathfrak{P}}(A)$ and the claim is proved. Otherwise, let $A \setminus \{a\} \subsetneq B$.

Clearly, it results that $a \notin B$. Furthermore, it results $\pi_{\mathfrak{P}}(B) \preceq \pi_{\mathfrak{P}}(A \setminus \{a\})$. Suppose by contradiction that $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(A \setminus \{a\})$. Then, we have $\pi_{\mathfrak{P}}(B) \preceq \pi_{\mathfrak{P}}(A)$, so

$$u \equiv_B u' \implies u \equiv_A u' \implies F(u,a) = F(u',a) \implies u \equiv_{B\cup\{a\}} u'.$$

In other terms, we have shown that $\pi_{\mathfrak{P}}(B) \preceq \pi_{\mathfrak{P}}(B \cup \{a\})$. Nevertheless, we also have $\pi_{\mathfrak{P}}(B \cup \{a\}) \preceq \pi_{\mathfrak{P}}(B)$, that is, $\pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(B \cup \{a\})$. This contradicts the maximality of B. Thus, $a \in C_{\mathfrak{P}}(A)$.

(ii) Let $a \in C_{\mathfrak{P}}(M_{\mathfrak{P}}(A))$ and suppose by contradiction that $a \notin A$. Therefore, we have $A \subseteq M_{\mathfrak{P}}(A) \setminus \{a\} \subseteq M_{\mathfrak{P}}(A)$ and, hence, that $\pi_{\mathfrak{P}}(M_{\mathfrak{P}}(A)) \preceq \pi_{\mathfrak{P}}(M_{\mathfrak{P}}(A) \setminus \{a\}) \preceq \pi_{\mathfrak{P}}(A)$. Since $\pi_{\mathfrak{P}}(M_{\mathfrak{P}}(A)) = \pi_{\mathfrak{P}}(A)$, we conclude that $\pi_{\mathfrak{P}}(M_{\mathfrak{P}}(A) \setminus \{a\}) = \pi_{\mathfrak{P}}(M_{\mathfrak{P}}(A))$, contradicting the fact that $a \in C_{\mathfrak{P}}(A)$. □

4. Representation results in the finite case

In this section we assume that Ω be a finite set and $\sigma \in \mathrm{OP}(\Omega)$. The main result of this section is the following representation theorem, where we represent three different very general classes of set operators by means of the set operators induced by pairings.

THEOREM 4.1.

(i) $\sigma \in \mathrm{CLO}(\Omega)$ *if and only if there exists* $\mathfrak{P} \in \mathrm{PAIR}(\Omega)$ *such that* $\sigma = M_{\mathfrak{P}}$.

(ii) $\sigma \in \mathrm{NSO}(\Omega)$ *if and only if there exists* $\mathfrak{P} \in \mathrm{PAIR}(\Omega)$ *such that* $\sigma = C_{\mathfrak{P}}$.

(iii) $\sigma \in \mathrm{ANSO}(\Omega)$ *if and only if there exists* $\mathfrak{P} \in \mathrm{PAIR}(\Omega)$ *such that* $\sigma = C_{\mathfrak{P}}$ *and* $M_{\mathfrak{P}} \in \mathrm{MCLO}(\Omega)$.

PROOF. (i) Let $\mathfrak{P} \in \mathrm{PAIR}(\Omega)$ such that $\sigma = M_{\mathfrak{P}}$. In view of (ii) of Theorem 3.1, $\sigma = M_{\mathfrak{P}} \in \mathrm{CLO}(\Omega)$.

On the other hand, let $\sigma \in \mathrm{CLO}(\Omega)$, $\mathfrak{S} := \mathrm{Fix}(\sigma) \in \mathrm{CLSY}(\Omega)$, $E \in \mathfrak{S}$ be the minimum element in $\mathfrak{S}$ and let $C_1, C_2, \ldots, C_k$ be all maximal chains from E to the top Ω, where C_i is the chain in $\mathfrak{S}$ given by $A_{i_0} = E \subsetneq A_{i_1} \subsetneq \cdots A_{i_{l_i}} = \Omega$, for $i = 1, 2, \ldots, k$. Furthermore, we set $m_0 = 0$, $m_i := \sum_{s=1}^{i} l_s$, and $m := m_k$. We now define a pairing

$$\mathfrak{P} = \mathfrak{P}(\mathfrak{S}) := \langle U_{\mathfrak{S}}, \Omega, F_{\mathfrak{S}}, \mathbb{N}\rangle \in \mathrm{PAIR}(\Omega)$$

whose maximum partitioner lattices is exactly $\mathfrak{S}$. First of all, we consider the set $U_{\mathfrak{S}} = \{u_1, u_2, \ldots, u_m\}$. On the other hand, we must define the function $F_{\mathfrak{S}} : U_{\mathfrak{S}} \times \Omega \longrightarrow \mathbb{N}$. In this regard, we set $F_{\mathfrak{S}}(u_s, a_j) := 1$ if $a_j \in E$, for any $u_s \in U_{\mathfrak{S}}$. Let now $a_j \in \Omega \setminus E$. For each $i \in \{1, 2, \ldots, k\}$ we define t_{ij} as the smallest positive integer such that $a_j \in A_{i_{t_{ij}}} \setminus A_{i_{t_{ij}-1}}$. Finally, we set

$$F_{\mathfrak{S}}(u_s, a_j) := \begin{cases} 1 & \text{if } s = 1, \\ F_{\mathfrak{S}}(u_{s-1}, a_j) & \text{if } m_{i-1} + 2 \le s \le m_i - t_{ij} + 1, \\ F_{\mathfrak{S}}(u_{s-1}, a_j) + 1 & \text{otherwise.} \end{cases} \tag{4.1}$$

Let us first show that if $A \in \mathfrak{S}$ then $A \in \mathrm{MAXP}(\mathfrak{P})$. If $A = \Omega$ there is nothing to prove. Hence, assume $A \neq \Omega$. Let $C_i : A_{i_0} = E \subsetneq A_{i_1} \subsetneq \cdots A_{i_{l_i}} \subsetneq \Omega$ be a chain from E to

