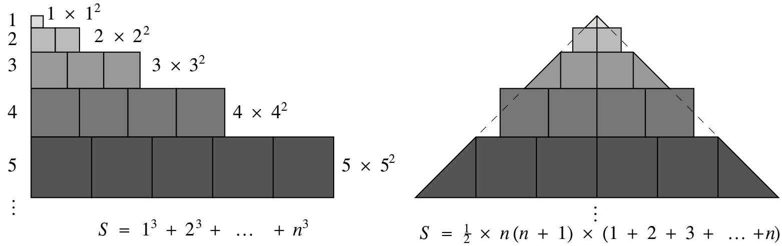
Notes

108.36 Visualising the sums of cubes by cutting and pasting

Here we present a visual demonstration of the well-known result that $1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2$. There are several of these arguments presented in the first two of Nelsen's books ([1, 2]), and they divide into two kinds: those which use a three-dimensional representation and those which stick to two dimensions ([3, 4, 5]). The following demonstration is clearly of the second type.



In the left-hand diagram, the cubes are represented as sums of squares, i.e. r^3 is shown as r squares of side r. If r is even, then the string of r squares are split into two strings of equal length, each string contains $\frac{1}{2}r$ squares. But, if r is odd, then the string of r squares are split into two equal strings by splitting the $\frac{1}{2}(r + 1)$-th square of the string diagonally.

In the right-hand diagram, the strings of the left-hand diagram are rearranged which neatly 'join up' so that the whole figure is a triangle of height $1 + 2 + 3 + \ldots + n$ and base $n(n + 1)$.

References

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BIKASH CHAKRABORTY
Nevanlinna Lab,
Department of Mathematics,
Ramakrishna Mission Vivekananda Centenary College,
Rahara, West Bengal 700 118, India
e-mail: *bikashchakraborty.math@yahoo.com,*
bikashchakrabortyy@gmail.com