J. Austral. Math. Soc. (*Series A*) **48** (1990), 87–88

GROWTH SEQUENCE OF GLOBALLY IDEMPOTENT SEMIGROUPS

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(Received 7 December 1988)

Communicated by H. Lausch

Abstract

The n-th member of the growth sequence of a globally idempotent finite semigroup without identity element is at least 2^n. (This had been conjectured by J. Wiegold.)

1980 *Mathematics subject classification* (*Amer. Math. Soc.*) (1985 *Revision*): 20 M 99.

In [2] Wiegold proved that the growth sequence of a semigroup S without identity is exponential, and put forward the conjecture that it increases at least as fast as 2^n. Moreover, he showed that this holds, if $S \cdot S \neq S$. In this note we prove the estimate for the remaining case.

We recall the definition of growth sequence.

DEFINITION. Let S be a finite semigroup, and let $d(S^n)$ denote the minimum number of generators of the nth direct power of S. The sequence $(d(S), d(S^2), \ldots)$ is called the *growth sequence* of S.

Recall also that S is said to be globally idempotent if $S \cdot S = S$. For other definitions we refer to [1].

THEOREM. *If S is a globally idempotent finite semigroup without identity then $d(S^n) \geq 2^n$.*

This work was partially supported by the Hungarian National Foundation for Scientific Research, grant No. 1813.

Proof. First let S be completely (0-) simple. Then S is the (0-) direct union of (0-) minimal left as well as of (0-) minimal right ideals. Since S is not a group (with 0), at least one of these unions has more than one member. Suppose that $S = L_1 \cup \cdots \cup L_k$, $k > 1$, each L_i being a left ideal, let $e = (\varepsilon_1, \ldots, \varepsilon_n) \in \{1, \ldots, k\}^n$, and put $\Lambda_e = \prod_{i=1}^n (L_{\varepsilon_i} \setminus \{0\})$. If G is a system of generators of S^n then a product $g_1 \cdots g_r$ ($g_j \in G$) is contained in Λ_e only if g_r is, because $S^n \setminus \Lambda_e$ is a left ideal in S^n. As $\Lambda_e \cap \Lambda_f = \varnothing$ if $e \neq f$, we have $|G| \geq k^n$ (even that $|G \setminus \{0\}| \geq k^n|$).

Next consider the case of a 0-direct union of groups: $S = \bigcup_{i=1}^k H_i \cup \{0\}$, $k > 1$. Let G and e be as above, and put $\Gamma_e = \prod_{i=1}^n H_{\varepsilon_i}$. Then $g_1 \cdots g_r \in \Gamma_e$ if and only if $g_i \in \Gamma_e$ for each i; thus, we have again at least k^n generators.

Finally, suppose that S has a non-trivial ideal. If M is a maximal ideal in S then S/M is globally idempotent and hence completely simple. Let J be a maximal ideal such that S/J is not a group with 0 if such an ideal exists and put $J = \bigcap_{i=1}^k M_i$ in the opposite case, where $M_1, \ldots, M_k$ is a full list of maximal ideals of S. Sure enough, $k > 1$ in this latter case, for otherwise S has an identity element, and S/J is a 0-direct union of k groups with 0 (in the other case, that is, if S/J is completely simple but not a group, k (> 1) will denote the number of 0-minimal left (or right) ideals of S/J).

Now let G be a system of generators of S^n and put $G' = G \cap (S^n \setminus J^n)$. Then $G' \cup \{J\}$ generates S^n/J^n. But $S^n/J^n \cong (S/J)^n$, whence $|G'| \geq k^n$, which proves the assertion.

The author is indebted to the sisters Szendrei for calling his attention to a gap in the proof in the original version of this paper.

References

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