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Ω and $t \in \{1, 2, \ldots, l_i\}$ be such that $A = A_{i_t}$. Set moreover $s := m_i - t$. Let us prove now that $u_s \equiv_A u_{s+1}$ but, for each choice of $a_j \in \Omega \setminus A$, it holds $F_{\mathfrak{S}}(u_s, a_j) \neq F_{\mathfrak{S}}(u_{s+1}, a_j)$. Let us notice that if $a_j \in A$, then $t_{ij} \leq t$, so $m_{i-1} + 1 \leq s \leq m_i - t_{ij}$. By (4.1), it follows that $F_{\mathfrak{S}}(u_s, a_j) = F_{\mathfrak{S}}(u_{s+1}, a_j)$, therefore we have shown that $u_s \equiv_A u_{s+1}$. On the other hand, if $a_j \notin A$, then $t_{ij} > t$, so $s = m_i - t > m_i - t_{ij}$. Again by (4.1), it follows that $F_{\mathfrak{S}}(u_{s+1}, a_j) = F_{\mathfrak{S}}(u_s, a_j) + 1$, that is, $u_s \not\equiv_A u_{s+1}$. This proves that $A \in \mathrm{MAXP}(\mathfrak{P})$.

Let now $A \in \mathrm{MAXP}(\mathfrak{P})$. We will now prove that $A \in \mathfrak{S}$. In this regard, let us consider the minimum element B in $\mathfrak{S}$ such that $A \subseteq B$. We now prove that $A \approx_{\mathfrak{P}} B$. Let us notice that, for any $B' \in \mathfrak{S}$ such that $B' \subsetneq B$, it results that $B' \subsetneq A$. In fact, let $a_k \in B'$ and $a_h \in A \setminus B' \subseteq B \setminus B'$. Let s be such that $u_{s-1} \equiv_A u_s$. Thus, $F_{\mathfrak{S}}(u_s, a_h) = F_{\mathfrak{S}}(u_{s-1}, a_h)$, so $m_{i-1} + 2 \leq s \leq m_i - t_{ih} + 1$. But both the conditions $a_h \in B \setminus B'$ and $a_k \in B'$ imply that $t_{ih} > t_{ik}$, so $m_{i-1} + 2 \leq s \leq m_i - t_{ik} + 1$ and, then, $F_{\mathfrak{S}}(u_s, a_k) = F_{\mathfrak{S}}(u_{s-1}, a_k)$. Since $A \in \mathrm{MAXP}(\mathfrak{P})$, it holds that $a_k \in A$ and thus that $B' \subsetneq A$. Without loss in generality, we can assume that B' is maximal among the proper subsets of B. Let now C_i be a chain through B. Then, there exists t such that $B' := A_{i_{t-1}} \subsetneq A \subseteq A_{i_t} = B$ and let $s \in \{1, 2, \ldots, m\}$ such that $u_{s-1} \equiv_A u_s$. If $a_k \in B$ and $a_h \in A \setminus B'$, then $t_{ik} < t_{ih}$, so, since $u_{s-1} \equiv_A u_s$, it holds that $F_{\mathfrak{S}}(u_s, a_k) = F_{\mathfrak{S}}(u_{s-1}, a_k)$, thus $a_k \in M_{\mathfrak{P}}(A) = A$. This proves that $A = B \in \mathfrak{S}$ and, thus, $\sigma = M_{\mathfrak{P}}$.

(ii) Let $\mathfrak{P} \in \mathrm{PAIR}(\Omega)$ such that $\sigma = C_{\mathfrak{P}}$. By (iv) of Theorem 3.1, $\sigma = C_{\mathfrak{P}} \in \mathrm{NSO}(\Omega)$.

On the other hand, let $\sigma \in \mathrm{NSO}(\Omega)$. Let us consider the set operator $\varphi : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ defined as follows:

$$\varphi_\sigma(A) := A \cup A^*,$$

where $A^* := \{b \in \Omega : \ b \in \Omega \setminus (A \cup \sigma(A \cup \{b\}))\}$. We will prove that φ is a closure operator on Ω. Clearly, φ satisfies (OP1). Let now $A \subseteq B$. It suffices to show that $A^* \subseteq \varphi_\sigma(B)$. Let $b \in A^*$. Assume that $b \in \Omega \setminus B$. Then, $A \cup \{b\} \subseteq B \cup \{b\}$, therefore, by (OP5), we have that $\sigma(A \cup \{b\}) \supseteq \sigma(B \cup \{b\}) \cap (A \cup \{b\})$. But since $b \in \Omega \setminus \sigma(A \cup \{b\})$, it follows that $b \in \Omega \setminus ((A \cup \{b\}) \cap \sigma(B \cup \{b\}))$. Clearly, it implies that $b \in \Omega \setminus \sigma(B \cup \{b\})$ and this proves that $b \in \Omega \setminus (B \cup \sigma(B \cup \{b\})) = B^*$, that is, $b \in \varphi_\sigma(B)$. On the other hand, if $b \in B$, it is obvious that $b \in \varphi_\sigma(B)$. This proves (OP4).

We now show that φ is idempotent. By (OP1), it follows that $\varphi_\sigma(A) \subseteq \varphi_\sigma(\varphi_\sigma(A))$. Clearly, if $(\varphi_\sigma(A))^* = \emptyset$, then $\varphi_\sigma(\varphi_\sigma(A)) \subseteq \varphi_\sigma(A)$. Assume now that $(\varphi_\sigma(A))^* \neq \emptyset$ and let $b \in (\varphi_\sigma(A))^*$. Then, $b \in \Omega \setminus (\varphi_\sigma(A) \cup \sigma(\varphi_\sigma(A) \cup \{b\})) = \Omega \setminus ((A \cup A^*) \cup \sigma(A \cup A^* \cup \{b\}))$. Let us prove that the condition $b \in \Omega \setminus A \cup A^*$ implies $b \in \sigma(A \cup A^* \cup \{b\})$.

