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Equivalence Relations and Reductions

1.1 Generalities on Equivalence Relations

Let E be an equivalence relation on a set X. If $A \subseteq X$, we let $E \restriction A = E \cap A^2$ be its **restriction** to A. We also let $[A]_E = \{x \in X : \exists y \in A(xEy)\}$ be its ***E*-saturation**. The set A is ***E*-invariant** if $A = [A]_E$. In particular, for each $x \in X$, $[x]_E$ is the **equivalence class**, or ***E*-class**, of x. A function $f : X \to Y$ is ***E*-invariant** if $xEy \implies f(x) = f(y)$. Finally, $X/E = \{[x]_E : x \in X\}$ is the **quotient space** of X modulo E.

Suppose that E, F are equivalence relations on sets X, Y, respectively, and $f : (X/E)^n \to Y/F$, $n \geq 1$, is a function. A **lifting** of f is a function $\tilde{f} : X^n \to Y$ such that $f(([x_i]_E)_{i<n}) = [\tilde{f}((x_i)_{i<n})]_F, \forall x \in X$. Similarly if $R \subseteq (X/E)^n$, its lifting is $\tilde{R} \subseteq X^n$, where $(x_i)_{i<n} \in \tilde{R} \iff ([x_i]_E)_{i<n} \in R$.

If E_i, $i \in I$, is a family of equivalence relations, with E_i living on X_i, we define the **direct sum** $\bigoplus_i E_i$ to be the equivalence relation on $\bigoplus_i X_i = \{(x, i) : x \in X_i\}$ defined by

$$(x, j) \bigoplus_i E_i \ (y, k) \iff j = k \mathbin{\&} xE_j y.$$

In particular, we let for $n \geq 1$, $nE = \bigoplus_{i<n} E$. Also let $\mathbb{N}E = \bigoplus_{i \in \mathbb{N}} E$.

We define the **direct product** $\prod_i E_i$ to be the equivalence relation on the space $\prod_i X_i$ defined by

$$(x_j) \prod_i E_i \ (y_j) \iff \forall j (x_j E_j y_j).$$

In particular, we let for $n \geq 1$, $E^n = \prod_{i<n} E$. Also let $E^{\mathbb{N}} = \prod_{i \in \mathbb{N}} E$.

If E, F are equivalence relations on X, then $E \subseteq F$ means that E is a subset of F, when these are viewed as subsets of X^2, i.e., E is finer than F or equivalently F is coarser than E. The **index** of F over E, in symbols $[F : E]$, is the supremum of the cardinalities of the sets of E-classes contained in an

F-class. Thus $[F : E] \leq \aleph_0$ means that every F-class contains only countably many E-classes.

We denote by $\Delta_X = \{(x, y) : x = y\}$ the equality relation on a set X, and we also let $I_X = X^2$. Note that if $E_y = E$, $y \in Y$, where E is an equivalence relation on a set X, then $\bigoplus_y E_y = E \times \Delta_Y$.

If E_i, $i \in I$, are equivalence relations on X, we denote by $\bigwedge_i E_i = \bigcap_i E_i$ the largest (under inclusion) equivalence relation contained in all E_i and by $\bigvee_i E_i$ the smallest (under inclusion) equivalence relation containing each E_i. We call $\bigwedge_i E_i$ the **meet** and $\bigvee_i E_i$ the **join** of (E_i).

If E is an equivalence relation on X, a set $S \subseteq X$ is a **complete section** of E if S intersects every E-class. Moreover, if S intersects every E-class in exactly one point, then S is a **transversal** of E.

Consider now an action $a\colon G \times X \to X$ of a group G on a set X. We often write $g \cdot x = a(g, x)$, if there is no danger of confusion. Let $G \cdot x = \{g \cdot x : g \in G\}$ be the **orbit** of $x \in X$. The action a induces an equivalence relation E_a on X whose classes are the orbits, i.e., $xE_a y \iff \exists g(g \cdot x = y)$. When a is understood, sometimes the equivalence relation E_a is also denoted by E^X_G. The action a is **free** if $g \cdot x \neq x$ for every $x \in X, g \in G, g \neq 1_G$.