Let us observe that since $b \in \Omega \setminus \varphi_\sigma(A)$, then $b \in \sigma(A \cup \{b\})$. Therefore, let us fix an integer m and assume that $b \in \sigma(A \cup B \cup \{b\})$ for any $B \subseteq A^*$ such that $|B| = m - 1$. Let now $B \subseteq A^*$ such that $|B| = m$ and fix $c \in B$. We will prove that $c \in \Omega \setminus \sigma((A \cup (B \setminus \{c\})) \cup \{c\})$ and that $b \in \sigma((A \cup B \setminus \{c\}) \cup \{b\})$ so that it is possible to apply (OP6) that, in this case, yields $c \in \sigma((A \cup B \setminus \{c\}) \cup \{b, c\}) = \sigma(A \cup B \cup \{b\})$, showing the claim for B. In particular, the claim holds for $B = A^*$. Then, $b \in \sigma(A \cup A^* \cup \{b\})$, contradicting our choice of b. Necessarily, it must be $(\varphi_\sigma(A))^* = \emptyset$.

Hence, let us firstly show that $c \in \Omega \setminus \sigma((A \cup (B \setminus \{c\})) \cup \{c\})$. As a matter of fact, we have that $c \in A^*$, so $c \in \Omega \setminus (A \cup \{c\})$. Furthermore, to say that $c \in B$ means that

$A \cup \{c\} \subseteq A \cup B$ so, by (OP5), $\sigma(A \cup \{c\}) \supseteq \sigma(A \cup B) \cap (A \cup \{c\})$. In other terms, $c \in \Omega \setminus \sigma(A \cup B)$, that is, $c \in \Omega \setminus \sigma((A \cup (B \setminus \{c\})) \cup \{c\})$.

On the other hand, we have that $b \in \sigma((A \cup B \setminus \{c\}) \cup \{b\})$, in fact we can use the inductive hypothesis on $B \setminus \{c\} \subseteq A^*$ since $|B \setminus \{c\}| = m - 1$. The two hypotheses of condition (OP6) are satisfied and this proves that φ_σ satisfies (OP3), so $\varphi_\sigma \in \mathrm{CLO}(\Omega)$.

By part (i), there exists a pairing $\mathfrak{P} \in \mathrm{PAIR}(\Omega)$ such that $M_{\mathfrak{P}} = \varphi_\sigma$. To conclude, we have to show that $\sigma = C_{\mathfrak{P}}$.

Let $b \in C_{\mathfrak{P}}(A)$, then $b \in \Omega \setminus M_{\mathfrak{P}}(A \setminus \{b\}) = \varphi_\sigma(A \setminus \{b\})$. This entails that $b \in \Omega \setminus (A \setminus \{b\})^*$ or, equivalently, $b \in \sigma(A \setminus \{b\} \cup \{b\}) = \sigma(A)$.

On the other hand, let $b \in \sigma(A)$. By (OP1), $b \in A$. Furthermore, assume by contradiction that $b \in M_{\mathfrak{P}}(A \setminus \{b\})$. Then, $b \in (A \setminus \{b\})^*$, that is, $b \in \Omega \setminus \sigma(A \setminus \{b\} \cup \{b\}) = \Omega \setminus \sigma(A)$, contradicting our choice of b. This implies that $C_{\mathfrak{P}}(A) = \sigma(A)$ for any $A \in \mathcal{P}(\Omega)$ and concludes the proof of part (ii).

(iii) Let $\sigma \in \mathrm{ANSO}(\Omega)$. By the above part (ii), there exists $\mathfrak{P} \in \mathrm{PAIR}(\Omega)$ such that $\sigma = C_{\mathfrak{P}}$. Let us consider $b, c \in \Omega$ such that $\{b, c\} \subseteq \Omega \setminus M_{\mathfrak{P}}(A)$ and $b \in M_{\mathfrak{P}}(A \cup \{c\})$. Hence, by (3.2), we have that $b \in C_{\mathfrak{P}}(A \cup \{b\})$ but $b \in \Omega \setminus C_{\mathfrak{P}}(A \cup \{b\} \cup \{c\})$. Moreover, by (OP8), it follows that $C_{\mathfrak{P}}(A) \cup \{b\} \subseteq C_{\mathfrak{P}}(A \cup \{b\})$. Let us assume by contradiction that $c \in \Omega \setminus M_{\mathfrak{P}}(A \cup \{b\})$. Again by (3.2), it results that $c \in C_{\mathfrak{P}}(A \cup \{b\} \cup \{c\})$ and, by (OP8), we have $b \in C_{\mathfrak{P}}(A) \cup \{b\} \subseteq C_{\mathfrak{P}}(A \cup \{b\}) \cup \{c\} \subseteq C_{\mathfrak{P}}(A \cup \{b\} \cup \{c\})$, which is a contradiction. Hence, $M_{\mathfrak{P}}$ must satisfy (OP7).

On the other hand, let $\mathfrak{P} \in \mathrm{PAIR}(\Omega)$ be such that $C_{\mathfrak{P}} = \sigma$ and $M_{\mathfrak{P}}$ satisfies (OP7). By (v) of Theorem 3.1, $C_{\mathfrak{P}} \in \mathrm{NSO}(\Omega)$. Hence, we have to show that $C_{\mathfrak{P}}$ satisfies (OP8). In this regard, let $b \in C_{\mathfrak{P}}(A \cup \{b\})$ and let $c \in C_{\mathfrak{P}}(A) \cup \{b\}$ be such that

$$c \in \Omega \setminus C_{\mathfrak{P}}(A \cup \{b\}). \tag{4.2}$$

If $c = b$, we have that $c \in C_{\mathfrak{P}}(A \cup \{c\})$ by (OP8), contradicting (4.2). Hence, assume $c \neq b$, that is, $c \in C_{\mathfrak{P}}(A)$ or, equivalently, $c \in \Omega \setminus M_{\mathfrak{P}}(A \setminus \{c\})$. Equations (4.2) and (3.2) imply that $c \in M_{\mathfrak{P}}(A \setminus \{c\} \cup \{b\})$. Furthermore, to say that $b \in C_{\mathfrak{P}}(A \cup \{b\})$ means that $b \in \Omega \setminus M_{\mathfrak{P}}(A)$ and, *a fortiori*, $b \in \Omega \setminus M_{\mathfrak{P}}(A \setminus \{c\})$. Set $B := A \setminus \{c\}$. Then, it results that $\{b, c\} \subseteq \Omega \setminus M_{\mathfrak{P}}(B)$ and $c \in M_{\mathfrak{P}}(B \cup \{b\})$, therefore, by (OP7), it must necessarily be $b \in M_{\mathfrak{P}}(B \cup \{c\}) = M_{\mathfrak{P}}(A)$, contradicting the fact that $b \in \Omega \setminus M_{\mathfrak{P}}(A)$. Therefore, $C_{\mathfrak{P}}$ satisfies (OP8). □

We can now deduce some direct consequences of the above theorem.

Corollary 4.2. *$C_{\mathfrak{P}} \in \mathrm{ANSO}(\Omega)$ if and only if $M_{\mathfrak{P}} \in \mathrm{MCLO}(\Omega)$.*

Proof. Assume that $M_{\mathfrak{P}} \in \mathrm{MCLO}(\Omega)$. By Theorem 4.1(iii), it follows that $\sigma = C_{\mathfrak{P}} \in \mathrm{ANSO}(\Omega)$. On the other hand, let $C_{\mathfrak{P}} \in \mathrm{ANSO}(\Omega)$. Again by Theorem 4.1(iii), there exists a pairing $\mathfrak{P}'$ such that $C_{\mathfrak{P}} = C_{\mathfrak{P}'}$ and $M_{\mathfrak{P}'} \in \mathrm{MCLO}(\Omega)$. Now, by (3.2), it results that $M_{\mathfrak{P}} = M_{\mathfrak{P}'}$ and this concludes the proof. □

Corollary 4.3. *If $C_{\mathfrak{P}} \in \mathrm{ANSO}(\Omega)$, then $\mathrm{MINP}(\mathfrak{P})$ is a matroid.*

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Proof. By Corollary 4.2, it follows that $M_{\mathfrak{P}} \in \mathrm{MCLO}(\Omega)$. Now, the family

$$\mathcal{F} := \{A \in \mathcal{P}(\Omega) : \ a \notin M_{\mathfrak{P}}(A \setminus \{a\})\ \forall a \in A\}$$

coincides with the indipendent set family of a matroid on Ω (see [29]). By virtue of (3.4), we deduce that $\mathcal{F} = \mathrm{MINP}(\mathfrak{P})$, hence the claim has been showed. □

Corollary 4.4. *Let $\mathcal{M}$ be the independent set family of a matroid on Ω. Then, there exists a pairing $\mathfrak{P} \in \mathrm{PAIR}(\Omega)$ such that $\mathrm{MINP}(\mathfrak{P}) = \mathcal{M}$.*

Proof. Let σ be the closure operator associated with $\mathcal{M}$ (see [29]). Then, it is well-known that

$$\mathcal{M} = \{A \in \mathcal{P}(\Omega) : a \notin \sigma(A \setminus \{a\})\ \forall a \in A\}. \tag{4.3}$$

By (i) of Theorem 4.1, there exists a pairing $\mathfrak{P} \in \mathrm{PAIR}(\Omega)$ such that $M_{\mathfrak{P}} = \sigma$. Then, the thesis follows by (4.3) and by (3.4). □

Relatively to $\mathfrak{P} \in \mathrm{PAIR}(\Omega)$ and $A \subseteq \Omega$, we set $\mathrm{ESS}_{\mathfrak{P}}(A) := \mathrm{ESS}_{M_{\mathfrak{P}}}(A)$ and $\mathrm{BAS}_{\mathfrak{P}}(A) := \mathrm{BAS}_{M_{\mathfrak{P}}}(A)$. In this case, we speak respectively of $\mathfrak{P}$-*symmetry essentials* and $\mathfrak{P}$-*symmetry bases* of A.

Corollary 4.5. *Let $\sigma \in \mathrm{CLO}(\Omega)$ and $A \subseteq \Omega$. Then, there exists a pairing $\mathfrak{P} \in \mathrm{PAIR}(\Omega)$ such that $\mathrm{ESS}_{\sigma}(A) = \mathrm{ESS}_{\mathfrak{P}}(A)$ and $\mathrm{BAS}_{\sigma}(A) = \mathrm{BAS}_{\mathfrak{P}}(A)$.*

Proof. This is a direct consequence of (i) of Theorem 4.1. □

5. Symmetry bases and essentials induced by pairings

For a given pairing $\mathfrak{P} \in \mathrm{PAIR}(\Omega)$, in this section we characterize the $\mathfrak{P}$-symmetry essentials and the $\mathfrak{P}$-symmetry bases by means of a new set system which can be considered as a localization of the dissymmetry with respect to a fixed subset A of Ω.

Let therefore $\mathfrak{P} = \langle U, F, \Lambda\rangle \in \mathrm{PAIR}(\Omega)$. For any $u, v \in U$, we set

$$\Delta_A^{\mathfrak{P}}(u, v) := \{a \in A : \ F(u, a) \neq F(v, a)\}.$$

It is clear that we can interpret the subset $\Delta_A^{\mathfrak{P}}(u, v)$ as the subset of all elements $a \in A$ which break the symmetry between u and v. In this perspective, it is natural to call $\Delta_A^{\mathfrak{P}}(u, v)$ the A-*dissymmetry neighbourhood* of u and v.

In the next result we establish some useful properties of the dissymmetry neighbourhoods.