1.2 Morphisms

Let E, F be equivalence relations on spaces X, Y, resp. A map $f\colon X \to Y$ is a **homomorphism** from E to F if $xEy \implies f(x)Ff(y)$. In this case we write $f\colon (X, E) \to (Y, F)$ or just $f\colon E \to F$, if there is no danger of confusion. A homomorphism f is a **reduction** if moreover $xEy \iff f(x)Ff(y)$. We denote this by $f\colon (X, E) \leq (Y, F)$ or just $f\colon E \leq F$. Note that a homomorphism as above induces a map from X/E to Y/F, which is an injection if f is a reduction. In other words, a homomorphism is a lifting of a map from X/E to Y/F, and a reduction is a lifting of an injection of X/E into Y/F. An **embedding** is an injective reduction. This is denoted by $f\colon (X, E) \sqsubseteq (Y, F)$ or just $f\colon E \sqsubseteq F$. An **invariant embedding** is an injective reduction whose range is an F-invariant subset of Y. This is denoted by $f\colon (X, E) \sqsubseteq^i (Y, F)$ or just $f\colon E \sqsubseteq^i F$. Finally, an **isomorphism** is a surjective embedding. This is denoted by $f\colon (X, E) \cong (Y, F)$ or just $f\colon E \cong F$.

If a, b are actions of a group G on spaces X, Y, resp., a **homomorphism** from a to b is a map $f\colon X \to Y$ such that $f(g \cdot x) = g \cdot f(x), \forall g \in G, x \in X$. If f is injective, we call it an **embedding** of a to b.

1.3 The Borel Category

We are interested here in studying (classes of) Borel equivalence relations on **standard Borel spaces** (i.e., Polish spaces with the associated Borel structure). If X is a standard Borel space space and E is an equivalence relation on X, then E is Borel if E is a Borel subset of X^2.

Given a class of functions Φ between standard Borel spaces, we can restrict the above notions of morphism to functions in Φ, in which case we use the subscript Φ in the above notation (e.g., $f\colon E \to_\Phi F$, $f\colon E \leq_\Phi F$, etc.). In particular if Φ is the class of Borel functions, we write $f\colon E \to_B F, f\colon E \leq_B F, f\colon E \sqsubseteq_B F, f\colon E \sqsubseteq^i_B F, f\colon E \cong_B F$ to denote that f is a Borel morphism of the appropriate type. Similarly when we consider the underlying topology, we use the subscript c in the case where Φ is the class of continuous functions between Polish spaces and write $f\colon E \to_c F, f\colon E \leq_c F, f\colon E \sqsubseteq_c F, f\colon E \sqsubseteq^i_c F, f\colon E \cong_c F$.

We say that E is **Borel reducible** to F if there is a Borel reduction from E to F. In this case we write $E \leq_B F$. If $E \leq_B F$ and $F \leq_B E$, then E, F are **Borel bireducible**, in symbols $E \sim_B F$. Finally we let $E <_B F$ if $E \leq_B F$ but $F \not\leq_B E$. Similarly we define the notions of E being **Borel embeddable** to F and E being **Borel invariantly embeddable** to F, for which we use the notations $E \sqsubseteq_B F$ and $E \sqsubseteq^i_B F$, respectively. Also we use $E \simeq_B F, E \simeq^i_B F$ for the corresponding notions of being **Borel biembeddable** and **Borel invariantly biembeddable** and $E \sqsubset_B F$ and $E \sqsubset^i_B F$ for the corresponding strict notions. More generally, if Φ is as above, we analogously define $E \leq_\Phi F, E \sqsubseteq_\Phi F$, etc.

Finally E, F are **Borel isomorphic**, in symbols $E \cong_B F$, if there is a Borel isomorphism from E to F. Note that by the usual (Borel) Schröder–Bernstein argument, E, F are Borel isomorphic if and only if they are Borel invariantly biembeddable, i.e., $\simeq^i_B = \cong_B$.