Proposition 5.1. *Let $D \subseteq A$ and $u, u' \in U$. Then, the following conditions hold:*

(i) $D = \Delta_A^{\mathfrak{P}}(u, u') \Longrightarrow u \equiv_{A \setminus D} u'$;

(ii) $u \equiv_{A \setminus D} u' \Longrightarrow \Delta_A^{\mathfrak{P}}(u, u') \subseteq D$;

(iii) $\Delta_A^{\mathfrak{P}}(u, u') \cap D = \emptyset \Longleftrightarrow u \equiv_D u'$.

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Proof. (i) Let $a \in A \setminus D$. Then, we have $F(v,a) = F(w,a)$, therefore $v \equiv_{A\setminus D} w$.

(ii) By hypothesis, $F(v,a) = F(w,a)$ for all $a \in A \setminus D$. This means that the set of elements of A for which the values of $F(v,a)$ and $F(w,a)$ are different, that is, $\Delta_A^{\mathfrak{P}}(v,w)$, stays in the complement of $A \setminus D$. Therefore, $\Delta_A^{\mathfrak{P}}(v,w) \subseteq D$.

(iii) Let $\Delta_A^{\mathfrak{P}}(v,w) \cap D = \emptyset$. This is equivalent to require that for all $c \in D$, $F(v,c) = F(w,c)$ or, equivalently, $v \equiv_D w$. □

We consider now the following set system on A:

$$\mathrm{DIS}_{\mathfrak{P}}(A) := \{\Delta_A^{\mathfrak{P}}(u,v) \neq \emptyset\},$$

which we call the $\mathfrak{P}$-*dissymmetry set system* of A.

In the next result, we show that the $\mathfrak{P}$-symmetry essentials of A are exactly the minimal elements of the $\mathfrak{P}$-dissymmetry set systems of A. This result therefore also provides a constructive method to determine all symmetry essentials of A.

Theorem 5.2. *Let $B \subseteq A$. Then, the following conditions are equivalent:*

(i) $\pi_{\mathfrak{P}}(A \setminus B) \neq \pi_{\mathfrak{P}}(A)$ *and* $\pi_{\mathfrak{P}}(A \setminus B') = \pi_{\mathfrak{P}}(A)$ *for all* $B' \subsetneq B$*;*

(ii) $A \setminus B$ *covers* A *in the lattice* $\mathbb{M}_{\mathfrak{P}}(A)$*;*

(iii) $B \in \min(\mathrm{DIS}_{\mathfrak{P}}(A))$*;*

(iv) $B \in \mathrm{ESS}_{\mathfrak{P}}(A)$.

Proof. (i) $\Longrightarrow$ (ii) Let $B \subseteq A$ such that $\pi_{\mathfrak{P}}(A \setminus B) \neq \pi_{\mathfrak{P}}(A)$ and $\pi_{\mathfrak{P}}(A \setminus B') = \pi_{\mathfrak{P}}(A)$ for all $B' \subsetneq B$. Let us show that $A \setminus B = M_{\mathfrak{P}}(A \setminus B) \cap A$. Clearly, $A \setminus B \subseteq M_{\mathfrak{P}}(A \setminus B) \cap A$. *Vice versa*, let $a \in M_{\mathfrak{P}}(A \setminus B) \cap A$ and assume that $a \notin A \setminus B$. Hence, $a \in B$. Let $u \equiv_{A\setminus B} u'$, then $F(u,c) = F(u',c)$ for any $c \in A \setminus B$ and, in particular, for any $c \in M_{\mathfrak{P}}(A \setminus B)$. Thus, $F(u,c) = F(u',c)$ for any $c \in (A \setminus B) \cup \{a\}$, that is, for any $c \in A \setminus (B \setminus \{a\})$. Set $B' := B \setminus \{a\}$. In other terms, we have shown that $u \equiv_{A\setminus B} u'$ implies $u \equiv_{A\setminus B'} u'$. But $\pi_{\mathfrak{P}}(A \setminus B') = \pi_{\mathfrak{P}}(A)$, so $u \equiv_{A\setminus B} u'$ implies $u \equiv_A u'$ or, equivalently, $\pi_{\mathfrak{P}}(A \setminus B) = \pi_{\mathfrak{P}}(A)$, contradicting the assumption on B. This proves that $A \setminus B \in \mathbb{M}_{\mathfrak{P}}(A)$. We must show that $A \setminus B$ covers A in the lattice $\mathbb{M}_{\mathfrak{P}}(A)$. Suppose by contradiction it were false. Then, there exists $A \setminus B \subsetneq C \subsetneq A$ such that C covers A in the lattice $\mathbb{M}_{\mathfrak{P}}(A)$. But, this ensures that $C = A \setminus B'$ for some nonempty $B' \subsetneq B$. Let us prove that $\pi_{\mathfrak{P}}(C) \neq \pi_{\mathfrak{P}}(A)$. Assume by contradiction that $\pi_{\mathfrak{P}}(C) = \pi_{\mathfrak{P}}(A)$. Since $C = A \cap M$ for some $M \in \mathrm{MAXP}(\mathfrak{P})$, so $C \subseteq M$ and $M_{\mathfrak{P}}(C) = M_{\mathfrak{P}}(A) \subseteq M$, that is, $A \subseteq M$. But, this implies that $A \cap M = C \supseteq A$ and this is impossible. So, $\pi_{\mathfrak{P}}(C) \neq \pi_{\mathfrak{P}}(A)$. But the existence of such a subset C contradicts our hypothesis on B.

(ii) $\Longrightarrow$ (i) Let B be a nonempty subset of A such that $A \setminus B$ covers A in the lattice $\mathbb{M}_{\mathfrak{P}}(A)$. Assume by contradiction that $\pi_{\mathfrak{P}}(A \setminus B) = \pi_{\mathfrak{P}}(A)$, that is, $M_{\mathfrak{P}}(A \setminus B) = M_{\mathfrak{P}}(A)$. Since $A \setminus B = A \cap C$ for some $C \in \mathrm{MAXP}(\mathfrak{P})$, we have that

$$M_{\mathfrak{P}}(A \cap C) = M_{\mathfrak{P}}(A \setminus B) = M_{\mathfrak{P}}(A) \subseteq M_{\mathfrak{P}}(A) \cap M_{\mathfrak{P}}(C) = M_{\mathfrak{P}}(A) \cap C,$$

that is, $M_{\mathfrak{P}}(A) \subseteq C$. In other terms, we showed that $A = A \cap M_{\mathfrak{P}}(A) \subseteq A \cap C = A \setminus B$ or, equivalently $A = A \setminus B$, which is a contradiction. Hence, $\pi_{\mathfrak{P}}(A) \neq \pi_{\mathfrak{P}}(A \setminus B)$.

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Furthermore, let $B' \subsetneq B$ and assume that $\pi_{\mathfrak{P}}(A \setminus B') \neq \pi_{\mathfrak{P}}(A)$. Then, $A \setminus B \subsetneq A \setminus B' \subseteq M_{\mathfrak{P}}(A \setminus B') \cap A$. In particular, we also have that $M_{\mathfrak{P}}(A \setminus B') \cap A \subsetneq A$, otherwise $A \subseteq M_{\mathfrak{P}}(A \setminus B')$, that is, $M_{\mathfrak{P}}(A) = M_{\mathfrak{P}}(A \setminus B')$, contradicting our assumption. In this way, we showed that $A \setminus B$ does not cover A in the lattice $\mathbb{M}_{\mathfrak{P}}(A)$, which is a contradiction. This proves (i).

(i) $\Longrightarrow$ (iii) Let $B \subseteq A$ be such that $\pi_{\mathfrak{P}}(A \setminus B) \neq \pi_{\mathfrak{P}}(A)$ and $\pi_{\mathfrak{P}}(A \setminus B') = \pi_{\mathfrak{P}}(A)$ for all $B' \subsetneq B$. Hence, there exist two distinct elements $v, w \in U$ such that $v \equiv_{A \setminus B} w$ and $v \not\equiv_B w$. Equivalently, we can express the previous condition by saying that $A \setminus B \subseteq A \setminus \Delta_A^{\mathfrak{P}}(v, w)$, that is, $\Delta_A^{\mathfrak{P}}(v, w) \subseteq B$. We now claim that $\Delta_A^{\mathfrak{P}}(v, w) = B$. Indeed, if $b \in B$ and $B' := B \setminus \{b\} \subsetneq B$, we deduce that $v \not\equiv_{A \setminus B'} w$ by our assumption on B. Therefore $b \in \Delta_A^{\mathfrak{P}}(v, w)$. By the arbitrariness of $b \in B$, it follows that $\Delta_A^{\mathfrak{P}}(v, w) = B$. This proves that $B \in \mathrm{DIS}_{\mathfrak{P}}(A)$. Moreover, we proved that whenever two elements $v, w \in U$ satisfy the relation $\Delta_A^{\mathfrak{P}}(v, w) \subseteq B$, then $\Delta_A^{\mathfrak{P}}(v, w) = B$. This means that $B \in \min(\mathrm{DIS}_{\mathfrak{P}}(A))$.

(iii) $\Longrightarrow$ (i) Let now $B = \Delta_A^{\mathfrak{P}}(v, w) \neq \emptyset$, for some $v, w \in U$, be minimal in the poset $(\mathrm{DIS}_{\mathfrak{P}}(A), \subseteq)$. Since B is nonempty, by (iii) of Proposition 5.1, it follows that $v \not\equiv_A w$. Moreover, by (i) of Proposition 5.1 we also obtain $v \equiv_{A \setminus B} w$. Then, we have $\pi_{\mathfrak{P}}(A \setminus B) \neq \pi_{\mathfrak{P}}(A)$.

Let now $B' \subsetneq B$. B minimal in $\mathrm{DIS}_{\mathfrak{P}}(A)$ implies that, for all $u, u' \in U$ such that $\Delta_A^{\mathfrak{P}}(u, u') \neq \emptyset$, $\Delta_A^{\mathfrak{P}}(u, u') \nsubseteq B'$. We claim that $\pi_{\mathfrak{P}}(A \setminus B') = \pi_{\mathfrak{P}}(A)$. It is obvious that $u \equiv_A u'$ implies $u \equiv_{A \setminus B'} u'$; furthermore suppose that $u \equiv_{A \setminus B'} u'$ and assume by contradiction that $u \not\equiv_A u'$. Then, we have $\Delta_A^{\mathfrak{P}}(u, u') \subseteq B'$, which is an absurd. Hence, $u \equiv_A u'$ if and only if $u \equiv_{A \setminus B'} u'$, so $\pi_{\mathfrak{P}}(A \setminus B') = \pi_{\mathfrak{P}}(A)$. In this way, we have shown that B satisfies also the second condition of (i).

(i) if and only if (iv) In view of the definition of $\mathrm{ESS}_{\mathfrak{P}}(A)$, we immediately get the equivalence between (i) and (iv). $\square$

Relatively to the $\mathfrak{P}$-symmetry bases of A, in the next result we show that they coincide with the minimal transversal of $\mathrm{DIS}_{\mathfrak{P}}(A)$. In this regard, we recall that in the literature the *hypergraph transversal problem* for a hypergraph (that is, a finite set system) $\mathcal{F}$ is the problem of generating all the elements of $\mathrm{Tr}(\mathcal{F})$. In general, this is an important mathematical problem which has many applications both in mathematics and in computer science [20]. In [18], a complete geometric classification has been provided of the subgraphs induced by the members of $\mathrm{BAS}_G(\Omega)$ when G is the Petersen graph and Ω is its vertex set.

THEOREM 5.3. *Let $B \subseteq A$. Then, the following conditions are equivalent:*

(i) $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(B)$ *and* $\pi_{\mathfrak{P}}(A) \neq \pi_{\mathfrak{P}}(B')$ *for all* $B' \subsetneq B$*;*
(ii) $B \in \min([A]_{\approx_{\mathfrak{P}}})$*;*
(iii) $M_{\mathfrak{P}}(B) = M_{\mathfrak{P}}(A)$ *and* $C_{\mathfrak{P}}(B) = B$*;*
(iv) $B \in \mathrm{Tr}(\min(\mathrm{DIS}_{\mathfrak{P}}(A)))$*;*
(v) $B \in \mathrm{BAS}_{\mathfrak{P}}(A)$.

PROOF. (i) $\Longrightarrow$ (ii) Let $B \subseteq A$ be such that $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(B)$ and $\pi_{\mathfrak{P}}(A) \neq \pi_{\mathfrak{P}}(B')$ for all $B' \subsetneq B$. Then, $B \in [A]_{\approx_{\mathfrak{P}}}$ and, by our assumptions on B, there is no $D \in [A]_{\approx_{\mathfrak{P}}}$

such that $D \subsetneq B$. Hence, B is necessarily minimal in the subset family $[A]_{\approx_{\mathfrak{P}}}$. Hence, $B \in \{X : X \in \min([A]_{\approx_{\mathfrak{P}}}),\ X \subseteq A\}$.

(ii) $\Longrightarrow$ (i) Let $B \in \min([A]_{\approx_{\mathfrak{P}}})$. Then, $\pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(A)$. Let now $B' \subsetneq B$. Since B is minimal in $[A]_{\approx_{\mathfrak{P}}}$, it follows that $B' \notin [A]_{\approx_{\mathfrak{P}}}$, so that $\pi_{\mathfrak{P}}(B') \neq \pi_{\mathfrak{P}}(A)$. This shows that B satisfies the conditions in (i).

(i) $\Longrightarrow$ (iii) Let $B \subseteq A$ be such that $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(B)$ and $\pi_{\mathfrak{P}}(A) \neq \pi_{\mathfrak{P}}(B')$ for all $B' \subsetneq B$. Then, we have that $M_{\mathfrak{P}}(A) = M_{\mathfrak{P}}(B)$. Furthermore, if we take $B' = B \setminus \{b\}$, it follows that $\pi_{\mathfrak{P}}(B') \neq \pi_{\mathfrak{P}}(A)$. By the arbitrariness of b, we deduce that $C_{\mathfrak{P}}(B) = B$.

(iii) $\Longrightarrow$ (i) Let $B \subseteq A$ be such that $M_{\mathfrak{P}}(B) = M_{\mathfrak{P}}(A)$ and $C_{\mathfrak{P}}(B) = B$. To say that $M_{\mathfrak{P}}(B) = M_{\mathfrak{P}}(A)$, implies that $A \approx_{\mathfrak{P}} B$, that is, $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(B)$. Furthermore, let $b \in B$ and set $B' := B \setminus \{b\}$. Since $C_{\mathfrak{P}}(B) = B$, it follows that $b \notin M_{\mathfrak{P}}(B')$, so $M_{\mathfrak{P}}(B') \neq M_{\mathfrak{P}}(B) = M_{\mathfrak{P}}(A)$, that is, $\pi_{\mathfrak{P}}(B') \neq \pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(A)$ and this proves the claim.

(i) if and only if (iv) We claim that B as in (i) must be a transversal of $\mathrm{DIS}_{\mathfrak{P}}(A)$ and, *vice versa*, that a transversal of $\mathrm{DIS}_{\mathfrak{P}}(A)$ must be as in (i). By definition of $\mathrm{DIS}_{\mathfrak{P}}(A)$ it results that D is nonempty and that there exist two distinct elements $u, u' \in U$ such that $D = \{a \in A : F(u, a) \neq F(u', a)\}$. Since D contains at least one element, we deduce that $u \not\equiv_A u'$, and this also implies $u \not\equiv_B u'$ from the hypothesis $\equiv_B = \equiv_A$. Hence, there exists an element $b \in B$ such that $F(u, b) \neq F(u', b)$, that is, $b \in B \cap D$. This shows that $B \cap D \neq \emptyset$, therefore B is a transversal of $\mathrm{DIS}_{\mathfrak{P}}(A)$.

On the other hand, let us assume that $B \subseteq A$ is a transversal of $\mathrm{DIS}_{\mathfrak{P}}(A)$ and let u, u' be two any distinct elements in U. If $u \equiv_A u'$ it is obvious that we also have $u \equiv_B u'$. We can assume therefore that $u \not\equiv_A u'$. Then, it follows that the $D := \Delta_A^{\mathfrak{P}}(u, u')$ is nonempty, so that $D \in \mathrm{DIS}_{\mathfrak{P}}(A)$. Since B is a transversal of $\mathrm{DIS}_{\mathfrak{P}}(A)$ we have that $B \cap D \neq \emptyset$. Let $b \in B \cap D$. Then, there exists $b \in B$ such that $F(u, b) \neq F(u', b)$, and this implies that $u \not\equiv_B u'$. Hence, $\equiv_B = \equiv_A$. This proves that $\pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(A)$ if and only if B is a transversal of $\mathrm{DIS}_{\mathfrak{P}}(A)$. Clearly, it suffices to be transversal of $\min(\mathrm{DIS}_{\mathfrak{P}}(A))$ in order to be also a transversal of $\mathrm{DIS}_{\mathfrak{P}}(A)$. Hence, we showed that $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(B)$ if and only if B is a transversal of $\min(\mathrm{DIS}_{\mathfrak{P}}(A))$.

Now, assume that $B \subseteq A$ satisfies (i). By what we showed above, B is a transversal of $\mathrm{DIS}_{\mathfrak{P}}(A)$. Moreover, if $b \in B$, we have that $\pi_{\mathfrak{P}}(B \setminus \{b\}) \neq \pi_{\mathfrak{P}}(A)$, therefore $B \setminus \{b\}$ is not a transversal of $\mathrm{DIS}_{\mathfrak{P}}(A)$. Hence, B is a minimal transversal of $\mathrm{DIS}_{\mathfrak{P}}(A)$.

On the other hand, let B be a minimal transversal of $\mathrm{DIS}_{\mathfrak{P}}(A)$, then, by what we proved above, it follows that $\pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(A)$. Moreover, if $b \in B$ the subset $B \setminus \{b\}$ is not a transversal of $\mathrm{DIS}_{\mathfrak{P}}(A)$ by virtue of the minimality of B, therefore, it results that $\pi_{\mathfrak{P}}(B \setminus \{b\}) \neq \pi_{\mathfrak{P}}(A)$. Hence, $B \in \mathrm{BAS}_{\mathfrak{P}}(A)$ and the thesis follows.

(i) if and only if (v) Since

$$\mathrm{BAS}_{\mathfrak{P}}(A) = \{B \in \mathcal{P}(A) : M_{\mathfrak{P}}(B) = M_{\mathfrak{P}}(A),\ M_{\mathfrak{P}}(B') \subsetneq M_{\mathfrak{P}}(A)\ \forall B' \subsetneq B\},$$

we immediately deduce the equivalence between (i) and (v). □

We conclude by showing some further basic properties of the A-symmetry bases. In particular, we relate the A-symmetry bases to the set operator $C_{\mathfrak{P}}$.

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Theorem 5.4.

(i) $\mathrm{BAS}_{\mathfrak{P}}(A) \subseteq \max(\mathrm{MINP}_{\mathfrak{P}}(A))$ *for any* $A \in \mathcal{P}(\Omega)$.

(ii) $C_{\mathfrak{P}}(A) := \bigcap\{C : C \in \mathrm{BAS}_{\mathfrak{P}}(A)\}$.

(iii) *Let* $A \in \mathrm{MAXP}(\mathfrak{P})$. *Then,* $a \in A \setminus C_{\mathfrak{P}}(A)$ *if and only if there exists* $B \in \mathcal{P}(\Omega)$ *such that* $a \notin B$ *and* $a \in M_{\mathfrak{P}}(B) \subseteq M_{\mathfrak{P}}(A)$.

Proof. (i) Let $B \in \mathrm{BAS}_{\mathfrak{P}}(A)$, then $B \subseteq A$ and $B \in \min([A]_{\approx}) \subseteq \mathrm{MINP}(\mathfrak{P})$ by part (i). This proves that $B \in \mathrm{MINP}_{\mathfrak{P}}(A)$. Suppose by contradiction that there exists $C \in \mathrm{MINP}_{\mathfrak{P}}(A)$ such that $B \subsetneq C$. Then, $\pi_{\mathfrak{P}}(C) \preceq \pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(A)$. Moreover, we have that $C \in \mathrm{MINP}(\mathfrak{P})$, hence there exists $B \in \mathrm{MAXP}(\mathfrak{P})$ such that $C \in \min([B]_{\approx})$. Thus, $\pi_{\mathfrak{P}}(C) = \pi_{\mathfrak{P}}(B)$ and, since $C \subseteq A$, it results that $\pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(C) = \pi_{\mathfrak{P}}(B)$. Hence, $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(B)$, that is, $A = B$. Therefore, $C \in \min([A]_{\approx_{\mathfrak{P}}})$, so we conclude that $C \in \mathrm{BAS}_{\mathfrak{P}}(A)$, which is an absurd since it contains B. So $B \in \max(\mathrm{MINP}_{\mathfrak{P}}(A))$.

(ii) Let $a \in \bigcap\{C : C \in \mathrm{BAS}_{\mathfrak{P}}(A)\}$ and assume by contradiction that $a \notin C_{\mathfrak{P}}(A)$. This implies that $\pi_{\mathfrak{P}}(A) = \pi_{\mathfrak{P}}(A \setminus \{a\})$ and, in particular, the existence of a subset $B \subseteq A \setminus \{a\}$ belonging to $\mathrm{BAS}_{\mathfrak{P}}(A \setminus \{a\}$. But we also have $B \in \mathrm{BAS}_{\mathfrak{P}}(A)$, contradicting the fact that $a \in \bigcap\{C : C \in \mathrm{BAS}_{\mathfrak{P}}(A)\}$.

On the other hand, let $a \in C_{\mathfrak{P}}(A)$. Let $B \in \mathrm{BAS}_{\mathfrak{P}}(A)$ and assume by contradiction that $a \notin B$. This implies $B \subseteq A \setminus \{a\}$, so $\pi_{\mathfrak{P}}(A) \preceq \pi_{\mathfrak{P}}(A \setminus \{a\}) \preceq \pi_{\mathfrak{P}}(B) = \pi_{\mathfrak{P}}(A)$, but this would mean that $a \notin C_{\mathfrak{P}}(A)$, leading us to a contradiction.

(iii) Let $a \in A \setminus C_{\mathfrak{P}}(A)$. Then, just consider $B \in \mathrm{BAS}_{\mathfrak{P}}(A)$ such that $a \notin B$. Nevertheless, we have $a \in M_{\mathfrak{P}}(B) = M_{\mathfrak{P}}(A)$.

On the other hand, assume that $a \in A$ and let B not containing a and such that $a \in M_{\mathfrak{P}}(B) \subseteq M_{\mathfrak{P}}(A)$. Now, set $C := A \setminus \{a\}$. Clearly, $M_{\mathfrak{P}}(C) \subseteq M_{\mathfrak{P}}(A) = A$. Moreover, we have that $B \subseteq C$ and $a \in M_{\mathfrak{P}}(B) \subseteq M_{\mathfrak{P}}(C)$ so, we conclude that $A \subseteq M_{\mathfrak{P}}(C)$ and, hence $A = M_{\mathfrak{P}}(C)$. In this way, we have shown the existence of a subset $C \in [A]_{\approx_{\mathfrak{P}}}$ not containing a. Now, let us observe that $a \notin D$ for any $D \in \mathrm{BAS}_{\mathfrak{P}}(C)$. *A fortiori*, since $\mathrm{BAS}_{\mathfrak{P}}(C) \subseteq \mathrm{BAS}_{\mathfrak{P}}(A)$ and in view of the above part (ii), it follows that $a \in A \setminus C_{\mathfrak{P}}(A)$. □

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